A note on Poisson-Lie algebroids (I)

Liviu Popescu

Abstract. In this paper we generalize the linear contravariant connection on Poisson manifolds to Lie algebroids and study its tensors of torsion and curvature. A Poisson connection which depends only on the Poisson bivector and structural functions of the Lie algebroid is given. The notions of complete and horizontal lifts are introduced and their compatibility conditions are pointed out.

M.S.C. 2000: 53D17, 17B66, 53C05.
Key words: Poisson manifolds, Lie algebroids, contravariant connection, complete and horizontal lifts.

1 Introduction

Poisson manifolds were introduced by A. Lichnerowicz in his famous paper [11] and their properties were later investigated by A. Weinstein [19]. The Poisson manifolds are the smooth manifolds equipped with a Poisson bracket on their ring of functions. The Lie algebroid [12] is a generalization of a Lie algebra and integrable distribution. In fact, a Lie algebroid is a vector bundle with a Lie bracket on his space of sections whose properties are very similar to those of a tangent bundle. We remark that the cotangent bundle of a Poisson manifold has a natural structure of a Lie algebroid [18]. In the last years the various aspects of these subjects have been studied in the different directions of research ([18], [15], [3], [4] [13], [10], [1], [9], [2]). In [16] the author has investigated the properties of connections on a Lie algebroid and together with D. Hrimiuc [8] studied the nonlinear connections of its dual. In [3] the linear contravariant Poisson connections on vector bundle are pointed out.
The purpose of this paper is to study some aspects of the geometry of the Lie algebroids endowed with a Poisson structure, which generalize some results on Poisson manifolds to Lie algebroids. The paper is organized as follows. In the section 2 we recall the Cartan calculus and the Schouten-Nijenhuis bracket to the level of a Lie algebroids and introduce the Poisson structure on Lie algebroid. We investigate the properties of linear contravariant connection and its tensors of torsion and curvature. In the last part of this section we find a Poisson connection which depends only on the Poisson bivector and structural functions of Lie algebroid which generalize the

Balkan Journal of Geometry and Its Applications, Vol.14, No.2, 2009, pp. 79-89.

results of R. Fernandes from [3].
The section 3 deals with the prolongation of Lie algebroid [7] over the vector bundle projection. We study the properties of the complete lift of a Poisson bivector and introduce the notion of horizontal lift. The compatibility conditions of these bivectors are investigated. We remark that in the particular case of the Lie algebroid $(E = TM, \sigma = Id)$ some results of G. Mitric and I. Vaisman [15] are obtained.

2 Lie algebroids

Let us consider a differentiable, n-dimensional manifold M and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot], \sigma)$ where $\pi : E \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]$ and $\sigma : E \to TM$ is a bundle map (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\chi(M)$, satisfying the Leibnitz rule

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2,$$

for every $f \in C^\infty(M)$ and s_1, $s_2 \in \Gamma(E)$. Therefore, we get

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

If $\omega \in \bigwedge^k(E^*)$ then the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, ..., s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, ..., \hat{s}_i, ..., s_{k+1}) + \\ &+ \sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([s_{i,}s_j], s_1, ..., \hat{s}_i, ..., \hat{s}_j, ...s_{k+1}). \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it results that $(d^E)^2 = 0$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to ξ by

$$\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi,$$

where i_ξ is the contraction with ξ.
If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta}s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

are called the *structure functions* of the Lie algebroid and satisfy the *structure equations* on Lie algebroid

$$\sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0.$$

Locally, if $f \in C^\infty(M)$ then $d^E f = \frac{\partial f}{\partial x^i}\sigma^i_\alpha s^\alpha$, where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d^E\theta = (\sigma^i_\alpha \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2}\theta_\gamma L^\gamma_{\alpha\beta})s^\alpha \wedge s^\beta,$$

Particularly, we get

$$d^E x^i = \sigma^i_\alpha s^\alpha, \quad d^E s^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}s^\beta \wedge s^\gamma.$$

The Schouten-Nijenhuis bracket on E is given by [18]

$$[X_1 \wedge ... \wedge X_p, Y_1 \wedge ... \wedge Y_q] = (-1)^{p+1}\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_p \wedge \wedge Y_1 \wedge ... \wedge \hat{Y}_j \wedge ... \wedge Y_q$$

where $X_i, Y_j \in \Gamma(E)$.

2.1 Poisson structures on Lie algebroids

Let us consider the bivector (i.e. contravariant, skew-symmetric, 2-section) $\Pi \in \Gamma(\wedge^2 E)$ given by the expression

$$\Pi = \frac{1}{2}\pi^{\alpha\beta}(x)s_\alpha \wedge s_\beta. \tag{2.1}$$

Definition 2.1 The bivector Π is a Poisson bivector on E if and only if the relation

$$[\Pi, \Pi] = 0,$$

is fulfilled.

Proposition 2.1 *Locally, the condition* $[\Pi, \Pi] = 0$ *is expressed as*

$$\sum_{(\alpha,\varepsilon,\delta)}(\pi^{\alpha\beta}\sigma^i_\beta \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} + \pi^{\alpha\beta}\pi^{\gamma\delta}L^\varepsilon_{\beta\gamma}) = 0 \tag{2.2}$$

If Π is a Poisson bivector then the pair (E, Π) is called a *Lie algebroid with Poisson structure*. A corresponding Poisson bracket on M is given by

$$\{f_1, f_2\} = \Pi(d^E f_1, d^E f_2), \quad f_1, f_2 \in C^\infty(M).$$

We also have the bundle map $\pi^\# : E^* \to E$ defined by

$$\pi^\# \rho = i_\rho \Pi, \quad \rho \in \Gamma(E^*).$$

Let us consider the bracket

$$[\rho, \theta]_\pi = \mathcal{L}_{\pi^\#\rho}\theta - \mathcal{L}_{\pi^\#\theta}\rho - d^E(\Pi(\rho, \theta)),$$

where $\mathcal{L}$ is Lie derivative and $\rho, \theta \in \Gamma(E^*)$. With respect to this bracket and the usual Lie bracket on vector fields, the map $\widetilde{\sigma} : E^* \to TM$ given by

$$\widetilde{\sigma} = \sigma \circ \pi^{\#},$$

is a Lie algebra homomorphism

$$\widetilde{\sigma}[\rho, \theta]_\pi = [\widetilde{\sigma}\rho, \widetilde{\sigma}\theta].$$

The bracket $[.,.]_\pi$ satisfies also the Leibnitz rule

$$[\rho, f\theta]_\pi = f[\rho, \theta]_\pi + \widetilde{\sigma}(\rho)(f)\theta,$$

and it results that $(E^*, [.,.]_\pi, \widetilde{\sigma})$ is a Lie algebroid [13]. Next, we can define the contravariant exterior differential $d^\pi : \bigwedge^k(E^*) \to \bigwedge^{k+1}(E^*)$ by

$$\begin{aligned} d^\pi\omega(s_1, ..., s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\widetilde{\sigma}(s_i)\omega(s_1, ..., \hat{s}_i, ..., s_{k+1}) + \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j}\omega([s_i, s_j]_\pi, s_1, ..., \hat{s}_i, ..., \hat{s}_j, ...s_{k+1}). \end{aligned}$$

Accordingly, we get the cohomology of Lie algebroid E^* with the anchor $\widetilde{\sigma}$ and the bracket $[.,.]_\pi$ which generalize the Poisson cohomology of Lichnerowicz for Poisson manifolds [11].
In the following we deal with the notion of contravariant connection on Lie algebroids, which generalize the similar notion on Poisson manifolds [18], [3].

Definition 2.2 If $\rho, \theta \in \Gamma(E^*)$ and $\Phi, \Psi \in \Gamma(E)$ then the linear contravariant connection on a Lie algebroid is an application $D : \Gamma(E^*) \times \Gamma(E) \to \Gamma(E)$ which satisfies the relations:
i) $D_{\rho+\theta}\Phi = D_\rho\Phi + D_\theta\Phi$,
ii) $D_\rho(\Phi + \Psi) = D_\rho\Phi + D_\rho\Psi$,
iii) $D_{f\rho}\Phi = fD_\rho\Phi$,
iv) $D_\rho(f\Phi) = fD_\rho\Phi + \widetilde{\sigma}(\rho)(f)\Phi, \quad f \in C^\infty(M)$.

Definition 2.3 The torsion and curvature of the linear contravariant connection are given by

$$T(\rho, \theta) = D_\rho\theta - D_\theta\rho - [\rho, \theta]_\pi,$$

$$R(\rho, \theta)\mu = D_\rho D_\theta\mu - D_\theta D_\rho\mu - D_{[\rho,\theta]_\pi}\theta,$$

where $\rho, \theta, \mu \in \Gamma(E^*)$.

In the local coordinates we define the Christoffel symbols $\Gamma_\gamma^{\alpha\beta}$ considering

$$D_{s^\alpha}s^\beta = \Gamma_\gamma^{\alpha\beta}s^\gamma,$$

and under a change of coordinates $x^{i'} = x^{i'}(x^i)$, $i, i' = \overline{1, n}$ on M, and $y^{\alpha'} = A_\alpha^{\alpha'}y^\alpha$, $\alpha, \alpha' = \overline{1, m}$ on E, corresponding to a new base $s^{\alpha'} = A_\alpha^{\alpha'}s^\alpha$, these symbols transform according to

$$\Gamma^{\alpha'\beta'}_{\gamma'} = A^{\alpha'}_{\alpha}A^{\beta'}_{\beta}A^{\gamma}_{\gamma'}\Gamma^{\alpha\beta}_{\gamma} + A^{\alpha'}_{\alpha}A^{\gamma}_{\gamma'}\sigma^{i}_{\varepsilon}\frac{\partial A^{\beta'}_{\gamma}}{\partial x^{i}}\pi^{\alpha\varepsilon}. \tag{2.3}$$

Proposition 2.2 *The local components of the torsion and curvature of the linear contravariant connection on a Lie algebroid have the expressions*

$$T^{\alpha\beta}_{\varepsilon} = \Gamma^{\alpha\beta}_{\varepsilon} - \Gamma^{\beta\alpha}_{\varepsilon} - \pi^{\alpha\gamma}L^{\beta}_{\gamma\varepsilon} + \pi^{\beta\gamma}L^{\alpha}_{\gamma\varepsilon} - \sigma^{i}_{\varepsilon}\frac{\partial\pi^{\alpha\beta}}{\partial x^{i}},$$

$$R^{\alpha\beta\gamma}_{\delta} = \Gamma^{\alpha\varepsilon}_{\delta}\Gamma^{\beta\gamma}_{\varepsilon} - \Gamma^{\beta\varepsilon}_{\delta}\Gamma^{\alpha\gamma}_{\varepsilon} + \pi^{\alpha\varepsilon}\sigma^{i}_{\varepsilon}\frac{\partial\Gamma^{\beta\gamma}_{\delta}}{\partial x^{i}} - \pi^{\beta\varepsilon}\sigma^{i}_{\varepsilon}\frac{\partial\Gamma^{\alpha\gamma}_{\delta}}{\partial x^{i}} + (\pi^{\beta\nu}L^{\alpha}_{\nu\varepsilon} - \pi^{\alpha\nu}L^{\beta}_{\nu\varepsilon} - \sigma^{i}_{\varepsilon}\frac{\partial\pi^{\alpha\beta}}{\partial x^{i}})\Gamma^{\varepsilon\gamma}_{\delta}.$$

The contravariant connection induces a contravariant derivative $D_{\alpha} : \Gamma(E) \to \Gamma(E)$ such that the following relations are fulfilled

$$D_{f_1\alpha_1+f_2\alpha_2} = f_1D_{\alpha_1} + f_2D_{\alpha_2}, \quad f_i \in C^{\infty}(M), \quad \alpha_i \in \Gamma(E^*),$$

$$D_{\rho}(f\Phi) = fD_{\rho}\Phi + \widetilde{\sigma}(\rho)(f)\Phi, \quad f \in C^{\infty}(M), \quad \rho,\theta \in \Gamma(E^*).$$

Let T be a tensor of type (r, s) with the components $T^{i_1\ldots i_r}_{j_1\ldots j_s}$ and $\theta = \theta_{\alpha}s^{\alpha}$ a section of E^*. The local coordinates expression of the contravariant derivative is given by

$$D_{\theta}T = \theta_{\alpha}T^{i_1\ldots i_r}_{j_1\ldots j_s}/^{\alpha}s_{i_1}\otimes\cdots\otimes s_{i_r}\otimes s^{j_1}\otimes\cdots\otimes s^{j_s},$$

where

$$T^{i_1\ldots i_r}_{j_1\ldots j_s}/^{\alpha} = \pi^{\alpha\varepsilon}\sigma^{i}_{\varepsilon}\frac{\partial T^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial x^{i}} + \sum_{a=1}^{r}\left(\Gamma^{i_a\alpha}_{\varepsilon}T^{i_1\ldots\varepsilon\ldots i_r}_{j_1\ldots j_s}\right) - \sum_{b=1}^{s}\left(\Gamma^{\varepsilon\alpha}_{j_b}T^{i_1\ldots i_r}_{j_1\ldots\varepsilon\ldots j_s}\right),$$

and $/$ denote the *contravariat derivative operator*.
We recall that a tensor field T on E is called *parallel* if and only if $DT = 0$.

Definition 2.4 A contravariant connection D is called a Poisson connection if the Poisson bivector is parallel with respect to D.

Let us consider a contravariant connection D with the Cristoffel symbols $\Gamma^{\alpha\beta}_{\gamma}$ and associated contravariant derivative. We obtain:

Proposition 2.3 *The contravariant connection* $\overline{D}$ *with the coefficients given by*

$$\overline{\Gamma}^{\alpha\beta}_{\gamma} = \Gamma^{\alpha\beta}_{\gamma} - \frac{1}{2}\pi_{\gamma\varepsilon}\pi^{\alpha\varepsilon}/^{\beta}, \tag{2.4}$$

is a Poisson connection.

Proof. Considering $\overline{/}$ the contravariant derivative operator with respect to the contravariant connection $\overline{D}$, we get

$$\pi^{\beta\gamma}\overline{/}^{\alpha} = \pi^{\alpha\varepsilon}\sigma^{i}_{\varepsilon}\frac{\partial\pi^{\beta\gamma}}{\partial x^{i}} + \overline{\Gamma}^{\beta\alpha}_{\varepsilon}\pi^{\varepsilon\gamma} + \overline{\Gamma}^{\gamma\alpha}_{\varepsilon}\pi^{\beta\varepsilon} =$$

$$= \pi^{\alpha\varepsilon}\sigma_\varepsilon^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i} + \left(\Gamma_\varepsilon^{\beta\alpha} - \frac{1}{2}\pi_{\varepsilon\tau}\pi^{\beta\tau}/^\alpha\right)\pi^{\varepsilon\gamma} + \left(\Gamma_\varepsilon^{\gamma\alpha} - \frac{1}{2}\pi_{\varepsilon\tau}\pi^{\gamma\tau}/^\alpha\right)\pi^{\beta\varepsilon} = 0,$$

which concludes the proof. □

Remark 2.1 *Considering the expression*

$$\Gamma_\gamma^{\alpha\beta} = \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i}, \tag{2.5}$$

in (2.4) we obtain a Poisson connection $\overline{D}$ with the coefficients

$$\overline{\Gamma}_\gamma^{\alpha\beta} = \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} - \frac{1}{2}\pi_{\gamma\varepsilon}\pi^{\alpha\varepsilon}/^\beta,$$

which depends only on the Poisson bivector and structural functions of the Lie algebroid.

Proof. Under a change of coordinates, the structure functions σ_α^i change by the rule [17], [14]

$$\sigma_{\alpha'}^{i'} A_\alpha^{\alpha'} = \frac{\partial x^{i'}}{\partial x^i}\sigma_\alpha^i,$$

and, by direct computation, follows that the coefficients (2.5) satisfy the transformation law (2.3). □

Theorem 2.1 *If the following relation*

$$\sum_{(\alpha,\varepsilon,\delta)} \pi^{\alpha\beta}\pi^{\gamma\delta}L_{\beta\gamma}^\varepsilon = 0,$$

is true, then the connection D with the coefficients

$$\Gamma_\gamma^{\alpha\beta} = \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i},$$

is a Poisson connection on Lie algebroid.

Proof. Using relation (2.2) we obtain that $\pi^{\beta\gamma}/^\alpha = 0$ if and only if the required relation is fulfilled. □

Proposition 2.4 *The set of Poisson connections on Lie algebroid is given by*

$$\overline{\Gamma}_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta} + \Omega_{\gamma\nu}^{\alpha\varepsilon}X_\varepsilon^{\nu\beta},$$

where

$$\Omega_{\gamma\nu}^{\alpha\varepsilon} = \frac{1}{2}\left(\delta_\nu^\alpha\delta_\gamma^\varepsilon - \pi_{\gamma\nu}\pi^{\alpha\varepsilon}\right),$$

and $D(\Gamma_\gamma^{\alpha\beta})$ is a Poisson connection with $X_\varepsilon^{\delta\beta}$ an arbitrary tensor.

Proof. By straightforward computation it results

$$\pi^{\beta\gamma}\overline{/}^{\alpha} = \pi^{\alpha\varepsilon}\sigma^i_\varepsilon \frac{\partial \pi^{\beta\gamma}}{\partial x^i} + \overline{\Gamma}^{\beta\alpha}_\varepsilon \pi^{\varepsilon\gamma} + \overline{\Gamma}^{\gamma\alpha}_\varepsilon \pi^{\beta\varepsilon} =$$

$$\pi^{\beta\gamma}/^{\alpha} + \frac{1}{2}\pi^{\varepsilon\gamma}(\delta^\beta_\nu \delta^\theta_\varepsilon - \pi_{\varepsilon\nu}\pi^{\beta\theta})X^{\nu\alpha}_\theta + \frac{1}{2}\pi^{\beta\varepsilon}(\delta^\gamma_\nu \delta^\theta_\varepsilon - \pi_{\varepsilon\nu}\pi^{\gamma\theta})X^{\nu\alpha}_\theta =$$

$$\pi^{\beta\gamma}/^{\alpha} + \frac{1}{2}\pi^{\theta\gamma}X^{\beta\alpha}_\theta - \frac{1}{2}\pi^{\beta\theta}X^{\gamma\alpha}_\theta + \frac{1}{2}\pi^{\beta\theta}X^{\gamma\alpha}_\theta - \frac{1}{2}\pi^{\theta\gamma}X^{\beta\alpha}_\theta = 0,$$

because $\pi^{\beta\gamma}/^{\alpha} = 0$, which ends the proof. □

3 The prolongation of Lie algebroid over the vector bundle projection

Let (E, π, M) be a vector bundle. For the projection $\pi : E \to M$ we can construct the prolongation of E (see [7], [14], [10], [16]). The associated vector bundle is $(\mathcal{T}E, \pi_2, E)$ where $\mathcal{T}E = \cup_{w\in E}\mathcal{T}_w E$ with

$$\mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w\pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection $\pi_2(u_x, v_w) = \pi_E(v_w) = w$, where $\pi_E : TE \to E$ is the tangent projection. The canonical projection $\pi_1 : \mathcal{T}E \to E$ is given by $\pi_1(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \to TE$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over manifold E. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1. We will denote $(V\mathcal{T}E, \pi_{2|_{V\mathcal{T}E}}, E)$ the vertical bundle of $(\mathcal{T}E, \pi_2, E)$. If $f \in C^\infty(M)$ we will denote by f^c and f^v the *complete and vertical lift* to E of f defined by

$$f^c(u) = \sigma(u)(f), \quad f^v(u) = f(\pi(u)), \quad u \in E.$$

For $s \in \Gamma(E)$ we can consider the *vertical lift* of s given by $s^v(u) = \varphi(s(\pi(u)))$, for $u \in E$, where $\varphi : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism. There exists a unique vector field s^c on E, the *complete lift* of s satisfying the two following conditions:
i) s^c is π-projectable on $\sigma(s)$,
ii) $s^c(\overset{\wedge}{\alpha}) = \widehat{\mathcal{L}_s\alpha}$,
for all $\alpha \in \Gamma(E^*)$, where $\overset{\wedge}{\alpha}(u) = \alpha(\pi(u))(u)$, $u \in E$ (see [5] [6]).
Considering the prolongation $\mathcal{T}E$ of E over the projection π, we may introduce the *vertical lift* $s^{\mathbf{v}}$ and the *complete lift* $s^{\mathbf{c}}$ of a section $s \in \Gamma(E)$ as the sections of $\mathcal{T}E \to E$ given by (see [14])

$$s^{\mathrm{v}}(u) = (0, s^v(u)), \quad s^{\mathrm{c}}(u) = (s(\pi(u)), s^c(u)), \quad u \in E.$$

Another canonical object on $\mathcal{T}E$ is the *Euler section* C, which is the section of $\mathcal{T}E \to E$ defined by $C(u) = (0, \varphi(u))$ for all $u \in E$.
The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where

$$\mathcal{X}_\alpha(u) = \left(s_\alpha(\pi(u)), \sigma^i_\alpha \frac{\partial}{\partial x^i}\bigg|_u\right), \quad \mathcal{V}_\alpha(u) = \left(0, \frac{\partial}{\partial y^\alpha}\bigg|_u\right),$$

and $(\partial/\partial x^i, \partial/\partial y^\alpha)$ is the local basis on TE. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha},$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0.$$

The vertical lift of a section $\rho = \rho^\alpha s_\alpha$ and the corresponding vector field are $\rho^{\mathrm{v}} = \rho^\alpha \mathcal{V}_\alpha$ and $\sigma^1(\rho^{\mathrm{v}}) = \rho^\alpha \frac{\partial}{\partial y^\alpha}$. The expression of the complete lift of a section ρ is

$$\rho^{\mathrm{c}} = \rho^\alpha \mathcal{X}_\alpha + (\dot{\rho}^\alpha - L^\alpha_{\beta\gamma}\rho^\beta y^\gamma)\mathcal{V}_\alpha,$$

and therefore

$$\sigma^1(\rho^{\mathrm{c}}) = \rho^\alpha \sigma^i_\alpha \frac{\partial}{\partial x^i} + (\sigma^i_\gamma \frac{\partial \rho^\alpha}{\partial x^i} - L^\alpha_{\beta\gamma}\rho^\beta) y^\gamma \frac{\partial}{\partial y^\alpha}.$$

In particular

$$s^{\mathrm{v}}_\alpha = \mathcal{V}_\alpha, \quad s^{\mathrm{c}}_\alpha = \mathcal{X}_\alpha - L^\beta_{\alpha\gamma} y^\gamma \mathcal{V}_\beta.$$

The coordinate expressions of C and $\sigma^1(C)$ are

$$C = y^\alpha \mathcal{V}_\alpha, \quad \sigma^1(C) = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

The local expression of the differential of a function L on $\mathcal{T}E$ is $d^E L = \sigma^i_\alpha \frac{\partial L}{\partial x^i}\mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha}\mathcal{V}^\alpha$, where$\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ and therefore, we have $d^E x^i = \sigma^i_\alpha \mathcal{X}^\alpha$ and $d^E y^\alpha = \mathcal{V}^\alpha$. The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E \mathcal{V}^\alpha = 0.$$

A nonlinear connection N on $\mathcal{T}E$ [17] is an $m-$dimensional distribution (called *horizontal distribution*) $N : u \in E \to H\mathcal{T}_u E \subset \mathcal{T}E$ that is supplementary to the vertical distribution. This means that we have the following decomposition $\mathcal{T}_u E = H\mathcal{T}_u E \oplus V\mathcal{T}_u E$, for $u \in E$. A connection N on $\mathcal{T}E$ induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E \to \mathcal{T}E$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E)$. We have

$$\mathrm{h} = \frac{1}{2}(id + N), \quad \mathrm{v} = \frac{1}{2}(id - N).$$

The sections

$$\delta_\alpha = (\mathcal{X}_\alpha)^{\mathrm{h}} = \mathcal{X}_\alpha - N^\beta_\alpha \mathcal{V}_\beta,$$

generate a basis of $H\mathcal{T}E$, where N^β_α are the coefficients of nonlinear connection. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called *adapted*. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - N^\alpha_\beta \mathcal{X}^\beta$. The Lie brackets of the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are [16]

$$[\delta_\alpha, \delta_\beta] = L^\gamma_{\alpha\beta}\delta_\gamma + \mathcal{R}^\gamma_{\alpha\beta}\mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta] = \frac{\partial \mathcal{N}^\gamma_\alpha}{\partial y^\beta}\mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0,$$

where

$$\mathcal{R}^{\gamma}_{\alpha\beta} = \delta_\beta(\mathcal{N}^{\gamma}_{\alpha}) - \delta_\alpha(\mathcal{N}^{\gamma}_{\beta}) + L^{\varepsilon}_{\alpha\beta}\mathcal{N}^{\gamma}_{\varepsilon}. \tag{3.1}$$

The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E$ is given by $\Omega = -\mathrm{N_h}$ where h is horizontal projector and $\mathrm{N_h}$ is the Nijenhuis tensor of h. In the local coordinates we have

$$\Omega = -\frac{1}{2}\mathcal{R}^{\gamma}_{\alpha\beta}\mathcal{X}^{\alpha}\wedge\mathcal{X}^{\beta}\otimes\mathcal{V}_{\gamma},$$

where $\mathcal{R}^{\gamma}_{\alpha\beta}$ are given by (3.1) and represent the local coordinate functions of the curvature tensor Ω in the frame $\bigwedge^2\mathcal{T}E^*\otimes\mathcal{T}E$ induced by $\{\mathcal{X}_\alpha,\mathcal{V}_\alpha\}$.

3.1 Compatible Poisson structures

Let us consider the Poisson bivector on Lie algebroid given by relation (2.1). We obtain:

Proposition 3.1 *The complete lift of* Π *on* $\mathcal{T}E$ *is given by*

$$\Pi^{\mathrm{c}} = \pi^{\alpha\beta}\mathcal{X}_\alpha\wedge\mathcal{V}_\beta + \left(\frac{1}{2}\sigma^i_\gamma\frac{\partial\pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta}L^{\alpha}_{\delta\gamma}\right)y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta. \tag{3.2}$$

Proof. Using the properties of vertical and complete lifts we obtain

$$\begin{aligned}\Pi^{\mathrm{c}} &= (\tfrac{1}{2}\pi^{\alpha\beta}s_\alpha\wedge s_\beta)^{\mathrm{c}} = (\tfrac{1}{2}\pi^{\alpha\beta})^{\mathrm{c}}(s_\alpha\wedge s_\beta)^{\mathrm{v}} + (\tfrac{1}{2}\pi^{\alpha\beta})^{\mathrm{v}}(s_\alpha\wedge s_\beta)^{\mathrm{c}} = \\ &= \tfrac{1}{2}\dot{\pi}^{\alpha\beta}s^{\mathrm{v}}_\alpha\wedge s^{\mathrm{v}}_\beta + \tfrac{1}{2}\pi^{\alpha\beta}(s^{\mathrm{c}}_\alpha\wedge s^{\mathrm{v}}_\beta + s^{\mathrm{v}}_\alpha\wedge s^{\mathrm{c}}_\beta) = \tfrac{1}{2}\dot{\pi}^{\alpha\beta}\mathcal{V}_\alpha\wedge\mathcal{V}_\beta + \\ &+ \tfrac{1}{2}\pi^{\alpha\beta}\left((\mathcal{X}_\alpha - L^{\delta}_{\alpha\gamma}y^\gamma\mathcal{V}_\delta)\wedge\mathcal{V}_\beta + \mathcal{V}_\alpha\wedge(\mathcal{X}_\beta - L^{\delta}_{\beta\gamma}y^\gamma)\mathcal{V}_\delta\right) = \\ &= \pi^{\alpha\beta}\mathcal{X}_\alpha\wedge\mathcal{V}_\beta + \left(\tfrac{1}{2}\sigma^i_\gamma\tfrac{\partial\pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta}L^{\alpha}_{\delta\gamma}\right)y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta.\end{aligned}$$

□

Proposition 3.2 *The complete lift* Π^{c} *is a Poisson bivector on* $\mathcal{T}E$.

Proof. Using the relations (3.2) and (2.2), by straightforward computation, we obtain that $[\Pi^{\mathrm{c}},\Pi^{\mathrm{c}}] = 0$, which ends the proof. □

Proposition 3.3 *The Poisson structure* Π^{c} *has the following property*

$$\Pi^{\mathrm{c}} = -\mathcal{L}_C\Pi^{\mathrm{c}},$$

which means that $(\mathcal{T}E,\Pi^{\mathrm{c}})$ *is a homogeneous Poisson manifold.*

Proof. A direct computation in local coordinates yields

$$\begin{aligned}\mathcal{L}_C\Pi^c &= \mathcal{L}_{y^\varepsilon\mathcal{V}_\varepsilon}\left(\pi^{\alpha\beta}\mathcal{X}_\alpha\wedge\mathcal{V}_\beta + (\tfrac{1}{2}\sigma^i_\gamma\tfrac{\partial\pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta}L^{\alpha}_{\delta\gamma})y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta\right) \\ &= \mathcal{L}_{y^\varepsilon\mathcal{V}_\varepsilon}(\pi^{\alpha\beta}\mathcal{X}_\alpha)\wedge\mathcal{V}_\beta - \pi^{\alpha\beta}\mathcal{X}_\alpha\wedge\mathcal{V}_\beta + \mathcal{L}_{y^\varepsilon\mathcal{V}_\varepsilon}(\tfrac{1}{2}\sigma^i_\gamma\tfrac{\partial\pi^{\alpha\beta}}{\partial x^i}y^\gamma\mathcal{V}_\alpha)\wedge\mathcal{V}_\beta \\ &- \tfrac{1}{2}\sigma^i_\gamma\tfrac{\partial\pi^{\alpha\beta}}{\partial x^i}y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta - (\mathcal{L}_{y^\varepsilon\mathcal{V}_\varepsilon}\pi^{\delta\beta}L^{\alpha}_{\delta\gamma}y^\gamma\mathcal{V}_\alpha)\wedge\mathcal{V}_\beta + \pi^{\delta\beta}L^{\alpha}_{\delta\gamma}y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta \\ &= -\pi^{\alpha\beta}\mathcal{X}_\alpha\wedge\mathcal{V}_\beta - \tfrac{1}{2}\sigma^i_\gamma\tfrac{\partial\pi^{\alpha\beta}}{\partial x^i}y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta + \pi^{\delta\beta}L^{\alpha}_{\delta\gamma}y^\gamma\mathcal{V}_\alpha\wedge\mathcal{V}_\beta \\ &= -\Pi^c\end{aligned}$$

□

Definition 3.1 Let us consider a Poisson bivector on E given by (2.1) then the horizontal lift of Π to $\mathcal{T}E$ is the bivector defined by

$$\Pi^H = \frac{1}{2}\pi^{\alpha\beta}(x)\delta_\alpha \wedge \delta_\beta.$$

Proposition 3.4 *The horizontal lift Π^H is a Poisson bivector if and only if Π is a Poisson bivector on E and the following relation*

$$\pi^{\alpha\beta}\pi^{\gamma\delta}\mathcal{R}^{\varepsilon}_{\beta\gamma} = 0,$$

is fulfilled.

Proof. The Poisson condition $[\Pi, \Pi] = 0$ leads to the relation (2.2) and equation $[\Pi^H, \Pi^H] = 0$ yields

$$\sum_{(\varepsilon,\delta,\alpha)} \left(\pi^{\alpha\beta}\pi^{\gamma\delta}L^{\varepsilon}_{\beta\gamma} + \pi^{\alpha\beta}\sigma^i_\beta \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} \right) \delta_\varepsilon \wedge \delta_\alpha \wedge \delta_\delta + \pi^{\alpha\beta}\pi^{\gamma\delta}\mathcal{R}^{\varepsilon}_{\beta\gamma}\mathcal{V}_\varepsilon \wedge \delta_\alpha \wedge \delta_\gamma = 0,$$

which ends the proof. □

We recall that two Poisson structures are called compatible if the bivectors ω_1 and ω_2 satisfy the condition

$$[\omega_1, \omega_2] = 0.$$

By straightforward computation in local coordinates we get:

Proposition 3.5 *The Poisson bivector Π^H is compatible with the complete lift Π^c if and only if the following relations hold*

$$\pi^{r\beta}\pi^{\alpha s}\left(\frac{\partial \mathcal{N}^{\gamma}_r}{\partial y^s} - \frac{\partial \mathcal{N}^{\gamma}_s}{\partial y^r}\right) - \pi^{r\gamma}\pi^{s\alpha}L^{\beta}_{sr} = 0,$$

$$\pi^{rs}\left(\delta_r(a^{\alpha\beta}) - a^{l\alpha}\frac{\partial N^{\beta}_r}{\partial y^l} + a^{l\beta}\frac{\partial N^{\alpha}_r}{\partial y^l} - \right.$$

$$-\pi^{\theta\beta}\mathcal{R}^{\alpha}_{r\theta} + (\pi^{\varepsilon\beta}L^{\theta}_{\varepsilon\gamma} - \pi^{\varepsilon\theta}L^{\beta}_{\varepsilon\gamma})y^\gamma\frac{\partial N^{\alpha}_r}{\partial y^\theta} +$$

$$\left. + \sigma^i_r\frac{\partial \pi^{\varepsilon\beta}}{\partial x^i}y^\gamma L^{\alpha}_{\varepsilon\gamma} + \pi^{\varepsilon\beta}L^{\alpha}_{\varepsilon\gamma}N^{\gamma}_r - \pi^{\varepsilon\beta}\sigma^i_r\frac{\partial L^{\alpha}_{\varepsilon\gamma}}{\partial x^i}y^\gamma \right) = 0.$$

where we have denoted

$$a^{\alpha\beta} = \sigma^i_\varepsilon\frac{\partial \pi^{\alpha\beta}}{\partial x^i}y^\varepsilon + N^{\alpha}_\varepsilon\pi^{\varepsilon\beta} - N^{\beta}_\varepsilon\pi^{\varepsilon\alpha}.$$

Acknowledgements. The author wishes to express his thanks to the organizers of the International Conference DGDS-2008 and MENP-5, August 28 - September 2, 2008, where this paper has presented. Also, many thanks to the referee for useful remarks concerning this paper.

References

[1] M. Anastasiei, *Geometry of Lagrangians and semisprays on Lie algebroids*, Proc. Conf. Balkan Society of Geometers, Mangalia, 2005, BSGP, (2006), 10-17.

[2] Carlo Cattani, *Harmonic wavelet solution of Poisson's problem*, Balkan Journal of Geometry and Its Applications, 13, 1 (2008), 27-37.

[3] R. L. Fernandes, *Connections in Poisson geometry I: holonomy and invariants*, J. Differential Geometry, 54 (2000), 303-365.

[4] R. L. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Advances in Mathematics, 170 (2002), 119-179.

[5] J. Grabowski, P. Urbanski, *Tangent and cotangent lift and graded Lie algebra associated with Lie algebroids*, Ann. Global Anal. Geom. 15, (1997), 447-486.

[6] J. Grabowski, P. Urbanski, *Lie algebroids and Poisson-Nijenhuis structures*, Rep. Math. Phys., 40 (1997), 195-208.

[7] P. J. Higgins, K. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, Journal of Algebra 129 (1990), 194-230.

[8] D. Hrimiuc, L. Popescu, *Nonlinear connections on dual Lie algebroids*, Balkan Journal of Geometry and Its Applications, 11, 1 (2006), 73-80.

[9] M. Ivan, Gh. Ivan, D. Opris, *The Maxwell-Block equations on fractional Leibniz algebroids*, Balkan Journal of Geometry and Its Applications, 13, 2 (2008), 50-58.

[10] M. de Leon, J. C. Marrero, E. Martinez, *Lagrangian submanifolds and dynamics on Lie algebroids* J. Phys. A: Math. Gen. 38 (2005), 241-308.

[11] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differential Geometry, 12 (1977), 253-300.

[12] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, Cambridge, 124, 1987.

[13] K. Mackenzie, Ping Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. Journal, 73, 2 (1994), 415-452.

[14] E. Martinez, *Lagrangian mechanics on Lie algebroids*, Acta Appl. Math., 67, (2001), 295-320.

[15] G. Mitric, I. Vaisman, *Poisson structures on tangent bundles*, Diff. Geom. and Appl., 18 (2003), 207-228.

[16] L. Popescu, *Geometrical structures on Lie algebroids*, Publ. Math. Debrecen 72, 1-2 (2008), 95-109.

[17] P. Popescu, *On the geometry of relative tangent spaces*, Rev. Roumain Math. Pures and Applications, 37, 8 (1992), 727-733.

[18] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Math., 118, Birkhäuser, Berlin, 1994.

[19] A. Weinstein, *The local structure of Poisson manifolds,* J. Differential Geometry, 18 (1983), 523-557

Author's address:

Liviu Popescu
University of Craiova,
Dept. of Applied Mathematics in Economy
13 A.I.Cuza st., 200585, Craiova, Romania
e-mail: `liviupopescu@central.ucv.ro, liviunew@yahoo.com`