SOLUTIONS TO A PARTIAL DIFFERENTIAL EQUATION APPEARED IN MECHANICS

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Abstract. The solutions, classical and generalized have been constructed and analyzed for a partial differential equation which appears as a mathematical model of many different phenomena.

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1. Introduction

Equation
\[
\frac{\partial^4}{\partial \xi^4} u(t, \xi) + \lambda \frac{\partial^2}{\partial \xi^2} u(t, \xi) + \frac{\partial^2}{\partial t^2} u(t, \xi) = 0, \quad 0 < \xi < 1
\]

appears in mathematical models for many different phenomena subject to different boundary or initial conditions (cf. for example [1],[3],[4],[5],[6],[8]). It is well-known that a solution to (1.1) is \( u(t, \xi) = Y(\xi)T(t) \), where \( Y \) and \( T \) have the analytical form:

\[
Y(\xi) = C_1 \cos hr_1 \xi + C_2 \sin hr_1 \xi + C_3 \cos r_2 \xi + C_4 \sin r_2 \xi
\]
\[ T(t) = C_5 \cos \Omega t + C_6 \sin \Omega t, \quad \Omega^2 \in \mathbb{R}_+, \]  
(1.3)

where
\[ r_1 = \sqrt{\frac{\sqrt{\lambda^2 + 4\Omega^2} - \lambda}{2}}; \quad r_2 = \sqrt{\frac{\sqrt{\lambda^2 + 4\Omega^2} + \lambda}{2}} \]  
(1.4)

(cf.[1],[2]). For \( \Omega \) any complex number cf. [10]. Our aim is to analyse solutions, classical and generalized to equation (1.1).

2. The corresponding equation to (1.1) in \( \mathcal{D}'(\mathbb{R}^2) \) and its solutions

2.1 corresponding equation to (1.1) in \( \mathcal{D}'(\mathbb{R}^2) \)

Suppose that there exists \( u(t, \xi) \in C_t^{(2)}(\mathbb{R}_+, \mathbb{R}) \) such that:

1. \( u(t, \xi) \) is a solution to (1.1),
2. there exist
\[ \lim_{t \to 0^+} u(t, \xi) = u_1(\xi) \in \mathcal{C}(\mathbb{R}) \]  
(2.1)
\[ \lim_{t \to 0^+} u_t^{(1)}(t, \xi) = u_2(\xi) \in \mathcal{C}(\mathbb{R}). \]  
(2.2)

Let \( [Hu] \) denote the regular distribution defined by the function \( H(t)u(t, \xi) \), where \( H \) is the Heaviside function (\( H(t) = 0, t < 0; H(t) = 1, t \geq 0 \)).

We show by a simple manner the relation between the second partial derivative in the sense of distributions, \( D_t^2[Hu] \), and the regular distribution, \( [\frac{\partial^2}{\partial t^2}u(t, \xi)] \):

\[ D_t^2[Hu] = \left[u_t^{(2)}(t, \xi)_0\right] - [u_2(\xi)] \otimes \delta(t) - [u_1(\xi)] \otimes \delta^{(1)}(t), \]  
(2.3)

where \( u_t^{(2)}(t, \xi)_0 = \frac{\partial^2}{\partial t^2}u(t, \xi), (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}; u_t^{(2)}(t, \xi)_0 = 0, (t, \xi) \in \mathbb{R}_- \times \mathbb{R} \) and \( u_t^{(2)}(t, \xi)_0 \) is not defined for \( (t, \xi) \in \{0\} \times \mathbb{R} \).

This is only a special case of a general theorem which gives the relation between partial derivatives in the sense of distributions and the classical ones.

Proof of (2.3). By definition of the derivative in \( \mathcal{D}'(\mathbb{R}^2) \), for \( \varphi \in \mathcal{D}(\mathbb{R}^2) \)

\[ < D_t[Hu], \varphi(t, \xi) > = < [Hu], (-1)\varphi_t^{(1)}(t, \xi) > \]  
\[ = - \int_{\mathbb{R}} \int_{\mathbb{R}_+} H(t)u(t, \xi)\varphi_t^{(1)}(t, \xi)dt \]  
\[ = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{\epsilon}^{\infty} u(t, \xi)\varphi_t^{(1)}(t, \xi)dt \]
\[
\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^{(1)}_t(t, \xi) \varphi(t, \xi) dt + \int_{-\infty}^{\infty} u(0, \xi) \varphi(0, \xi) d\xi = <u^{(1)}_t(t, \xi)_0, \varphi(t, \xi)> + <u(0, \xi) \otimes \delta(t), \varphi(t, \xi)>
\]

It follows that
\[
D_t[H] = [u^{(1)}_t(t, \xi)_0] + [u_1(\xi) \otimes \delta(t)],
\]

where
\[
u^{(1)}_t(t, \xi)_0 = \frac{\partial}{\partial t} u(t, \xi), \quad t > 0;
\]
\[
u^{(1)}_t(t, \xi) = 0, \quad t < 0 \quad \text{and} \quad \nu^{(1)}_t(t, \xi)_0
\]
is not defined in \(t = 0, \xi \in \mathbb{R}\). If we repeat the mode of proceeding to (2.4), then it follows (2.3). Now, to (1.1) it corresponds in \(D'(\mathbb{R}^2)\)
\[
D_t^2 + \lambda D_t^2 + D_t^2 \tilde{u} = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t),
\]
or
\[
(D_t^2 + P(D_\xi)) \tilde{u} = f,
\]

where
\[
P(D_\xi) = D^4_\xi + \lambda D^2_\xi, \quad f = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t) \quad \text{and} \quad \tilde{u} \in D'(\mathbb{R}^2).
\]

We seek for solutions to (2.5) with the property \(\text{supp } \tilde{u} \subset \mathbb{R}^+ \times \mathbb{R}\).

2.2. Solutions to (2.5)

By the lemma in [7, p. 30] the operator \(D_t^2 + P(D_\xi)\) is quasihyperbolic with respect to \(t\) if and only if the following condition is satisfied:
\[
\exists c > 0, d \in \mathbb{R}, \forall \xi \in \mathbb{R} : Re P(i\xi) - c(Im P(i\xi))^2 \geq d.
\]
In our case \(P(i\xi) = \xi^4 - \lambda \xi^2\). For every \(\xi \in \mathbb{R}, \xi^4 - \lambda \xi^2 \geq -\frac{\lambda^2}{4}\). Consequently the operator \(D_t^2 + P(D_\xi)\) is quasihyperbolic.

By Proposition 5 in [7, p. 32] the unique fundamental solution \(E\) of \(D_t^2 + P(D_\xi)\) with support in \(\mathbb{R}^+ \times \mathbb{R}\) and \(E \in e^{\alpha t} S'\) for an \(\alpha \in \mathbb{R}\) is given by
\[
E(t, \xi) = H(t) \mathcal{F}^{-1}_x \left( \frac{\sin(t \sqrt{P(2\pi i x)})}{\sqrt{P(2\pi i x)}} \right)(t, \xi), \quad (2.6)
\]
where \(\mathcal{F}^{-1}\) is the inverse Fourier transform.
Using Bochner’s formula (cf. [9,(VII,22)] or [7,p 19])

\[
E(t, |\xi|) = H(t)2\pi|\xi|^{1/2} \int_0^\infty \frac{\sin(t\sqrt{P(2\pi ix)})}{\sqrt{P(2\pi i)}} x^{1/2} J_{-1/2}(2\pi|\xi|x)dx,
\]

where \( J_v \) is the Bessel function.

Since

\[
J_{-1/2}(2\pi|\xi|x) = \frac{1}{\pi} \frac{\cos 2\pi|\xi|x}{\sqrt{|\xi|x}},
\]

we have

\[
E(t, \xi) = 2H(t) \int_0^\infty \frac{\sin(t\sqrt{P(2\pi ix)}) \cos(2\pi|\xi|x)}{\sqrt{P(2\pi i)}} \frac{1}{\sqrt{x}} dx.
\]

Suppose now that \( u_1(\xi) \) and \( u_2(\xi) \) in (2.5) have the properties that:

\[
([u_2(\xi)] \otimes \delta(t)) * [E(t, \xi)], ([u_1(\xi)] \otimes \delta(1)(t)) * [E(t, \xi)]
\]

exist, then there is a solution \( \tilde{u} \) to (2.5) in \( \mathcal{D}'(\mathbb{R}^2) \) with support in \( \mathbb{R}^2 \times \mathbb{R}^2 \)

\[
\tilde{u} = ([u_1(\xi)] \otimes \delta(1)(t)) + ([u_2(\xi)] \otimes \delta(t)) * [E(t, \xi)]
\]

\[
= [u_2(\xi)] * [E(t, \xi)] + [u_1(\xi)] * D_t[E(t, \xi)].
\]

This solution is unique in the vector space \( A \subset \mathcal{D}'(\mathbb{R}^2) \). \( A \) consists of all \( q \in \mathcal{D}'(\mathbb{R}^2) \) for which there exists \( E * q \) (cf. [12 chapter III, §11.3]). We proved the following

**Theorem 1.** Let \( E \) be given by (2.8) and let \( A \) be the vector space belonging to \( \mathcal{D}'(\mathbb{R}^2) \) such that for every \( g \in A \) there is \( [E] * g \).

Suppose that \( u_1(\xi) \) and \( u_2(\xi) \) are in \( C(\mathbb{R}) \) such that the convolutions (2.9) exist. Then

\[
\tilde{u} = [u_2(\xi)] * [E(t, \xi)] + [u_1(\xi)] * D_t[E(t, \xi)]
\]

is a solution to

\[
(D^4_\xi + \lambda D^2_\xi + D^2_t)\tilde{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}).
\]

But it is also the unique solution in the space \( A \subset \mathcal{D}'(\mathbb{R}^2) \) satisfying the initial condition in \( t \) in the sense that

\[
D^4_\xi + \lambda D^2_\xi + D^2_t\tilde{u} = [u_2(\xi)] \otimes \delta(t) + [u_1(\xi)] \otimes \delta(1)(t).
\]
Remarks: 1. If \( u_1(\xi) \) and \( u_2(\xi) \) also belong to \( C^4(\mathbb{R}) \), then by the property of convolution

\[
D_t^i \hat{u} = [u_2^{(i)}(\xi)] * [E(t, \xi)] + [u_1^{(i)}(\xi)] * D_t[E(t, \xi)], \quad i = 1, \ldots, 4.
\]

2. If we have two solutions \( u_1(t, \xi) \) and \( u_2(t, \xi) \) to (1.1) with some initial condition \( u_1(0, \xi) = u_2(0, \xi) \) and \( \frac{d}{dt} u_1(t, \xi)|_{t=0} = \frac{d}{dt} u_2(t, \xi)|_{t=0}, \xi \in \mathbb{R} \), then

\[
[u_2(t, \xi)] = [u_1(t, \xi)] + h,
\]

where \( h = 0 \) or \( h \notin \mathcal{A} \).

Let us prove it. The function \( U(t, \xi) = u_2(t, \xi) - u_1(t, \xi) \) satisfies (1.1) with initial condition \( U_t^{(i)}(t, \xi)|_{t=0} = 0, i = 0, 1, \xi \in \mathbb{R} \), consequently the regular distribution \([U(t, \xi)] \in \mathcal{D}'(\mathbb{R}^2)\) satisfies (2.5) with \( f = 0 \). Then \([U(t, \xi)] = h\), where \( h = 0 \) or \( h \notin \mathcal{A} \). Hence \([U(t, \xi)] = [u_2(t, \xi)] - [u_1(t, \xi)] = h\).

3. The well-known solution to (1.1) \( u(t, \xi) = Y(\xi)T(t) \), where \( Y \) and \( T \) have been given by (1.3) and (1.4), has not the convolution with \( E(t, \xi) \) in the sense of distributions, i.e., \([u(t, \xi)] * [E(t, \xi)] \) does not exist. If were true that \([u(t, \xi)] * [E(t, \xi)] \) exists, then by 1. and the property of convolution:

\[
[u(t, \xi)] = [u(t, \xi)] * \delta(t, \xi) = [u(t, \xi)] * (D_t^2 + P(D_\xi))[E(t, \xi)]
\]

\[
= \left( (D_t^2 + P(D_\xi))[u(t, \xi)] \right) * [E(t, \xi)]
\]

\[
= \left( \left( \frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} \right) u(t, \xi) \right) * [E(t, \xi)] = 0.
\]

Thus \( u(t, \xi) = 0, \ t > 0, \ \xi \in \mathbb{R} \).

4. If equation (2.5) with \( f = 0 \) has a solution belonging to \( \mathcal{D}'(\mathbb{R}^2) \), it does not belong to \( \mathcal{A} \).

Proof. A solution to (1.1) in \( \mathcal{D}'(\mathbb{R}^2) \) is \( u(t, \xi) \equiv 0, \ (t, \xi) \in \mathbb{R} \). By 2. if there is a solution to (1.1) belonging to \( \mathcal{D}'(\mathbb{R}^2) \) which is not identical zero, then it does not belong to \( \mathcal{A} \).

The solution \( u(t, \xi) = Y(\xi)T(t) \), where \( Y \) and \( T \) have been given by (1.2) and (1.3), respectively, is in fact a solution to

\[
(Y^{(4)}(\xi) + \lambda Y^{(2)}(\xi) + \omega^2 Y(\xi))T(t) + (T^{(2)}(t) - \omega^2 T(t))Y(\xi) = 0, \ t > 0, \ \xi \in \mathbb{R},
\]

(2.12)
for $\omega^2 \in \mathbb{R} \setminus \{0\}$. This equation can be written in the form
\[
\left( P\left( \frac{d}{d\xi} \right) + \frac{d^2}{dt^2} - \omega^2 \right) Y(\xi) T(t) = 0,
\]
where
\[
P\left( \frac{d}{d\xi} \right) = \frac{d^4}{d\xi^4} + \lambda \frac{d^2}{d\xi^2} + \omega^2.
\]

In the sequel we suppose that $\omega^2 > 0$. Since
\[
P(i\xi) = \xi^4 - \lambda \xi^2 + \omega^2 > 0, \quad \xi \in \mathbb{R}, \quad \omega^2 - \frac{\lambda^2}{4} > 0,
\]
by Proposition 6 in [7] there is the unique fundamental solution $E_\omega(t, \xi)$ of $P\left( \frac{d}{d\xi} \right) + \frac{d^2}{dt^2} - \omega^2$ with support in $\mathbb{R}_+ \times \mathbb{R}$ and belonging to $e^{\alpha t} S'$ for an $\alpha \in \mathbb{R}$. It has the following representation
\[
E_\omega(t, \xi) = E(t, \xi) - \omega H(t) \int_0^t \frac{\tau}{\sqrt{\tau^2 - \xi^2}} J_1(\omega \sqrt{\tau^2 - \xi^2}) E(\tau, \xi) d\tau,
\]
where $E(t, \xi)$ is given by (2.8).

**Theorem 2.** If in Theorem 1 instead of $E(t, \xi)$ we take $E_\omega(t, \xi)$, given by (2.14), then we obtain another form of solutions to
\[
\left( P\left( \frac{d}{d\xi} \right) + \frac{d^2}{dt^2} - \omega^2 \right) [u(t, \xi)] = 0
\]
with
\[
P\left( \frac{d}{d\xi} \right) = \frac{d^4}{d\xi^4} + \lambda \frac{d^2}{d\xi^2} + \omega^2,
\]
where $\omega^2 - \frac{\lambda^2}{4} > 0$, $\omega^2 > 0$.

**2.3. A convolutor to $E(t, \xi)$**

At the end of Part 2 we give a sufficient condition for a regular distribution to have convolution with $E(t, \xi)$, such that this convolution is also a regular distribution.

**Lemma 1.** If $f(\xi, t)$ has the property that
\[
|f(\xi, t)| \leq H(t) \alpha(t) \beta(\xi), \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}_+,
\]
where $\alpha(t)$ and $\beta(\xi)$ are positive functions, then
\[
f(t, \xi) \in S', \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}_+.
\]
where \( \alpha(t) \in L_{\text{loc}}([0, \infty)) \) and \( \beta(\xi) \in L^1(\mathbb{R}) \), then \( f(\xi, t) \) defines a regular distribution \( [f(\xi, t)] \) such that \( [f(\xi, t)] \ast [E(\xi, t)] \) exists and is also a regular distribution defined by the function \( (f(\xi, t) \ast E(\xi, t))(\xi, t) \) which is bounded in \( \xi \in \mathbb{R} \), for every \( t \geq 0 \).

**Proof.** It is enough to prove that there exists the convolution of two functions \( f(\xi, t) \ast E(\xi, t) \) and that this convolution is a locally integrable function on \((\mathbb{R} \times \mathbb{R}_+)\)

\[
|f(\xi, t) \ast E(\xi, t)| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi - x, t - \tau) E(x, \tau) dx d\tau \right|
\]

\[
\leq \int_{-\infty}^{\infty} dx \int_{0}^{t} \alpha(t - \tau) \beta(\xi - x) |E(x, \tau)| d\tau
\]

\[
\leq H(t) \int_{-\infty}^{\infty} \beta(x) dx \int_{0}^{t} \alpha(t - \tau) B(\tau) d\tau,
\]

where

\[
B(\tau) = \sup_{\xi \in \mathbb{R}} |E(\xi, t)| = \sup_{\xi \in \mathbb{R}} \left| \int_{-\infty}^{\infty} e^{2\pi i \xi \tau} \frac{\sin(2\pi \tau |\xi| \sqrt{4\pi^2 x^2 - \lambda})}{2\pi |x| \sqrt{4\pi^2 x^2 - \lambda}} dx \right|
\]

\[
\leq \sup_{\xi \in \mathbb{R}} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{-\sqrt{\lambda}/2\pi} + \int_{-\sqrt{\lambda}/2\pi}^{\sqrt{\lambda}/2\pi} + \int_{\sqrt{\lambda}/2\pi}^{-\sqrt{\lambda}/2\pi} e^{2\pi i \xi \tau} \frac{\sin(2\pi \tau |\xi| \sqrt{4\pi^2 x^2 - \lambda})}{2\pi |x| \sqrt{4\pi^2 x^2 - \lambda}} dx \right|.
\]

In the first and the second integral we can use the inequality

\[
\left| e^{2\pi i \xi \tau} \frac{\sin(2\pi \tau |x| \sqrt{4\pi^2 x^2 - \lambda})}{2\pi |x| \sqrt{4\pi^2 x^2 - \lambda}} \right| \leq \frac{1}{2\pi |x| \sqrt{4\pi^2 x^2 - \lambda}}, \quad t \geq 0, \quad |x| \geq \frac{\sqrt{\lambda}}{2\pi}.
\]

The third integral is:

\[
\int_{-\infty}^{-\sqrt{\lambda}/2\pi} + \int_{-\sqrt{\lambda}/2\pi}^{\sqrt{\lambda}/2\pi} + \int_{\sqrt{\lambda}/2\pi}^{-\sqrt{\lambda}/2\pi} e^{2\pi i \xi \tau} \frac{1}{2\pi |x| \sqrt{\lambda - (2\pi x)^2}} \sin h(\tau 2\pi x \sqrt{\lambda - (2\pi x)^2}) dx
\]

\[
\leq \frac{1}{2\pi |x| \sqrt{\lambda - (2\pi x)^2}} \sin h(\tau 2\pi x \sqrt{\lambda - (2\pi x)^2}) dx
\]
\[
\sqrt{\frac{x}{2\pi}} = \int_{-\sqrt{\frac{x}{2\pi}}}^{\sqrt{\frac{x}{2\pi}}} e^{2\pi ix}\xi f(x, \tau)dx.
\]

The function \(f(x, \tau)\) is not defined for \(x = 0\). But since there exists
\[
\lim_{x \to 0} f(x, \tau) = \tau, \quad \tau \geq 0,
\]
this function can be extended to \((-\sqrt{\frac{x}{2\pi}}, \sqrt{\frac{x}{2\pi}})\) as a continuous function. Thus
\[
\left| \int_{-\sqrt{\frac{x}{2\pi}}}^{\sqrt{\frac{x}{2\pi}}} e^{2\pi ix}\xi f(x, \tau)dx \right| \leq \int_{-\sqrt{\frac{x}{2\pi}}}^{\sqrt{\frac{x}{2\pi}}} |f(x, \tau)|dx.
\]

Consequently, \(B(\tau) \in \mathcal{L}_{loc}([0, \infty))\) and the Lemma is proved.

**Remark.** If \(u_1(\xi) \equiv 0\) and \(u_2(\xi) \in L^1(\mathbb{R})\) then by Lemma we proved, it follows that the solution (2.10) is a regular distribution defined by the function \(u_2(\xi) \ast E(t, \xi)\), with support in \(\mathbb{R}_+ \times \mathbb{R}\) and bounded in \(\xi \in \mathbb{R}\), for every \(t \geq 0\).

3. Special case of equation (1.1)

3.1. Fourier’s method

In Part 2 the solutions to equation (2.5) have been limited by the space \(\mathcal{A}\). Now we consider equation (1.1) in case \(\lambda = 0\) without this limitation. A detailed discussion of the mentioned case by Fourier’s method separation of variables one can find in [3]. Transverse vibrations of a homogeneous rod has been given by
\[
\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0.
\]

Five various types of boundary conditions have been considered for a solution supposed in the form \(u(x, t) = v(x) g(t)\)

1. \(v''(x) = v'''(x) = 0\), for \(x = 0\) and \(x = \pi\);
2. \(v(x) = v''(x) = 0\), for \(x = 0\) and \(x = \pi\);
3. \(v(x) = v'(x) = 0\), for \(x = 0\) and \(x = \pi\);
4. \(v'(x) = v'''(x)\) for \(x = 0\) and \(x = \pi\);
5. \(v^{(i)}(0) = v^{(i)}(\pi), \quad i = 0, 1, 2, 3\).
We would like to analyse the existence of other solutions (generalized or classical) to equation (3.1). Therefore we use a method explained in [11] and the Laplace Transform. We hope that not only our solution (3.28) but also the Comments at the end can be interesting for applications.

3.2 Distributions and the Laplace transform

We repeat some definitions and facts related to the space $S'$ of tempered distributions and to the Laplace transform (in short LT) of them (cf. [12] and [13]).

Let $\Gamma$ be a closed convex acute cone in $\mathbb{R}^n$, $\Gamma^* = \{ t \in \mathbb{R}^n, tx \equiv t_1x_1 + \ldots + t_nx_n \geq 0, \forall x \in \Gamma \}$ and $C = int\Gamma^*$. Let $K$ be a compact set in $\mathbb{R}^n$.

By $S'(\Gamma + K)$ is denoted the space of tempered distributions defined on the close set $\Gamma + K \subset \mathbb{R}^n$. Then $S'(\Gamma +)$ is defined by way of

$$S'(\Gamma +) = \bigcup_{K \subset \mathbb{R}^n} S'(\Gamma + K). \quad (3.2)$$

The set $S'(\Gamma +)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $\ast$.

If $\Gamma + K$ is convex, as it will be in our case, then the LT of $f \in S'(\Gamma +)$ is defined by

$$\hat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt} \rangle, \quad z \in C + i\mathbb{R}^n. \quad (3.3)$$

If $\sigma \geq 0$, $f \in S'(\Gamma +)$, then

$$\mathcal{L}(e^{\sigma t}f)(z) = \langle f(t), e^{-(z-\sigma)t} \rangle, \quad Rez > \sigma, \quad (3.4)$$

where $t = (t_1, \ldots, t_n)$, $z = (z_1, \ldots, z_n)$ and $zt = z_1t_1 + \ldots + z_nt_n$. It is one to one operation.

For the properties of so defined LT one can consult [13]. We shall cite only some of them, we use in the sequel:

1) $\mathcal{L}\left( \frac{\partial^m}{\partial t^m} f \right)(z) = (zi)^m \mathcal{L}(f)(z)$.

2) If $f \in S'(\Gamma +)$ and $g \in S'(\Gamma +)$, then $\mathcal{L}(f \times g)(z, s) = \mathcal{L}(f)(z)\mathcal{L}(g)(s)$, $z \in C_1 + i\mathbb{R}^n$, $s \in C_2 + i\mathbb{R}^n$.

3) If $f, g \in S'(\Gamma +)$, then $f \ast g \in S'(\Gamma +)$ and

$$\mathcal{L}(f \ast g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z), \quad z \in C + i\mathbb{R}^n.$$
4) \( \mathcal{L}(\delta(t-t_0))(z) = e^{-zt_0} \).

5) \( \mathcal{L}(f)(z+a) = \mathcal{L}(e^{-at}f)(z), \ Re a > 0. \)

6) If \( f \in \mathcal{L}_{\text{loc}}(\mathbb{R}^n_+) \) and \( |f(x)| \leq Ce^{qx}, \ x \geq x_0 > 0, \) then \( f(x)e^{-qx} \in \mathcal{S}'(\mathbb{R}^n_+) \) and

\[
\int_{\mathbb{R}^n_+} e^{-(z+q)t} f(t) dt = \int_{\mathbb{R}^n_+} e^{-zt} e^{-qt} f(t) dt = \mathcal{L}(e^{-qt}f)(z).
\]

Let \( \mathcal{H}_{(\alpha,\beta)}(\mathbb{C}), \ \alpha \geq 0, \ \beta \geq 0, \ a \geq 0, \) denote the sets of holomorphic functions on \( C + i\mathbb{R}^n \) which satisfy the following growth condition

\[
|f(z)| \leq Me^{a|x|}(1 + |z|^2)^{\alpha/2}(1 + \Delta^{-\beta}(x, \partial C)), \ z = x + iy \in C + i\mathbb{R}^n, \ (3.5)
\]

where \( \partial C \) is the boundary of \( C \) and \( \Delta(x, \partial C) \) is the distance between \( x \) and \( \partial C \). We set

\[
\mathcal{H}_a(C) = \bigcup_{\alpha \geq 0, \beta \geq 0} \mathcal{H}_{(\alpha,\beta)}(\mathbb{C}) \quad \text{and} \quad \mathcal{H}_+(C) = \bigcup_{a \geq 0} \mathcal{H}_a(C).
\]

**Proposition A.** ([13] p.191). The algebras \( \mathcal{H}_+(C) \) and \( \mathcal{S}'(C^*+) \) and also their subalgebras \( \mathcal{H}_0(C) \) and \( \mathcal{S}'(C^*) \) are isomorphic. This isomorphism is accomplished via the LT.

A property of the defined LT which can be used in a practical way is the following:

Let \( f \in \mathcal{S}'(\mathbb{R}^n_+ + \mathcal{P}) \). The LT of \( f \), \( \mathcal{L}(f) \), can be obtained by one after the other applications of the LT-s \( \mathcal{L}_1(f), \ldots, \mathcal{L}_n(f) \), \( \mathcal{L}(f) = \mathcal{L}_1(f) \circ \cdots \circ \mathcal{L}_n(f) \).

If \( \sigma \geq 0, \ f \in \mathcal{S}'(C^*+) \) and \( g = e^{\sigma t}f \) then by definition \( \mathcal{L}(g)(s) = (f(t), e^{-(s-\sigma)t}), \ Res > \sigma \).

Let \( F(s) \) be a function holomorphic for \( Re s > \sigma \). The function \( F(\xi + \sigma) \) is holomorphic for \( Re \xi > 0 \). If \( F(\xi + \sigma) \in \mathcal{H}(\mathbb{R}^n_+) \), then there exists \( f \in \mathcal{S}'(\mathbb{R}^n_+) \) such that \( \mathcal{L}(e^{\sigma t}f)(s) = F(s), \ Res > \sigma \).

We shall quote some auxiliary formulas for the classical Laplace Transform we need. Let \( H \) denotes the Heaviside function, \( H(t) = 0, \ t < 0; H(t) = 1, \ t \geq 0. \)

1. \( \mathcal{L}^{-1}_z\left(\frac{1}{z + a\sqrt{s}}\right) = H(x)e^{-ax\sqrt{s}}. \)

2. \( \mathcal{L}^{-1}_s\left(\frac{1}{\sqrt{s}e^{-ax\sqrt{s}}}\right) = \frac{H(t)}{\sqrt{\pi t}} e^{-(ax)^2/(4t)}, \ x > 0, \ Re a > 0. \)
3. Solution to (3.1) in $\mathcal{D}'(\mathbb{R}_+^2)$

We consider the equation (3.1) with initial conditions

$$u(0, t) = \frac{\partial}{\partial x}(0, t) = 0, \quad t \geq 0,$$

$$\frac{\partial^k}{\partial x^k} u(0, t) = A_k(t), \quad k = 2, 3, \quad t \geq 0,$$

$$u(x, 0) = B_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = B_1(x), \quad x \geq 0,$$

where $[H(t)A_k(t)] \in e^{\sigma t}S'(\mathbb{R}_+), k = 2, 3, p > 0$ and $[H(x)B_i(x)] \in e^{\sigma x}S'(\mathbb{R}_+), i = 0, 1, q > 0, \sigma > 0$.

To find an equation in $\mathcal{D}'(\mathbb{R}_+^2)$ which corresponds to (3.1) for $x > 0, t > 0$ we need the following relations between derivatives in the sense of distributions and the classical ones.

Let $H^2(x_1, x_2) = H(x_1)H(x_2)$, where $H$ is the Heaviside function. For a function $f$ with continuous partial derivatives on $\mathbb{R}^2$, $[H^2 f]$ is the distribution, defined by $H^2 f$, belonging to $\mathcal{D}'(\mathbb{R}^2)$ and to $\mathcal{D}'(\mathbb{R}_+^2)$, as well. Let $\left( \frac{\partial^p}{\partial x_i^p} f \right)_0$ denote the function equal to $\frac{\partial^p}{\partial x_i^p} f$ on the $\mathbb{R}_+^2$ and equal zero on $\mathbb{R}^2 \setminus \mathbb{R}_+^2$, but is not defined for $(x_1, x_2) \in \{(0, x_2) \cup (x_1, 0); x_1 \geq 0, x_2 \geq 0\}$.

With the notation as above we have (cf. [11])

$$D_{x_i}^p[H^2 f] = \left[ H^2 \left( \frac{\partial^p}{\partial x_i^p} f \right)_0 \right] + R_p(f), \quad p \in \mathbb{N},$$

where

$$R_p(f) = \left[ H^2 \left( \frac{\partial^{p-1}}{\partial x_i^{p-1}} f(x) \right)_{x_i=0} \right. \times \delta(x_i) + \ldots + \left. [H^2 f(x)|_{x_i=0}] \times \delta^{(p-1)}(x_i) \right].$$
To equation (3.1) with initial condition (3.6) it corresponds in $\mathcal{D}'(\mathbb{R}^2_+)$

$$\frac{\partial^4}{\partial x^4}[u(x, t)] + \frac{\partial^2}{\partial t^2}[u(x, t)] = [H(t)A_2(t)] \times \delta^{(1)}(x)$$

$$+ [H(t)A_3(t)] \times \delta(x) + [H(x)B_1(x)] \times \delta(t) + [H(x)B_0(x)] \times \delta^{(1)}(t).$$

Applying the LT we have

$$(z^4 + s^2)\mathcal{L}(u)(z, s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s,$$

or

$$\mathcal{L}(u)(z, s) = \frac{Q(z, s)}{z^4 + s^2},$$

with

$$Q(z, s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s.$$  \hspace{1cm} (3.8)

Since

$$\frac{1}{z^4 + s^2} = \frac{1}{2is} \left( \frac{1}{z^2 - is} - \frac{1}{z^2 + is} \right),$$

$$\frac{Q(z, s)}{z^4 + s^2} = \frac{Q(z, s)}{2is} \left( \frac{1}{z^2 - is} - \frac{1}{z^2 + is} \right).$$  \hspace{1cm} (3.10)

By Proposition A in [11] $\frac{Q(z, s)}{z^4 + s^2}$ has to be holomorphic in $\{(z, s) \in \mathbb{C}^2; \text{Re}z > w_1 > 0, \text{Res} > w_2 > 0\}$. Since $z^4 + s^2 = (z - z_1)(z + z_1)(z - z_2)(z + z_2)$, where $z_1 = e^{i\frac{\pi}{4}}\sqrt{s}$, $z_2 = e^{-i\frac{3\pi}{4}}\sqrt{s}$, it is necessary to have

$$Q(e^{i\frac{\pi}{4}}\sqrt{s}, s) = 0 \text{ and } Q(-e^{i\frac{3\pi}{4}}\sqrt{s}, s) = 0$$

or equivalently

$$Q(e^{i\frac{\pi}{4}}\sqrt{s}, s) = 0 \text{ and } Q(-e^{-i\frac{\pi}{4}}\sqrt{s}, s) = 0.$$  \hspace{1cm} (3.11)

*First step*

In the first step we consider the first addend in (3.10). Now we need (3.11) to be satisfied which gives:

$$\mathcal{L}(A_2)(s)e^{i\frac{\pi}{4}}\sqrt{s} + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s}) + s\mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}) = 0.$$  \hspace{1cm} (3.12)

Now we can express $\mathcal{L}(A_3)(s)$,

$$\mathcal{L}(A_3)(s) = -\mathcal{L}(A_2)(s)e^{i\frac{\pi}{4}}\sqrt{s} - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s}) - s\mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}).$$
With such expressed $\mathcal{L}(A_3)(s)$ the first addend in (3.10) is:

$$\frac{Q(z, s)}{2is(z^2 - is)} = \frac{\mathcal{L}(A_2)(s)(z - e^{i\frac{\pi}{4}}\sqrt{s})}{2is(z^2 - is)} +$$

$$+ \frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s}) + s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}))}{2is(z^2 - is)}$$

$$= \frac{\mathcal{L}(A_2)(s)}{2is(z + e^{i\frac{\pi}{4}}\sqrt{s})} + \frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}} +$$

$$+ \frac{\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}} \left( \frac{1}{z - e^{i\frac{\pi}{4}}\sqrt{s}} - \frac{1}{z + e^{i\frac{\pi}{4}}\sqrt{s}} \right).$$

By using the auxiliary formulas 1., 2. and 5. we quoted we find the LT of (3.13).

Let us consider the first addend in (3.13)

$$\mathcal{L}^{-1}\left( \frac{\mathcal{L}(A_2)(s)}{2is(z + e^{i\frac{\pi}{4}}\sqrt{s})} \right) = \mathcal{L}^{-1}_s \circ \left( \frac{1}{2is(z + e^{i\frac{\pi}{4}}\sqrt{s})} \mathcal{L}(A_2)(s) \right)$$

$$= \frac{1}{2i} \mathcal{L}^{-1}_s \left( \frac{1}{\sqrt{s}} e^{-e^{i\frac{\pi}{4}}\sqrt{s}} \right) \frac{1}{\sqrt{s}} \mathcal{L}(A_2)(s)$$

$$= \frac{H(x)H(t)}{2i\Gamma(1/2)} \chi(e^{i\frac{\pi}{4}}x, t) \ast \int_0^t (t - \tau)^{-1/2} A_2(\tau) d\tau. \tag{3.14}$$

The second addend in (3.13) is:

$$\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}} \left( \frac{1}{z - e^{i\frac{\pi}{4}}\sqrt{s}} - \frac{1}{z + e^{i\frac{\pi}{4}}\sqrt{s}} \right). \tag{3.15}$$

We shall start with

$$\mathcal{L}^{-1}\left( \frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z + e^{i\frac{\pi}{4}}\sqrt{s})} \right)$$

$$= \mathcal{L}^{-1}_z \circ \mathcal{L}^{-1}_s \left( \frac{\mathcal{L}(B_1)(z)}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z + e^{i\frac{\pi}{4}}\sqrt{s})} \right)$$

$$- \mathcal{L}^{-1}_s \circ \mathcal{L}^{-1}_z \left( \frac{\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z + e^{i\frac{\pi}{4}}\sqrt{s})} \right). \tag{3.16}$$
The first addend in (3.16) is
\[
\mathcal{L}_{z}^{-1}\left(B_1(z)\mathcal{L}_{s}^{-1}\left(\frac{1}{4(e^{i\pi \sqrt{s}})^3(e^{i\pi \sqrt{s}} + z)}\right)\right)
\]
\[
= \mathcal{L}_{z}^{-1}\left(B_1(z)\mathcal{L}_{s}^{-1}\left(\frac{1}{4e^{i\frac{3\pi}{4}}s} \cdot \frac{t}{\sqrt{s}} \cdot \mathcal{L}_{s}^{-1}\left(\frac{1}{z + e^{i\pi \sqrt{s}}}\right)\right)\right)
\]
\[
= \frac{1}{4e^{i\frac{3\pi}{4}}} \int_{0}^{t} \chi(e^{\frac{i\pi}{4}}x, \tau) d\tau * B_1(x).
\]

For the second addend in (3.16) we have
\[
-\mathcal{L}_{s}^{-1} \circ \mathcal{L}_{z}^{-1}\left(\frac{\mathcal{L}(B_1)(e^{i\pi \sqrt{s}})}{4i\sigma e^{i\frac{3\pi}{4}} \sqrt{s}(z + e^{i\pi \sqrt{s}})}\right)
\]
\[
= -\mathcal{L}_{s}^{-1}\left(\mathcal{L}(B_1)(e^{i\pi \sqrt{s}}) \cdot \frac{1}{4e^{i\frac{3\pi}{4}}s} \cdot \frac{1}{\sqrt{s}} \mathcal{L}_{z}^{-1}\left(\frac{1}{z + e^{i\pi \sqrt{s}}}\right)\right)
\]
\[
= -\mathcal{L}_{s}^{-1}\left(\frac{1}{4e^{i\frac{3\pi}{4}}} H(x) e^{-e^{i\pi \sqrt{s}}x/\sqrt{s}} \int_{0}^{\infty} e^{-e^{i\pi \sqrt{s}}\tau} B_1(\tau) d\tau\right)
\]
\[
= d - \frac{1}{4e^{i\frac{3\pi}{4}}} \int_{0}^{t} \frac{1}{\sqrt{s}} \mathcal{L}_{s}^{-1}\left(\frac{1}{\sqrt{s}} \int_{0}^{\infty} e^{-e^{i\pi \sqrt{s}}(x+\tau)/t} B_1(\tau) d\tau\right)
\]
\[
= -\frac{1}{4e^{i\frac{3\pi}{4}}} \int_{0}^{t} \int_{0}^{\infty} e^{-\frac{1}{4i}(x+\tau)^2/t} \frac{1}{\sqrt{\pi t}} B_1(\tau) d\tau
\]
\[
= -\frac{1}{4ie^{i\frac{3\pi}{4}}} \int_{0}^{t} du \int_{0}^{\infty} \chi(e^{\frac{i\pi}{4}}(x + \tau), u) B_1(\tau) d\tau.
\]

The first addend in (3.15) gives
\[
\mathcal{L}^{-1}\left(\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi \sqrt{s}})}{4i\sigma e^{i\pi} \sqrt{s}(z - e^{i\pi \sqrt{s}})}\right)
\]
\[
= \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(z)}{4ie^{i\frac{3\pi}{4}}s \sqrt{s}(z - e^{i\pi \sqrt{s}})} - \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(e^{i\pi \sqrt{s}})}{4ie^{i\frac{3\pi}{4}}s \sqrt{s}(z - e^{i\pi \sqrt{s}})}
\]
Solutions to a partial differential equation . . .

$$\begin{align*}
= \frac{1}{4ie^\frac{i\pi}{4}} \left( \mathcal{L}_s^{-1} \frac{1}{s \sqrt{s}} e^{e^\frac{i\pi}{4} x \sqrt{s} (x - B_1(x))} - \mathcal{L}_s^{-1} \frac{1}{s \sqrt{s}} e^{e^\frac{i\pi}{4} x \sqrt{s} u} B_1(u) \right) \\
= \frac{1}{4ie^\frac{i\pi}{4}} \left( \mathcal{L}_s^{-1} \frac{1}{s \sqrt{s}} \int_0^x e^{e^\frac{i\pi}{4} (x-u) \sqrt{s}} B_1(u) du - \mathcal{L}_s^{-1} \frac{1}{s \sqrt{s}} \int_0^\infty e^{e^\frac{i\pi}{4} (u-x) \sqrt{s}} B_1(u) du \right) \\
= \frac{-1}{4ie^\frac{i\pi}{4}} \mathcal{L}_s^{-1} \frac{1}{s \sqrt{s}} \int_x^\infty e^{-e^\frac{i\pi}{4} (u-x) \sqrt{s}} B_1(u) du \\
= \frac{-1}{4ie^\frac{i\pi}{4}} \int_0^t \int_x^\infty \chi(e^\frac{i\pi}{4} (u-x), \tau) B_1(u) du d\tau.
\end{align*}$$

(3.19)

If we collect all the results obtained in (3.16)–(3.19), then the inverse LT of (3.15) is a function denoted by $F(B_1, x, t, \frac{\pi}{4})$,

$$F(B_1, x, t, \frac{\pi}{4}) = -\frac{1}{4ie^\frac{i\pi}{4}} \int_0^t \int_x^\infty \chi(e^\frac{i\pi}{4} (u-x), \tau) B_1(u) du d\tau$$

$$- \frac{1}{4ie^\frac{i\pi}{4}} \int_0^t \int_x^\infty \chi(e^\frac{i\pi}{4} x, \tau) B_1(x) du$$

$$+ \frac{1}{4ie^\frac{i\pi}{4}} \int_0^t \int_0^\infty \chi(e^\frac{i\pi}{4} (x+\tau), u) B_1(\tau) du d\tau.$$ 

(3.20)

To find the inverse LT of (3.13), it is yet to be find the inverse LT of

$$s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^\frac{i\pi}{4} \sqrt{s})) \left( \frac{1}{z - e^\frac{i\pi}{4} \sqrt{s}} - \frac{1}{z + e^\frac{i\pi}{4} \sqrt{s}} \right).$$

(3.21)

If we compare (3.21) with (3.15), we can observe that in the structure of (3.21) we have additionally only a product by $s$. Since $F(B_0, x, 0, \frac{\pi}{4}) = 0$, the inverse LT of (3.21) is

$$\frac{\partial}{\partial t} F(B_0, x, t, \frac{\pi}{4}) = -\frac{1}{4ie^\frac{i\pi}{4}} \int_x^\infty \chi(e^\frac{i\pi}{4} (u-x), t) B_0(u) du$$

$$- \frac{1}{4ie^\frac{i\pi}{4}} \chi(e^\frac{i\pi}{4} x, t) \ast B_0(x).$$

(3.22)
\[
+ \frac{1}{4ie^{i\pi/4}} \int_{x}^{\infty} \chi(e^{i\pi/4}(x + \tau), t) B_0(\tau) d\tau.
\]

To finish the first step we collect the all obtained results which give
\[
L^{-1} \left( \frac{Q(z, s)}{2is(z^2 - is)} \right)(x, t) = \frac{1}{2i\Gamma(1/2)} \chi(e^{i\pi/4}x, t) \ast \int_{0}^{t} (t - \tau)^{-1/2} A_2(\tau) d\tau + F(B_1, x, t, \frac{\pi}{4}) + \frac{\partial}{\partial t} F(B_0, x, t, \frac{\pi}{4}),
\]
where \(F\) is given by (3.20).

**Second step**

In the second step we consider the second addend in (3.10). Now we need (3.11) to be satisfied:
\[
L(A_2)(s)e^{-i\pi/4} \sqrt{s} + L(A_3)(s) + L(B_1)(e^{-i\pi/4} \sqrt{s}) + sL(B_0)(e^{-i\pi/4} \sqrt{s}) = 0.
\]

The procedure to find the inverse LT of
\[
\frac{Q(z, s)}{2is(z^2 + is)} = \frac{Q(z, s)}{4ie^{-i\pi/4}s\sqrt{s}} \left( \frac{1}{z - e^{-i\pi/4} \sqrt{s}} - \frac{1}{z + e^{-i\pi/4} \sqrt{s}} \right)
\]
is just the same as for the first addend in (3.10), which we applied in the first step. Consequently because the \(Re e^{-i\pi/4} > 0\), we have
\[
L^{-1} \left( \frac{Q(z, s)}{2is(z^2 + is)} \right)(x, t) = \frac{1}{2i\Gamma(1/2)} \chi(e^{-i\pi/4}x, t) \ast \int_{0}^{t} (t - \tau)^{-1/2} A_2(\tau) d\tau + F(B_1, x, t, -\frac{\pi}{4}) + \frac{\partial}{\partial t} F(B_0, x, t, -\frac{\pi}{4}).
\]

**Third step**

It remains to find the solution \(u(x, t)\) to equation (3.1). This can be done, now, by taking the inverse LT of (3.8) or in fact of (3.10).
By (3.23) and (3.26) we have

\[
\begin{align*}
    u(x, t) &= \frac{1}{2i\Gamma(1/2)}e^{i\pi x / 4}t^* \int_{0}^{t} (t - \tau)^{-1/2} A_2(\tau) d\tau \\
    &\quad + F(B_1, x, t, \frac{\pi}{4}) + \frac{\partial}{\partial t} F(B_0, x, t, \frac{\pi}{4}) \\
    &\quad - \frac{1}{2i\Gamma(1/2)}e^{-i\pi x / 4}t^* \int_{0}^{t} (t - \tau)^{-1/2} A_2(\tau) d\tau \\
    &\quad - F(B_1, x, t, -\frac{\pi}{4}) - \frac{\partial}{\partial t} F(B_0, x, t, -\frac{\pi}{4}).
\end{align*}
\]

(3.27)

Now we can apply properties of \( \chi \) 3.- 5. in Section 3.2 to (3.27):

\[
\begin{align*}
    u(x, t) &= \frac{1}{\Gamma(1/2)} \frac{1}{\sqrt{\pi t}} \sin \frac{x^2}{4t} t^* \int_{0}^{t} (t - \tau)^{-1/2} A_2(\tau) d\tau \\
    &\quad + \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \int_{0}^{t} \int_{x}^{\infty} \left( \cos \frac{(u - x)^2}{4\tau} + \sin \frac{(u - x)^2}{4\tau} \right) B_1(u) du d\tau \\
    &\quad + \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \int_{0}^{t} \left( \cos \frac{x^2}{4\tau} + \sin \frac{x^2}{4\tau} \right) t^* B_1(x) \\
    &\quad - \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \int_{0}^{t} \int_{0}^{\infty} \left( \cos \frac{(x + \tau)^2}{4u} + \sin \frac{(x + \tau)^2}{4u} \right) B_1(\tau) d\tau du \\
    &\quad + \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \int_{x}^{\infty} \left( \cos \frac{(u - x)^2}{4t} + \sin \frac{(u - x)^2}{4t} \right) B_0(u) du
\end{align*}
\]
\[ + \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \left( \cos \frac{x^2}{4t} + \sin \frac{x^2}{4t} \right) B_0(x) \]

\[ - \frac{\sqrt{2}}{4} \frac{1}{\sqrt{\pi t}} \int_0^\infty \left( \cos \frac{(x + \tau)^2}{4t} + \sin \frac{(x + \tau)^2}{4t} \right) B_0(\tau) d\tau. \]  

\[ (3.28) \]

Comments

1. Functions \( A_2 \) and \( A_3 \) we can express by \( B_0 \) and \( B_1 \) using (3.12) and (3.24).

2. If we have also some boundary conditions, then we try to settle \( B_0 \) and \( B_1 \) in such a way that they are satisfied, if it is possible.

3. If in the initial conditions (3.6)

\[ A_2(t) = -2 \nu^2 C_1 g(t), \quad A_3(t) = -2 \nu^3 C_2 g(t) \]  

\[ B_0(x) = K_1 v(x) \quad \text{and} \quad B_1(x) = \mu K_2 v(x), \]

then it follows by (3.8) that \( u(x, t) = v(x) g(t) \) is a solution to (3.1), as well, with initial condition (3.6), where

\[ v(x) = C_1 \cos \nu x + C_2 \sin \nu x + C_3 \cos h\nu x + C_4 \sin h\nu x \]  

\[ g(t) = K_1 \cos \mu t + K_2 \sin \mu t; \]

\( \nu = \sqrt{\omega^2}, \quad \mu = \sqrt{\omega^2}, \quad \omega > 0; \quad K_1, K_2, C_i, \ i = 1, ..., 4, \) are constants.

Proof. To prove that \( u(x, t) = v(x) g(t) \) is a solution to (3.1) with (3.6) and (3.29) which satisfies (3.8) we use the known properties of \( f \) and \( g \) (cf. [1]):

\[ \hat{v}(z) = -\frac{2}{z^4 - \omega^2} (\nu^2 C_1 z + \nu^3 C_2) \]

\[ \hat{g}(s) = \frac{1}{s^2 + \omega_2^2} (K_1 s + \mu K_2). \]

Then

\[ \mathcal{L}(u)(z, s) = -\frac{2(\nu^2 C_1 z + \nu^3 C_2 \hat{g}(s))}{z^4 + s^2} \]

\[ + \frac{(K_1 s + \mu K_2) \hat{v}(z)}{z^4 + s^2} = \frac{\hat{v}(z) \hat{g}(s)}{z^4 + s^2} = \hat{v}(z) \hat{g}(s). \]
4. There exists one and only one solution \( u(x, t) \) to (3.1) for \( x > 0, \ t > 0 \) which satisfies (3.6)\(_1^1\) and (3.6)\(_3^3\) with fixed \( B_0, B_1 \), such that
\[
[H_2(x, t)u(x, t)] \in e^{\sigma(x+t)}S'(\mathbb{R}^2_+).
\]

Proof. Suppose to have two solutions to (3.1), \( u_1 \) and \( u_2 \). Let \( U = u_1 - u_2 \) and \( a_i = A^1_i - A^2_i, \ i = 2, 3; \ A^1_i, A^2_i \) are given in (3.6)\(_2^2\) for \( u_1 \) and \( u_2 \), respectively. Then \( U \) satisfies
\[
\frac{\partial^4}{\partial x^4}[U(x, t)] + \frac{\partial^2}{\partial t^2}[U(x, t)] = [H(t)a_2(t)] \times \delta^{(1)}(x) + [H(t)a_3(t)] \times \delta(x).
\]

Because of (3.11) \( a_i(t) = 0, \ i = 2, 3 \). Therefore by (3.8), \( U(x, t) = 0, \ x \geq 0, t \geq 0 \).

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