Tzitzeica theory - opportunity for reflection in Mathematics

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Abstract

Our aim is to emphasize the topical interest, richness and vitality of Tzitzeica theory. Rather than aspiring to a comprehensive treatise, this Note contains the mathematical discussion of only those topics which are in connection with our concerns. That is why, we have in view four aspects :

· to recall the symmetry groups associated to Tzitzeica PDEs;

· to emphasize again that the Tzitzeica PDEs are Euler -Lagrange equations;

· to reveal some physical roots of Tzitzeica theory;

· to underline new properties of Tzitzeica PDEs and their connection to Painlevé ODEs.

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1 Introduction

Gheorghe Tzitzeica is known as one of the founders of the *centro-affine differential geometry*. He introduced a class of surfaces and a class of curves that today carry his name. Also, he realized the *curve net theory* based on a partial derivative equation of Laplace type. The work of Gheorghe Tzitzeica is a permanent incitation to mathematical reflection. It inspired many mathematicians, but the most faithful continuers have been the Romanians Alexandru Myller, Octav Mayer, Gheorghe Theodor Gheorghiu and Gheorghe Călugăreanu. The Tzitzeica ideas have reappeared in recent times due to their relation with groups of symmetries, variational equations, electrons theory, soliton theory, etc.

2 PDEs of Tzitzeica type

In 1907, Gheorghe Tzitzeica [9] introduced a famous class of surfaces, that now carries his name. Locally, these surfaces can be considered as graphs of functions $u = u(x, y)$ which satisfy the partial derivative equation

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$$u_{xx}u_{yy} - u_{xy}^2 = c(xu_x + yu_y - u)^4,\ c = const \neq 0. \tag{2.1}$$

In 1923, still Gheorghe Tzitzeica [10] has described the surfaces that carry his name as solutions of a completely integrable PDEs system

$$r_{uu} = ar_u + br_v,\ r_{uv} = hr,\ r_{vv} = a''r_u + b''r_v, \tag{2.2}$$

where $r = x\bar{i} + y\bar{j} + z\bar{k}$.

More precisely:

1) *the ruled Tzitzeica surfaces* are defined as solutions of the PDEs system

$$r_{uu} = \frac{h_u}{h}r_u + \frac{\varphi(u)}{h}r_v,\ r_{uv} = hr,\ r_{vv} = \frac{h_v}{h}r_v, \tag{2.3}$$

where h is a solution of partial derivative equation (complete integrability condition)

$$(\ln h)_{uv} = h; \tag{2.4}$$

2) the *non-ruled Tzitzeica surfaces* are defined as solutions of the PDEs system

$$r_{uu} = \frac{h_u}{h}r_u + \frac{1}{h}r_v,\ r_{uv} = hr,\ r_{vv} = \frac{1}{h}r_u + \frac{h_v}{h}r_v, \tag{2.5}$$

where h is a solution of partial derivative equation (complete integrability condition)

$$(\ln h)_{uv} = h - \frac{1}{h^2}. \tag{2.6}$$

With a change of function ln $h = \omega$ the equations (2.4) and (2.6) rewrite respectively

$$\omega_{uv} = e^{\omega}, \tag{2.7}$$

and

$$\omega_{uv} = e^{\omega} - e^{-2\omega}. \tag{2.8}$$

Being impressed by the Tzitzeica genius and stimulated by the papers of Bobenko [2] and Wolf [18], as well as the debates with Romanian and foreigner [6], [8] geometers, together Nicoleta Bâlă, the author studied two problems [1],[11]:

1) the characterization of the symmetry groups associated to the equations (2.1)-(2.4);

2) showing that the equations (2.1), (2.3), (2.4) are equivalent to Euler-Lagrange equations.

The surprises where in line with expectations:

A. The symmetry group associated to the equation (2.1) is the *unimodular subgroup* of the *centro-affine group*. The Lie algebra of this group is generated by the vector fields

$$X_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, X_2 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, X_3 = y\frac{\partial}{\partial x}, X_4 = u\frac{\partial}{\partial x},$$

(2.9) $$X_5 = x\frac{\partial}{\partial y}, X_6 = u\frac{\partial}{\partial y}, X_7 = x\frac{\partial}{\partial u}, X_8 = y\frac{\partial}{\partial u}.$$

These infinitesimal generators have permitted the finding of group-invariant solutions of the equation (2.1). Also, we proved that the only partial derivative equation of Monge-Ampere-Tzitzeica type, invariant with respect to unimodular group (2.9), is the equation (2.1). The equation (2.1) is equivalent to an Euler-Lagrange equation produced by the second order Lagrangian

$$L_1(x, y, u^{(2)}) = \frac{u(u_{xy}^2 - u_{xx}u_{yy})}{(xu_x + yu_y - u)^4} - cu.$$

Automatically, it appears the variational symmetry group and adequate conservation laws. From this point of view, the problem is still open.

Open problem. There exists or not a first order Lagrangian producing the Tzitzeica equation?

B. The symmetry subgroup that acts on the space of independent variables and of the system (2.2) is generated by vector fields of the type

$$Z = \zeta(u)\frac{\partial}{\partial u} + \eta(v)\frac{\partial}{\partial v},$$

where the functions $\zeta(u), \eta(v)$ are solutions of a PDEs system. The symmetry subgroup which acts on the space of dependent variables x, y, z of the system (2.2) is the unimodular subgroup of the centro-affine group of generators

$$X_1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, X_2 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, X_3 = y\frac{\partial}{\partial x}, X_4 = z\frac{\partial}{\partial x},$$

$$X_5 = x\frac{\partial}{\partial y}, X_6 = z\frac{\partial}{\partial y}, X_7 = x\frac{\partial}{\partial z}, X_8 = y\frac{\partial}{\partial z}.$$

In fact, the previous result is found.

C. The general vector field which describes the infinitesimal symmetry algebra associated to the equation (2.7) is

$$W = f(u)\frac{\partial}{\partial u} + g(v)\frac{\partial}{\partial v} + (f'(u) + g'(v))\frac{\partial}{\partial u}.$$

The equation (2.7) is the Euler-Lagrange equation provided by the first order Lagrangian

$$L_2(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega}.$$

The Lie algebra of the variational symmetries for the action produced by L_2 is generated by the vector fields

$$W_1 = u\frac{\partial}{\partial u} - \frac{\partial}{\partial w}, W_2 = v\frac{\partial}{\partial v} - \frac{\partial}{\partial w}, W_3 = \frac{\partial}{\partial u}, W_4 = \frac{\partial}{\partial v}.$$

Automatically, adequate conservation laws have appeared. Also the equation (2.7) is conservative in the sense that the divergence of the momentum-energy tensor field

$$T^\alpha_\beta = \omega_\beta \frac{\partial L}{\partial \omega_\alpha} - L\delta^\alpha_\beta,\ \alpha, \beta = 1, 2,\ \omega_1 = \omega_u, \omega_2 = \omega_v$$

vanishes on the solutions of the equation.

D. The vector fields which generate the algebra of infinitesimal symmetries associated to the equation (2.8) are

$$U_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, U_2 = \frac{\partial}{\partial u}, U_3 = \frac{\partial}{\partial v}.$$

The equation (2.8) is the Euler-Lagrange equation produced by the first order Lagrangian

$$L_3(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^\omega - e^{-2\omega}.$$

The Lie algebra of variational symmetry group for the action produced by L_3 is the same with $\{U_1, U_2, U_3\}$. Automatically, one writes adequate conservation laws. We prefer to add that the equation (2.8) is conservative in the sense that the divergence of the momentum-energy tensor field vanishes on the solutions of the equation.

All the results underlined in the sections A-D confirm that Tzitzeica theory can be considered as essential part of variational principles on differential manifolds. It was a matter of course since the real world is governed, among other things, by optimum principles, and Gheorghe Tzitzeica realized indirectly that this is the clue of real problems.

3 Tzitzeica geometric dynamics

We find of interest to present shortly the *Tzitzeica geometric dynamics* introduced by us in [15]. Also we point out its connections with physical theories. It is well-known that the most simple *Tzitzeica surface* is that described by the equation $xyz = 1$ (see Fig.1). If we denote the *Tzitzeica potential* by $u(x, y, z) = xyz$, then its gradient lines are solutions of the first order ODEs system

$$\frac{dx}{dt} = yz, \frac{dy}{dt} = zx, \frac{dz}{dt} = xy.$$

Interpreting them as trajectories for the motion of a particle in R^3, we have well known properties:

1) if two of the numbers $x(0), y(0), z(0)$ are null, then the particle does not move;

2) if at least two of the initial values $x(0), y(0), z(0)$ are different from zero, then either the particle moves to infinity in a finite time (in future), or it comes from the infinity in a finite time (in the past).

This differential system together with Euclidean metric determine a geometric dynamics of Tzitzeica type described by the second order ODEs system

$$\frac{d^2x}{dt^2} = x(z^2 + y^2), \frac{d^2y}{dt^2} = y(x^2 + z^2), \frac{d^2z}{dt^2} = z(y^2 + x^2).$$

This conservative dynamics is characterized by the density of energy

$$2f(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$$

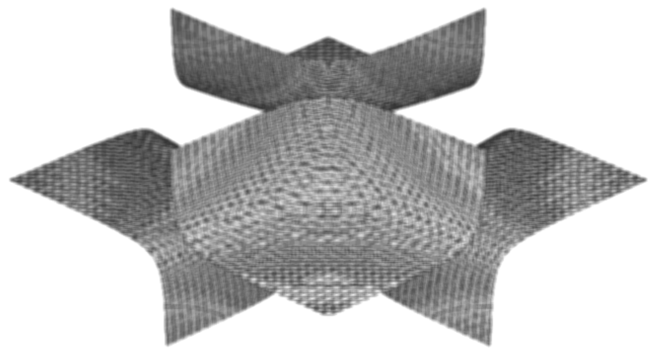

Figure 1: Tzitzeica surface

whose constant level sets are *totally h-geodesic* and hence *h-minimal* (see Fig.2). Surfaces with shapes similar to those in Fig. 2 were met as *Fermi surfaces* (surfaces of constant energy) in the theory of Wolfgang Pauli and Arnold Sommerfeld [7] for metal electrons (crystal meshes populated by electrons). For a complete theory of this type, see [12]-[17].

4 Tzitzeica law in economics

In the case of general economic equilibrium analysis *the demand and the supply functions* relative to each good are in principle functions of the prices of all goods, and not only of the price of the good to which they refer. That is why the problem of the static or dynamic stability of demand and supply requires the introduction of *excess demand vector field* E of components $E_i = E_i(p_1, ..., p_n), i = 1, ..., n$, where E_i is the excess demand for the $i - th$ good, and p_i are prices [3], [17].

From dynamical point of view, we must analyze the *excess demand flow*

$$\frac{dp_i}{dt} = E_i,\ i = 1, ..., n.$$

In this context, the *Walras law* $\delta^{ij} p_i E_j = 0$ means that $\delta^{ij} p_i p_j$ is a first integral of the excess demand flow. This condition is used for normalization and the completeness of the field E, because the constant level surfaces are spheres.

The *Tzitzeica law* $\sum p_1 ... p_{n-1} E_n = 0$ (*summation by cyclic permutations*), introduced in [3], [17], says that $p_1 ... p_n$ is a first integral of the excess demand flow. This reflects a constant *volume of the prices* since the tangent hyperplane to the hypersurface $p_1 ... p_n = C$, at an arbitrary point, determines together with the coordinate hyperplanes a hyper-tetrahedron of constant volume. Since this last condition represents a multiplicative effect, the individual prices act in series: the effect of process i with price p^i followed by process j with price p^j will be the process of price $p^i p^j$.

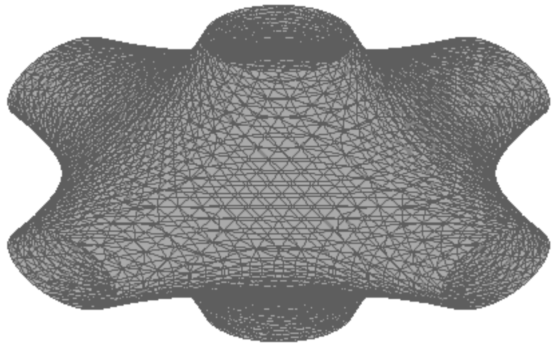

Figure 2: 0.5-Level set of energy density

Open problem. A function $f : R^n \to R$, $(x_1, ..., x_n) \to f(x_1, ..., x_n)$ whose value remains unchanged under any permutation of its independent variables is called *symmetric function*. Any rational symmetric function is a rational function of elementary symmetric polynomials

$$P_1 = \sum x_i,\ P_2 = \sum x_i x_j,\ P_3 = \sum x_i x_j x_k,\ ...,\ P_n = x_1 x_2 ... x_n,$$

where the summations are extended over all distinct products of distinct factors. If X is a C^∞ vector field of components X_i, then each condition of type

$$\sum X_i = 0,\ \sum (x_i X_j + X_i x_j) = 0,\ \sum (X_i x_j x_k + x_i X_j x_k + x_i x_j X_k) = 0,$$

$$...,\ \sum (X_1 x_2 ... x_n + x_1 X_2 x_3 ... x_n + ... + x_1 x_2 ... x_{n-1} X_n) = 0$$

generates a first integral of the flow

$$\frac{dx_i}{dt} = X_i(x).$$

If instead "equalities" we use the sign ≤ 0 or $<$, we obtain *Lyapunov functions*, respectively *strong Lyapunov functions*. Find the sense of such functions when the vector field X has a practical meaning.

5 Properties of Tzitzeica PDE

Let us use MAPLE simulations (PDEtools package) to find new properties of Tzitzeica PDE (2.1). The main result relates the Tzitzeica PDE to the second order Painlevé equations, $y''(x) = f(x, y(x), y'(x))$, i.e., equations where f is a rational function of y, y' with coefficients functions of x. The Painlevé equations have numerous applications to differential geometry, probability theory, soliton theory, topological field theory, and others [4], [5] [6], [8].

5.1 Decomposition of solutions

Theorem. *The Tzitzeica PDE (2.1) admits solutions of the form*

$$u(x,y) = \frac{a}{f(x)y} \ \ or\ u(x,y) = \frac{f(x)}{y}$$

if and only if the function f is solution of a second order Painlevé ODE.

Model 1, Proof.

$> PDE := diff(u(x,y),x,x) * diff(u(x,y),y,y) - diff(u(x,y),x,y)^2 = c*(x*diff(u(x,y),x) + y*diff(u(x,y),y) - u(x,y))^4;$

$> ansatz := u(x,y) = a/(f(x)*y);$

Use **pdetest** to simplify the PDE with regard to this ansatz.

$ans\_1 := pdetest(ansatz, PDE);$

$$ans\_1 := -\frac{a^2(-3f'^2f^4 + 2f''f^5 + cx^4a^2f'^4 + 8cx^3a^2f'^3f)}{f^8y^4}$$
$$-\frac{a^2(24cx^2a^2f'^2f^2 + 32cxa^2f'f^3 + 16ca^2f^4)}{f^8y^4}$$

The ansatz above separated the variables, so the solution of the PDE is determined by the function f which is a solution of a Painlevé equation.

$> ans\_f := dsolve(ans\_1, f(x));$

$$ans\_f := f =\_\ ae^{(\int\_\ b(\_a)d\_a+\_C1)}$$

$$\&\text{where}\left[\{\_b(\_a) = \frac{3(27a^2c -\_\ a^2)\_b(\_a)^3}{2\_a} + \frac{2(27a^2c -\_\ a^2)\_b(\_a)^2}{\_a^2}\right.$$

$$+\frac{3(18a^2c -\_\ a^2)\_b(\_a)}{2\_a^3} + \frac{6a^2c}{\_a^4} + \frac{a^2c}{2\_a^5\_b(\_a)}\},$$

$$\{\_a = \frac{f}{x}, \_\ b(\_a) = \frac{x}{f'x - f}\}$$

$$\left.\{f =\_\ ae^{(\int\_\ b(\_a)d\_a+\_C1)}, x =\_\ ae^{(\int\_\ b(\_a)d\_a+\_C1)}\}\right]$$

Model 2, Proof.

Another particular result can be obtained by separating the variables by product. We can use **HINT** option to obtain the general solution, inspired by the solution above.

$> struc1 := solve(PDE, HINT = f(x)/y);$

$struc1 := (u(x,y) = f/y)$

$$\&\text{where}\left[\{f'' = \frac{f'^2 - 8cx^3f'^3f + 24cx^2f'^2f^2 + cx^4f'^4 - 32cxf'f^3 + 16cf^4}{2f}\}\right]$$

$> struc2 := solve(PDE, HINT = f(x)/xy);$
$struc2 := (u(x, y) = f/xy)$

$$\&\text{where}\left[\{f'' = \frac{2ff'x - 3f^2 + f'^2x^2 - 12cx^3f'^3f}{2fx^2} + \frac{54cx^2f'^2f^2 - 108cxf'f^3 + cx^4f'^4 + 81cf^4}{2fx^2}\}\right]$$

In these two cases, the solution of the Tzitzeica PDE is determined by the function f which is a solution of a Painlevé equation.

Case c= -1: $> ode := diff(f(x), ‘\$‘(x, 2)) = 1/2 * (diff(f(x), x)^2 - x^4 * diff(f(x), x)^4 - 16 * f(x)^4 + 32 * x * diff(f(x), x) * f(x)^3 - 24 * x^2 * diff(f(x), x)^2 * f(x)^2 + 8 * x^3 * diff(f(x), x)^3 * f(x))/f(x);$

$> ans := dsolve(\{ode, f(0) = 1, D(f)(0) = 1\}, f(x), type = series);$
$ans := f = 1 + x - \frac{15}{4}x^2 - \frac{8}{3}x^3 - \frac{1}{3}x^4 - \frac{16}{5}x^5 + O(x^6)$

Case c= 1: $> ode1 := diff(f(x), ‘\$‘(x, 2)) = 1/2 * (diff(f(x), x)^2 + x^4 * diff(f(x), x)^4 + 16 * f(x)^4 - 32 * x * diff(f(x), x) * f(x)^3 + 24 * x^2 * diff(f(x), x)^2 * f(x)^2 - 8 * x^3 diff(f(x), x)^3 * f(x)/f(x);$

$> ans := dsolve(\{ode1, f(0) = 1, D(f)(0) = 1\}, f(x), type = series);$
$ans := f = 1 + x + \frac{17}{4}x^2 + \frac{8}{3}x^3 + \frac{1}{3}x^4 - \frac{16}{5}x^5 + O(x^6)$

Now let us refer to a boundary value problem with curvature c as unknown parameter:

$> ode2 := diff(f(x), ‘\$‘(x, 2)) = 1/2 * (2 * f(x) * diff(f(x), x) * x - 3 * f(x)^2 + diff(f(x), x)^2 * x^2 - 108 * c * x * diff(f(x), x) * f(x)^3 + 81 * c * f(x)^4 - 12 * c * x^3 * diff(f(x), x)^3 * f(x) + 54 * c * x^2 * diff(f(x), x)^2 * f(x)^2 + c * x^4 * diff(f(x), x)^4/f(x)/x^2, f(0.1) = 1, f(0.2) = 1.5, D(f)(0.1) = -2;$

$> dsol := dsolve(\{ode2\}, numeric);$
$dsol := proc(x_{bvp})...endproc$
$dsol(0.1);$
$[x = 0.1, f = 0.999999, f' = -1.999999, c = 0.112938]$

5.2 DEtools[symgen]

We look for a symmetry generator for previous ODEs.

Case c = -1

$> with(DEtools):$
$> PDEtools[declare](f(x), pime = x);$
$> ode3 := diff(f(x), ‘\$‘(x, 2)) = 1/2 * (diff(f(x), x)^2 - x^4 * diff(f(x), x)^4 - 16 * f(x)^4 + 32 * x * diff(f(x), x) * f(x)^3 - 24 * x^2 * diff(f(x), x)^2 * f(x)^2 + 8 * x^3 * diff(f(x), x)^3 * f(x))/f(x);$
$> odeadvisor(ode3);$
$> symgen(ode3);$
$[\_\xi = -x, \_\eta = f]$

Case c = 1

$> with(DEtools):$
$> PDEtools[declare](f(x), pime = x);$

$> ode4 := diff(f(x), `\$`(x, 2)) = 1/2 * (diff(f(x), x)^2$
$+x^4 * diff(f(x), x)^4 + 16 * f(x)^4 - 32 * x * diff(f(x), x) * f(x)^3 + 24 * x^2 * diff(f(x), x)^2 * f(x)^2 - 8 * x^3 * diff(f(x), x)^3 * f(x))/f(x);$
$> odeadvisor(ode4);$
$> symgen(ode4);$
$[\_\xi = -x, \_\eta = f]$

5.3 DEtools[buildsym]

Let us build the symmetry generator given a solution of an ODE.
$> with(DEtools, bildsym, equinv, symtest);$
We start with a pair of infinitesimals:
$sym := [\_\xi = -x, \_\eta = f]$;
The most general first order ODE invariant under the above flow is:
$> ODE := equinv(sym, f(x));$
$ODE := f' = \frac{\_F1(fx)}{x^2}$. This ODE can be solved using the following command:
$> ans := dsolve(ODE, Lie);$
$ans := f = \frac{RootOf\left(ln(x) - \int^{\_Z} \frac{1}{\_a + \_F1(\_a)} d\_a - \_C1\right)}{x}$
The infinitesimals can be reobtained from this solution:
$> buildsym(ans, f(x));$
$\left[\_\xi = 0, \_\eta = -\frac{\_F1(fx) + fx}{x}\right]$

5.4 convert/ODE

Let us convert the previous ode3 to the other ODE of different type:
$> convert(ode3, y\_x);$

$$x_{f,f} = \frac{1}{2}\frac{x(f)^4}{x_f f} - 4x(f)^3 + 12x_f f x(f)^2 - 16x_f^2 f^2 x(f) + 8f^3 x_f^3 - \frac{1}{2}\frac{x_f}{f}$$

Now, let us find the solution of a Cauchy problem attached to this second order ODE.
$> ode5 := diff(x(f), `\$`(f, 2)) = 1/2 * x(f)^4/(f * diff(x(f), f)) - 4 * x(f)^3 + 12 * diff(x(f), f) * f * x(f)^2 - 16 * diff(x(f), f)^2 * f^2 * x(f) + 8 * f^3 * diff(x(f), f)^3 - 1/2 * diff(x(f), f)/f;$
$> ans := dsolve(\{ode5, x(0) = 1, D(x)(0) = 1\}, x(f), type = series);$
$ans := x(f) = 1 + f - \frac{1}{2}f^2 + \frac{1}{2}f^3 - \frac{5}{8}f^4 + \frac{7}{8}f^5 + O(f^6)$

5.5 Tzitzeica PDE as hypersurface in second order jet space

Let us transpose the Tzitzeica PDE in the second order jet space of coordinates (x, y, z, p, q, r, s, t), i.e., $c(xp + yq - u)^4 = rt - s^2$. This represents a hypersurface of dimension 7 in an 8-dimensional space. To analyze this hypersurface we can use MAPLE:
$> f := c * (x * p + y * q - u)^4 - (r * t - s^2);$
$> f1 := diff(f, x); > f2 := diff(f, y); > f3 := diff(f, u);$

$> f4 := diff(f,p); > f5 := diff(f,q); > f6 := diff(f,r); > f7 := diff(f,s); > f8 := diff(f,t);$

It follows the critical points:

$> solve(\{f1 = 0, f2 = 0, f3 = 0, f4 = 0, f5 = 0, f6 = 0, f7 = 0, f8 = 0\}, \{x, y, u, p, q, r, s, t\});$

$\{p = p, q = q, s = 0, r = 0, t = 0, u = xp + yq, x = x, y = y\}$

To obtain information about the curvature, we can use:

$> with(linalg):$

$> H := hessian(f, [x, y, z, p, q, r, s, t]);$

$> D := det(H); D := -6144 * c^5 * (x * p + y * q - u)^{14}$

Consequently the sign of the curvature of the hypersurface $f = 0$ is opposite to the sign of the curvature c of the Tzitzeica surface $u = u(x, y)$.

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