ORTHOGONAL POLYNOMIALS RELATED TO THE
OSCILLATORY-CHEBYSHEV WEIGHT FUNCTION\textsuperscript{1}

G. V. MILOVANOVIĆ, A. S. CVETKOVIC

(Presented at the 8th Meeting, held on November 26, 2004)

Abstract. In this paper we discuss the existence question for polynomials orthogonal with respect to the moment functional

\[ L(p) = \int_{-1}^{1} p(x)x(1 - x^2)^{-1/2}e^{\imath \zeta x} dx, \; \zeta \in \mathbb{R}. \]

Since the weight function alternates in sign in the interval of orthogonality, the existence of orthogonal polynomials is not assured. A nonconstructive proof of the existence is given. The three-term recurrence relation for such polynomials is investigated and the asymptotic formulae for recursion coefficients are derived.

AMS Mathematics Subject Classification (2000): Primary 30C10, 33C47

Key Words: Orthogonal polynomials; Moments; Moment functional; Three-term recurrence relation; Oscillatory Chebyshev weight; Asymptotic formulae; Bessel functions

\textsuperscript{1}The authors were supported in part by the Serbian Ministry of Science, Technology and Development (Project #2002: Applied Orthogonal Systems, Constructive Approximation and Numerical Methods).
1. Introduction

Let \( P \) be the space of all algebraic polynomials and \( P_n \) be the linear space of all algebraic polynomials of degree at most \( n \).

Let a linear functional \( L \) be given on the linear space of all algebraic polynomials, i.e., let the functional \( L \) satisfy following equality, for each \( P, Q \in P \),

\[
L(\alpha P + \beta Q) = \alpha L(P) + \beta L(Q), \quad \alpha, \beta \in \mathbb{C}.
\]

The value of the linear functional \( L \) at every polynomial is known if the values of \( L \) are known at the set of all monomials, due to linearity. The corresponding values of the linear functional \( L \) at the set of monomials are called the moments and we denote them by \( \mu_k, k \in \mathbb{N}_0 \),

\[
L(x^k) = \mu_k, \quad k \in \mathbb{N}_0.
\]

In [3, p. 7], the following definition can be found.

**Definition 1** A sequence of polynomials \( \{P_n(x)\}_{n=0}^{+\infty} \) is called the polynomial sequence orthogonal with respect to the moment functional \( L \), provided for all nonnegative integers \( m \) and \( n \),

- \( P_n(x) \) is polynomial of degree \( n \),
- \( L(P_n(x)P_m(x)) = 0 \), if \( m \neq n \),
- \( L(P_n^2(x)) \neq 0 \).

If the sequence of orthogonal polynomials exists for a given linear functional \( L \), then \( L \) is called quasi-definite or regular linear functional. Under the condition \( L(P_n^2(x)) > 0 \), the functional \( L \) is called positive definite (see [3]).

Using only linear algebraic tools the following theorem can be stated (see [3, p. 11]).

**Theorem 1.** The necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials with respect to the linear functional \( L \) are that for each \( n \in \mathbb{N} \) the Hankel determinants

\[
\Delta_n = \begin{vmatrix}
\mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{n-1} \\
\mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\
\mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-2}
\end{vmatrix} \neq 0. \quad (1)
\]
Orthogonal polynomials related to the oscillatory-Chebyshev weight function

In this paper we consider the linear functional $L$ given by

$$L(p) = \int_{-1}^{1} p(x)x(1 - x^2)^{-1/2}e^{i\zeta x}dx,$$

(2)

where $\zeta \in \mathbb{R}$. In order to prove the existence of the corresponding orthogonal polynomials, we need to compute the sequence of moments $\mu_k$, $k \in \mathbb{N}_0$, and then to prove that all Hankel determinants are different from zero. The case of orthogonality with respect to the linear functional $L(p) = \int_{-1}^{1} p(x)xe^{im\pi x}dx$, $m \in \mathbb{N}$, was investigated in [8].

Denote the sequence of moments by $\mu_k(\zeta)$, $k \in \mathbb{N}_0$. Then, for each $k \in \mathbb{N}_0$, we can easily verify that

$$\mu_k(\zeta) = \int_{-1}^{1} x^{k+1}(1 - x^2)^{-1/2}e^{i\zeta x}dx = \int_{-1}^{1} x^{k+1}(1 - x^2)^{-1/2}e^{-i\zeta x}dx = \mu_k(-\zeta).$$

This means that we need only to discuss the case $\zeta > 0$, since the corresponding results for $\zeta < 0$ can be obtained by a simple conjugation. We exclude the case $\zeta = 0$ since for this case $\mu_0 = \Delta_0 = 0$, so that the linear functional $L$ is not regular. In what follows we assume $\zeta > 0$. Let $J_\nu$ be the Bessel function of the order $\nu$ defined by (cf. [14, p. 40])

$$J_\nu(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m(z/2)^{\nu+2m}}{m!\Gamma(\nu + m + 1)}.$$  

(3)

**Theorem 2.** The sequence of moments satisfy the following recurrence relation

$$\mu_{k+2} = -\frac{k+2}{i\zeta} \mu_{k+1} + \mu_k + \frac{k+1}{i\zeta} \mu_{k-1}, \quad k \in \mathbb{N},$$

(4)

with the initial conditions

$$\mu_0 = i\pi J_1(\zeta), \quad \mu_1 = \frac{\pi}{\zeta}(\zeta J_0(\zeta) - J_1(\zeta)), \quad \mu_2 = \frac{i\pi}{\zeta^2}(\zeta J_0(\zeta) + (\zeta^2 - 2)J_1(\zeta)).$$

**Proof:** We start with the following simple equality

$$\int_{-1}^{1} x^{k+1}(1 - x^2)(1 - x^2)^{-1/2}e^{i\zeta x}dx = \mu_k - \mu_{k+2}, \quad k \in \mathbb{N}_0.$$

If we apply integration by parts to the integral which appears in the previous equality, we get

$$\int_{-1}^{1} x^{k+1}\sqrt{1 - x^2}e^{i\zeta x}dx = \frac{1}{i\zeta} \left[ x^{k+1}\sqrt{1 - x^2}e^{i\zeta x} \right]_{-1}^{1} + \frac{1}{i\zeta} \int_{-1}^{1} \frac{x^{k+2}}{\sqrt{1 - x^2}}e^{i\zeta x}dx.$$
\[
-k + 1 + \int_{-1}^{1} \frac{x^k - x^{k+2}}{\sqrt{1-x^2}} e^{i\zeta x} \, dx
= \frac{1}{i\zeta} \mu_{k+1} - \frac{k + 1}{i\zeta} (\mu_{k-1} - \mu_{k+1}), \quad k \in \mathbb{N},
\]
so that we have
\[
\mu_{k+2} = -\frac{k + 2}{i\zeta} \mu_{k+1} + \mu_k + \frac{k + 1}{i\zeta} \mu_{k-1}, \quad k \in \mathbb{N}.
\]

In order to start the recursion we need to compute the moments \(\mu_0, \mu_1\) and \(\mu_2\). Because of the symmetry argument, we see that
\[
\mu_0 = i \int_{-1}^{1} x(1 - x^2)^{-1/2} \sin \zeta x \, dx,
\]
\[
\mu_1 = \int_{-1}^{1} x^2(1 - x^2)^{-1/2} \cos \zeta x \, dx,
\]
\[
\mu_2 = i \int_{-1}^{1} x^3(1 - x^2)^{-1/2} \sin \zeta x \, dx,
\]
The series expansion for the function \(e^{i\zeta x}\) is valid for \(|\zeta x| < +\infty\), and it converges uniformly everywhere in the complex plane (see [6]). This gives an opportunity to integrate term by term (see [13]), so that we have
\[
\mu_0 = \sum_{k=0}^{+\infty} \frac{i\zeta^k}{k!} \int_{-1}^{1} \frac{x^{k+1}}{\sqrt{1-x^2}} \, dx = \sqrt{\pi} \sum_{k=0}^{+\infty} \frac{i\zeta^k}{k!} \frac{1 + (-1)^{k+1}}{2} \frac{\Gamma((k+2)/2)}{\Gamma((k+3)/2)}
\]
\[
= i\pi \sum_{k=0}^{+\infty} (-1)^k \frac{\zeta^{2k+1}}{(2k+1)!} \frac{(2k+1)!!}{(2k+2)!!} = i\pi \sum_{k=0}^{+\infty} (-1)^k \frac{(\zeta/2)^{2k+1}}{k!(k+1)!} = i\pi J_1(\zeta),
\]
where \(J_1(\zeta)\) is the Bessel function of the order one. Here, we used the known expressions for the moments of the Chebyshev weight of the first kind given in [7] and the series expansion for the Bessel function given in (3). Also, we used the simple fact that
\[
\sqrt{\pi} \frac{1 + (-1)^{k+1}}{2} \frac{\Gamma((k+2)/2)}{\Gamma((k+3)/2)} = \frac{(2n+1)!!}{(2n+2)!!}, \quad k = 2n + 1,
\]
which can be verified by the induction argument. Using the same method we derive the corresponding expression for the moments \(\mu_1\) and \(\mu_2\). \(\square\)
To explore further the moment sequence we adopt the following notation

$$\mu_k = \frac{i\pi}{(i\zeta)^k} (P_k J_1 + \zeta Q_k J_0), \quad k \in \mathbb{N}_0.$$ (6)

We have the following statement.

**Theorem 3.** The polynomials $P_k$ and $Q_k$ in $\zeta^2$ with integer coefficients of degrees $2\lfloor k/2 \rfloor$ and $2\lfloor (k − 1)/2 \rfloor$, respectively, satisfy the following recurrence relation

$$y_{k+2} = -(k + 2)y_{k+1} - \zeta^2 y_k - (k + 1)\zeta^2 y_{k-1},$$

with initial conditions

$$P_0 = 1, \quad P_1 = -1, \quad P_2 = 2 - \zeta^2,$$
$$Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = -1.$$

The term with $\zeta^0$ in $P_k$ is equal to $(-1)^k k!, \quad k \in \mathbb{N}_0$.

**Proof:** Putting (6) into the recurrence relation for the moments (4), we get

$$\frac{i\pi}{(i\zeta)^{k+2}} (P_{k+2} J_1 + \zeta Q_{k+2} J_0) = -\frac{i\pi(k + 2)}{(i\zeta)^{k+2}} (P_{k+1} J_1 + \zeta Q_{k+1} J_0)$$

$$-\frac{i\pi\zeta^2}{(i\zeta)^{k+2}} (P_k J_1 + \zeta Q_k J_0) - \frac{(k + 1)i\pi\zeta^2}{(i\zeta)^{k+2}} (P_{k-1} J_1 + \zeta Q_{k-1} J_0).$$

Since the equation is valid for all real $\zeta \neq 0$ and the functions $J_0$ and $J_1$ are linearly independent, it is just enough to read term with the functions $J_0$ and $J_1$ to obtain the recurrence relation stated.

The initial conditions for the moments give $P_0 = 1, \quad P_1 = -1, \quad P_2 = 2 - \zeta^2$ and $Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = -1$, for polynomials $P_k$ and $Q_k$, respectively.

Now, obviously $P_0, \ P_1$ and $P_2$ are real polynomials with integer coefficients. According to the recurrence, $P_3$ is also real polynomial in $\zeta$ and has degree 2, i.e.,

$$P_3 = -(1 + 2)P_2 - \zeta^2 P_1 - \zeta^2 P_0 = -6 + 2\zeta^2.$$

Suppose that $P_{k-1}, \ P_k$ and $P_{k+1}$ are real polynomials in $\zeta$, with degrees $2\lfloor (k-1)/2 \rfloor, \ 2\lfloor k/2 \rfloor$ and $2\lfloor (k+1)/2 \rfloor$, respectively. Then, using the recurrence relation

$$P_{k+2} = -(k + 2)P_{k+1} - \zeta^2 P_k - (k + 1)\zeta^2 P_{k-1},$$
we deduce the following recurrence
\[ A_{2n+2} = -A_{2n}, \quad A_{2n+3} = -(2n+3)A_{2n+2} - A_{2n+1} - (2n+2)A_{2n}, \quad n \in \mathbb{N}_0, \]
for the leading coefficient in \( P_k, k \in \mathbb{N}_0 \), with initial conditions \( A_0 = 1, \quad A_1 = -1, \quad A_2 = -1 \). It can be checked by a direct calculation that we have the solution \( A_{2n} = (-1)^n, \quad A_{2n+1} = (-1)^{n+1}(n+1), \quad n \in \mathbb{N}_0 \).

For the \( Q \)-sequence we have
\[ Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = -1, \quad Q_3 = 3 - \zeta^2. \]

We can easily verify the degrees of these initial polynomials. Suppose that \( Q_{k-1}, Q_k \) and \( Q_{k+1} \) are polynomials with degrees \( 2[(k-2)/2], 2[(k-1)/2] \) and \( 2[k/2] \), respectively. Using the recurrence for these polynomials, we can obtain the corresponding recurrence for their leading coefficients. Namely, we have
\[ A_{2n+3} = -A_{2n+1}, \quad A_{2n+2} = -(2n+2)A_{2n+1} - A_{2n} - (2n+1)A_{2n-1}, \quad n \in \mathbb{N}_0, \]
with initial conditions \( A_0 = 0, \quad A_1 = 1, \quad A_2 = -1 \). It can be checked directly that the solution is \( A_{2n+1} = (-1)^n \) and \( A_{2n} = (-1)^{n+1}n, \quad n \in \mathbb{N}_0 \).

To prove the statement on the coefficient \( A_k \) with \( \zeta^0 \) in \( P_k, k \in \mathbb{N}_0 \), we use the initial conditions \( A_0 = 1, \quad A_1 = -1, \) and \( A_2 = 2 \). Then, by the recurrence relation, we obtain
\[ A_{k+2} = -(k+2)A_{k+1}, \quad k \in \mathbb{N}, \]
so that we conclude easily \( A_k = (-1)^k k!, \quad k \in \mathbb{N}_0 \). \( \Box \)

Can we say anything about the existence of orthogonal polynomials? To illustrate the problem, we can calculate the Hankel determinant \( \Delta_2 \) in the form
\[ \Delta_2 = \frac{\pi^2 J_1^2(\zeta)}{\zeta^2} \left( -\zeta^2 \frac{J_0^2(\zeta)}{J_1^2(\zeta)} + \zeta \frac{J_0(\zeta)}{J_1(\zeta)} + 1 - \zeta^2 \right). \]

It is easy to conclude that \( \Delta_2 = 0 \), provided
\[ \frac{J_0(\zeta)}{J_1(\zeta)} = \frac{1 \pm \sqrt{5 - 4\zeta^2}}{2\zeta}, \]
but any solution must be real so it must be \( |\zeta| < \sqrt{5}/2 \). A careful numerical inspection shows that we cannot find solution for this equation, and it seems that it does not exist in the set of real numbers.
However, this is not the case with $\Delta_3$. Using some computer algebra, it can be checked easily that

$$\Delta_3 = \frac{i\pi^3 J_1^3}{\zeta^6} \left( 7\zeta^3 \frac{J_0^3}{J_1^3} + (2\zeta^2 - 21)\zeta^2 \frac{J_0^2}{J_1^2} + \zeta (5\zeta^2 + 12) \frac{J_0}{J_1} + 2\zeta^4 - 15\zeta^2 + 4 \right).$$

The smallest positive solution for the equation $\Delta_3 = 0$ is given by

$$\zeta = 6.459008151994783455531721397032502543805710669120882 \ldots,$$

so that for this particular number, the sequence of orthogonal polynomials does not exist.

However, there is the way to ensure the existence of orthogonal polynomials. Choose $\zeta$ to be any positive zero of the Bessel function $J_0(\zeta)$. Then our sequence of moments becomes

$$\mu_k = \frac{i\pi}{(i\zeta)^k} P_k J_1(\zeta).$$

Because of the interlacing property of the positive zeros of the Bessel functions (see [14, p. 479]), we know that $J_1(\zeta) \neq 0$.

**Theorem 4.** Suppose $\zeta$ is a positive zero of the Bessel function $J_0$. Then the sequence of polynomials orthogonal with respect to the functional $L$, given by (2), exists.

**Proof:** We give the proof of this statement using the fact that all zeros of the Bessel function $J_0$ are transcendental numbers [10] (see also [11], [4], [12]). Our sequence of moments is given by

$$\mu_k = \frac{i\pi}{(i\zeta)^k} P_k J_1(\zeta), \quad k \in \mathbb{N}_0,$$

where we know the basic properties of the polynomials $P_k$, $k \in \mathbb{N}_0$, stated in Theorem 3.

Consider the Hankel determinants $\Delta_k$, $k \in \mathbb{N}$, given in (1). We have to prove that the determinants $\Delta_k \neq 0$, $k \in \mathbb{N}_0$. Then, according to Theorem 1, the sequence of orthogonal polynomials exists. Consider determinant $\Delta_k$ and extract from every of its rows the factor $i\pi J_1(\zeta)$. Denoting the obtained determinant with $\Delta'_k$, we have

$$\Delta_k = (i\pi J_1(\zeta))^{k+1} \Delta'_k.$$
Now, we can consider the determinant $\Delta'_k$ as a Hankel determinant for the sequence of moments $\mu'_\nu = (i\zeta)^{\nu} P_\nu, \nu \in \mathbb{N}_0$. If we now extract from every $\nu$-th row the factor $1/(i\zeta)^{\nu-1}$ and after that from $j$-th column the factor $1/(i\zeta)^{j-1}$, we obtain new determinant $\Delta''_k$ and equality

$$\Delta_k = (i\pi J_1(\zeta))^{k+1} \Delta'_k = \frac{(i\pi J_1(\zeta))^{k+1}}{(i\zeta)^{k(k+1)}} \Delta''_k.$$  

The determinant $\Delta'_k$ is the Hankel determinant for the sequence of moments $\mu'_\nu = P_\nu, \nu \in \mathbb{N}_0$. Hence, the value of $\Delta''_k$ is certain polynomial in $\zeta^2$, since all its elements are polynomials in $\zeta^2$. Since $\zeta$ is transcendental number, the polynomial with integer coefficients $\Delta''_k$ cannot be zero at $\zeta$, because all its zeros must be algebraic numbers. There is only one possibility for $\Delta''_k$ to have $\zeta$ as its zero, if the polynomial $\Delta''_k$ is identically zero.

Thus, we have to prove that $\Delta'_k$ is not identically zero. Since $\Delta'_k$ is a polynomial in $\zeta$ and all its coefficients are polynomials in $\zeta$, the term with $\zeta^0$ of the polynomial $\Delta''_k$ equals the Hankel determinant which elements are terms with $\zeta^0$ in the polynomials $P_\nu, \nu \in \mathbb{N}_0$. According to Theorem 3, we know that $P_\nu(0) = (-1)^{\nu} \nu!$, $\nu \in \mathbb{N}_0$, so that the term with $\zeta^0$ in the polynomial $\Delta''_k$ equals the Hankel determinant $\hat{\Delta}_k$ for the sequence of moments $\tilde{\mu}_\nu = (-1)^{\nu} \nu!$, $\nu \in \mathbb{N}_0$.

If we extract $-1$ from the rows $2\nu + 1, \nu = 0, 1, \ldots, 2[k/2]$, and from the columns $2j + 1, j = 0, 1, \ldots, 2[k/2]$, we get the Hankel determinant $\tilde{\Delta}_k$ for the sequence of moments $\tilde{\mu}_\nu = \nu!, \nu \in \mathbb{N}_0$, where the following equation holds

$$\tilde{\Delta}_k = \hat{\Delta}_k.$$  

Now, it is easy to recognize the sequence of moments $\tilde{\mu}_\nu = \nu!$, as the sequence of moments for the Laguerre measure (see [7]). But, then it is easy to compute $\tilde{\Delta}_k$,

$$\tilde{\Delta}_k = \hat{\Delta}_k = \prod_{\nu=0}^{k} (\nu!)^2.$$  

This means that $\Delta''_k$ is not a polynomial which is identically equal to zero. Hence $\Delta''_k \neq 0$, which implies that $\Delta_k \neq 0$. The previous discussion is valid for any $k \in \mathbb{N}$, which means that $\Delta_k \neq 0, k \in \mathbb{N}$. □

In the rest of this paper we denote by $p_n, n \in \mathbb{N}_0$, the orthonormal sequence of polynomials with respect to the linear functional $L$, and by $\pi_n, n \in \mathbb{N}_0$, we denote the monic version of this polynomial sequence.
2. Asymptotic formulae

First we prove one auxiliary result, explaining the asymptotic properties of polynomials $Q_n$ orthogonal with respect to the weight function $w$, defined by $w(x) = \chi_{[-1,1]}(x)(1 - x^2)^{-1/2} e^{i \zeta x}$.

Since this weight function is in Magnus class of the complex weight functions (see [5]), the polynomials $Q_n$, $n \in \mathbb{N}_0$, exist asymptotically. Also, their three-term recurrence coefficients have the asymptotic behavior of the class $M(0,1)$ introduced and studied in [9]. Denote the three-term recurrence coefficients for the corresponding orthonormal polynomials by $\alpha_n^Q$ and $\beta_n^Q$. Then we have

$$\lim_{n \to +\infty} \alpha_n^Q = 0, \quad \lim_{n \to +\infty} \beta_n^Q = \frac{1}{2}.$$ 

Actually, using some recent results, we know even more.

**Lemma 1.** For the monic polynomial $Q_n$, $n \in \mathbb{N}_0$, orthogonal with respect to the weight function $\omega(x) = \chi_{[-1,1]}(x)(1 - x^2)^{-1/2} \exp(i \zeta x)$, we have the following asymptotic formula

$$\frac{Q_{n+1}(0)}{Q_n(0)} = \frac{1}{2} \cosh \frac{\zeta - i(n + 1)\pi}{2} \cosh \frac{\zeta - in\pi}{2} + O(q^n),$$

where $0 < q < 1$.

**Proof.** We use the following result proved in [1]. Suppose $h$ is a complex function being analytic in some neighborhood of the interval $[-1,1]$, which is different from zero on the interval $[-1,1]$. Then, the monic orthogonal polynomials $Q_n$ with respect to $h(x) \chi_{[-1,1]}(x)(1 - x^2)^{-1/2}$ exist asymptotically, and

$$\gamma_n Q_n(x) = \varphi_+(x) + \varphi_-(x) + O(q^n), \quad x \in [-1,1], \quad 0 < q < 1,$n

where

$$\gamma_n^{-1} = 2^{-n} \exp \left( \frac{1}{2\pi} \int_{-1}^{1} \log h(x) (1 - x^2)^{-1/2} \, dx \right)$$

and

$$\varphi(z) = (z + (z^2 - 1)^{1/2})^n \exp \left( -\frac{1}{2\pi} (z^2 - 1)^{1/2} \int_{-1}^{1} \frac{\log h(x)}{z - x} \frac{dx}{(1 - x^2)^{1/2}} \right).$$
Here, \( \varphi_+ (x) \) is the limit of \( \varphi(z) \) as \( z \) approaches \( x \in [-1, 1] \) over the upper half-plane of the complex \( z \) plane and \( \varphi_- (x) \) is the limit of \( \varphi(z) \) as \( z \) approaches \( x \in [-1, 1] \) from the lower half-plane of the complex \( z \) plane. The square root is chosen such that it has cut along the interval \([-1, 1]\) and it behaves as \( z \) as \( z \) approaches \( \infty \).

We use \( x = \cos \theta \in [-1, 1], \theta \in [0, \pi] \). First, we calculate the integral which appears in the \( \varphi \) function for \( \text{Im}(z) > 0 \). So we have

\[
\int_{-1}^{1} \frac{i \zeta x}{z - x} \frac{dx}{(1 - x^2)^{1/2}} = -i \pi \zeta + i \zeta \int_{-1}^{1} \frac{1}{z - x} \frac{dx}{(1 - x^2)^{1/2}}
\]

\[
= -i \pi \zeta + i \zeta \int_{-1}^{1} \frac{dx}{(1 - x^2)^{1/2}} \sum_{k=0}^{+\infty} \frac{x^k}{z^k}
\]

\[
= -i \pi \zeta + i \zeta \sum_{k=0}^{+\infty} \frac{m_k^C}{z^k}
\]

\[
= -i \pi \zeta \left( 1 - \frac{1}{(1 - z^{-2})^{1/2}} \right),
\]

where \( m_k^C \) are the moments for Chebyshev weight of the first kind. This gives

\[
\varphi_+ (x) = e^{i n \theta} \exp \left( -\frac{\zeta}{2} (\sin \theta + i \cos \theta) \right)
\]

and

\[
\varphi_- (x) = e^{-i n \theta} \exp \left( \frac{\zeta}{2} (\sin \theta - i \cos \theta) \right).
\]

Using the mentioned result, we can calculate directly

\[
\gamma_n Q_n (x) = e^{i n \theta} \exp \left( -\frac{\zeta}{2} (\sin \theta + ix) \right) + e^{-i n \theta} \exp \left( \frac{\zeta}{2} (\sin \theta - ix) \right) + O(q^n)
\]

and also, for \( \theta = \pi/2 \), we have

\[
\frac{Q_{n+1}(0)}{Q_n(0)} = \frac{1}{2} \frac{\cosh \frac{\zeta - i(n + 1)\pi}{2}}{\cosh \frac{\zeta - i n \pi}{2}} + O(q^n). \quad \Box
\]

Using the polynomials \( Q_n, n \in \mathbb{N}_0 \), we can express the polynomials \( \pi_n, n \in \mathbb{N}_0 \), and the corresponding three-term recurrence coefficients. Thus, we have the following statement:
Theorem 5. Suppose the sequence of orthogonal polynomials $Q_n$ exists for $n > N_Q$. Then, for polynomials orthogonal with respect to the functional $L$ given by (2), we have

$$\pi_n(x) = \frac{1}{x} (Q_{n+1}(x) - \gamma_n Q_n(x)), \quad n > N_Q, \quad (7)$$

where

$$\gamma_n = \frac{Q_{n+1}(0)}{Q_n(0)}, \quad n > N_Q.$$

The recursion coefficients for the sequence $\pi_n$, $n \in \mathbb{N}_0$, in the recurrence relation can be expressed in the following form

$$\alpha_n = -\gamma_n - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}, \quad \beta_{n+1}^2 = \frac{\gamma_{n+1}}{\gamma_n} (\beta_{n+1}^Q)^2, \quad n > N_Q.$$

Proof. For the (monic) orthogonal polynomials $Q_k$, $z \in \mathbb{C}, Q_k(z) \neq 0$, $k \in \mathbb{N}$, the polynomials

$$\pi_n(x; z) = \frac{1}{x - z} \left( Q_{n+1}(x) - \frac{Q_{n+1}(z)}{Q_n(z)} Q_n(x) \right)$$

are known as the kernel polynomials (cf. [3]). Several results are known in the case when the point $z$ is not in the supporting set of the measure of orthogonality and provided the sequence $Q_n$ exists. In our case $z = 0$ and, therefore, we give the proof here.

Thus, supposing that the sequence of polynomials is given by (7), we have

$$\int_{-1}^{1} x^n \pi_n(x) \frac{xe^{i\zeta x}}{\sqrt{1 - x^2}} dx = \int_{-1}^{1} x^n (Q_{n+1}(x) - \gamma_n Q_n(x)) \frac{e^{i\zeta x}}{\sqrt{1 - x^2}} dx = 0,$$

provided $n > \nu$. According to the uniqueness property, up to a multiplicative constant, the polynomials (7) are orthogonal with respect to the weight function $x(1 - x^2)^{-1/2} \exp(i\zeta x) \chi_{[-1,1]}$. We note that $\gamma_n = Q_{n+1}(0)/Q_n(0)$.

If we assume that polynomials $Q_n$, $n > N_Q$, satisfy the following three-term recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n^Q) Q_n(x) - (\beta_n^Q)^2 Q_{n-1}(x), \quad n > N_Q + 1,$$

it is clear that

$$\gamma_{n+1} = -\alpha_{n+1}^Q - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}, \quad n > N_Q.$$
Also, for polynomials $\pi_n, n > N_Q$, we have

\[
x\pi_{n+1} = Q_{n+2} - \gamma_{n+1}Q_{n+1}
\]

\[
= (x - \alpha_{n+1}^Q)Q_{n+1} - (\beta_{n+1}^Q)^2Q_n - \left(-\alpha_{n+1}^Q - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)Q_{n+1}
\]

\[
= \left(x + \gamma_n + \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)Q_{n+1} - \gamma_nQ_{n+1} - (\beta_{n+1}^Q)^2Q_n
\]

\[
= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2Q_{n-1}
\]

\[
+ \left(-\gamma_n(x - \alpha_n^Q) - (\beta_{n+1}^Q)^2 + \gamma_n\left(x + \gamma_n + \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)\right)Q_n
\]

\[
= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2Q_{n-1} + \gamma_n\left(\alpha_n^Q + \gamma_n\right)Q_n
\]

\[
= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2Q_{n-1} - \frac{\gamma_n}{\gamma_{n-1}}(\beta_n^Q)^2Q_n
\]

\[
= (x - \alpha_n)x\pi_n - \frac{\gamma_n}{\gamma_{n-1}}(\beta_n^Q)^2x\pi_{n-1},
\]

wherefrom we read directly the corresponding expressions for the three-term recurrence coefficients. \(\square\)

**Theorem 6.** For the three-term recurrence coefficients of the polynomial sequence $p_n, n \in \mathbb{N}_0$, we have the following asymptotic formulae

\[
\alpha_{2n+k} \to \frac{i(-1)^k}{2} \left(\tanh \frac{\zeta}{2} - \coth \frac{\zeta}{2}\right), \quad n \to +\infty, \quad k = 0, 1,
\]

and

\[
\beta_{2n+k}^2 \to \frac{1}{4} \left(\tanh^2 \frac{\zeta}{2}\right)(-1)^k, \quad n \to +\infty, \quad k = 0, 1.
\]

**Proof.** This theorem is a direct consequence of Lemma 1 and Theorem 5. \(\square\)

**Theorem 7.** Let $J$ be the associated Jacobi operator created using three-term recurrence coefficients $\alpha_k$ and $\beta_k, k \in \mathbb{N}_0$. Then, we have

\[
\sigma_{\text{ess}}(J) = [-1, 1].
\]
Proof. According to Theorem presented in [2], we know that the essential spectrum of the Jacobi matrix, with periodic three-term recurrence coefficients of the basic period $m$, can be obtained as an inverse image of the interval $[-2, 2]$ of the mapping

$$h(x) = \frac{p_{2m-1}}{p_{m-1}}.$$ 

In our case we have $m = 2$. Suppose that we give the sequences of three-term recurrence coefficients $\alpha_{2n+k} = a_k$, $\beta_{2n+k} = b_k$, $k = 0, 1$. Then, we can express $h(x)$ as

$$b_0 b_1 h(x) = x^2 - (a_0 + a_1) x + a_0 a_1 - b_0^2 - b_1^2.$$ 

For our case

$$a_0 = \frac{i}{2} \left( \tanh \frac{\zeta}{2} - \coth \frac{\zeta}{2} \right) = -a_1$$
and

$$b_0 = \frac{1}{2} \tanh \frac{\zeta}{2} = \frac{1}{4b_1}.$$ 

We get easily $h(x) = 4(x^2 - 1/2)$. The inverse mappings of $h$ are the mappings

$$h_1^{-1} = \frac{(h + 2)^{1/2}}{2}, \quad h_2^{-1} = -\frac{(h + 2)^{1/2}}{2},$$
respectively.

If we let $h$ change in $[-2, 2]$ we get as a result exactly $[-1, 1]$. □

Acknowledgments. The authors would like to thank Professor Richard Askey, University of Wisconsin–Madison, WI, and Professor Michel Waldschmidt, Université Pierre et Marie Curie (Paris VI), for references on transcendentality of zeros of Bessel functions of a rational order.

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Department Mathematics
Faculty of Electronic Engineering
University of Niš
P. O. Box 73
18000 Niš
Serbia and Montenegro