The Tzitzeica surface as solution of PDE systems

Vladimir Balan and Ştefania-Alina Vîlcea

Abstract

The present note emphasizes that a classical family of Tzitzeica surfaces is provided as general solution for two certain PDE systems. The infinitesimal generators of the symmetry algebra of the second PDE system are explicitely determined.

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The classical family of Tzitzeica surfaces $\Sigma: \ xyz = c, \ c \neq 0$, viewed as plot of a Monge chart $z = u(x,y)$, where $u : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$,

$$u(x,y) = \frac{c}{xy}, \ \ \forall (x,y) \in \mathbb{R}^* \times \mathbb{R}^*. \tag{1.1}$$

It can be easily checked that this mapping satisfies the system of PDEs

$$\begin{cases} x^2 u_{xx} - 2u = 0 \\ y^2 u_{yy} - 2u = 0 \\ xy u_{xy} - u = 0. \end{cases} \tag{1.2}$$

The system can be rewritten in the terms of the *second prolongation* $u^{(2)} : D \to U^{(2)} = U \times U_1 \times U_2 \subset \mathbb{R}^6$ of the function u, where the coordinates in the *Cartesian* product space $U^{(2)}$ represent the derivatives of the function u of orders from 0 to 2,

$$u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}). \tag{1.3}$$

The total space $D \times U^{(2)}$, whose coordinates are the independent variables and the dependent variables up to order 2 is *the second order jet space* over $D \times U$ ([7]). Then the system (1.2) becomes

$$F(x, y, u^{(2)}) = 0, \tag{1.4}$$

where $F = (F_1, F_2, F_3) : D \times U^{(2)} \to \mathbb{R}^3$. The equation (1.4) is said to be of *maximal rank*, if the *Jacobian matrix* :

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$$[J_F(x,y,u^{(2)})] = (F_x, F_y; F_u; F_{u_x}, F_{u_y}; F_{u_{xx}}, F_{u_{xy}}, F_{u_{yy}}), \tag{1.5}$$

satisfies the following condition:

$$rank[J_F] = 3, \quad \text{whenever} \quad F(x,y,u^{(2)}) = 0. \tag{1.6}$$

Regarding the system (1.2), we have the following

Theorem 1. The system of PDEs (1.2) has maximal rank and its general solution is described by the family of mappings (1.1), with $c \in \mathbb{R}$.

Proof. One can easily check that

$$[J_F(x,y,u^{(2)})] = \begin{pmatrix} 2xu_{xx} & 0 & -2 & 0 & 0 & x^2 & 0 & 0 \\ 0 & 2yu_{yy} & -2 & 0 & 0 & 0 & 0 & y^2 \\ yu_{xy} & xu_{xy} & -1 & 0 & 0 & 0 & xy & 0 \end{pmatrix} \tag{1.7}$$

has maximal rank under the stated conditions, since last three columns provide a equal to $-x^3y^3$ nonzero minor. The first PDE is in fact an Euler equation in x, and its solutions are of the form $u_1(x,y) = c(y)x^2 + d(y)/x$; similarly, the second PDE has the general solution of the form $u_3(x,y) = e(x)y^2 + f(x)/y$. The identification of the solutions u_1 and u_2 leads to the family of functions $u(x,y) = \frac{\alpha x^3+\beta}{x}y^2 + \frac{\gamma x^3+\delta}{x}\frac{1}{y} = \frac{\alpha y^3+\gamma}{y}x^2 + \frac{\beta y^3+\delta}{y}\frac{1}{x}$. Then the condition that u satisfies the last PDE leads to $\alpha = \beta = \gamma = 0$ and hence for $c = \delta$, u has the form (1.1). □

Remark 1. For $u_{xy} \neq 0$, the PDE system

$$\begin{cases} uu_{xy} - u_x u_y = 0 \\ u_{xx}u_{yy} - 4u_{xy}^2 = 0 \end{cases} \tag{1.8}$$

is of maximal rank. Moreover, we have the following

Theorem 2. *The orbit of the family (1.1) under the action of the translations group of $\mathbb{R}^2$ on the domain coincides with the set of effectively (x,y)-dependent rational solutions $u \in \mathbb{R}(x,y)\backslash(\mathbb{R}(x) \cup \mathbb{R}(y))$ of the system of PDE system (1.8).*

Proof. The first PDE rewrites $\frac{u_x}{u_{xy}} = \frac{u}{u_y}$ whence integration for y leads to the general solution $u_0(x,y) = a(x)b(y)$, which obviously contains (1.1). The condition that u_0 satisfies the second PDE leads to the pair of ODEs

$$\frac{aa''}{4(a')^2} = \frac{(b')^2}{bb''} = k \in \mathbb{R} \Leftrightarrow \begin{cases} \frac{a''}{4a'} = k\frac{a'}{a} \\ \frac{b''}{b'} = \frac{1}{k}\frac{b'}{b} \end{cases}$$

where the accents denote derivatives w.r.t the corresponding variables. The solutions of the system reside in three classes, corresponding accordingly to $k = \frac{1}{4}, k = 1$ and $k \in \mathbb{R}\backslash\{\frac{1}{4}, 1\}$. Namely, $u \in S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \{ae^{bx}/\sqrt[3]{cy+d} \ \mid \ a,b,c,d \in \mathbb{R}, c^2+d^2 > 0\} \\ S_2 &= \{ae^{by}/\sqrt[3]{cx+d} \ \mid \ a,b,c,d \in \mathbb{R}, c^2+d^2 > 0\} \\ S_3 &= \{(ax+b)^{1/(1-4k)}(cy+d)^{k/(k-1)} \ \mid \ a,b,c,d \in \mathbb{R}\}. \end{aligned}$$

The rational solutions live in S_3 and the conditions $\frac{1}{1-4k} = m \in \mathbb{Z}$ and $\frac{k}{k-1} = n \in \mathbb{Z}$ lead to $3m = -1 + \frac{4}{3n+1} \in \mathbb{Z}$ and hence infer $n = 1 \Rightarrow m = \frac{1}{2} \notin \mathbb{Z}$ or $n = -1 \Rightarrow m = -1 \in \mathbb{Z}$, whence $k = \frac{1}{2}$. But then the solutions in S_3 have the form $u(x,y) = \frac{1}{(ax+b)(cy+d)}$. The effective dependence in x and y leads to $a^2 + c^2 > 0$, and an appropriate translation on $\mathbb{R}^2$ gives to u the form (1.1). $\square$

Remark 2. According to [10], the general Tzitzeica surface PDE

$$u_{xx}u_{yy} - u_{xy}^2 = k(xu_x + yu_y - u)^4, k \in \mathbb{R} \tag{1.9}$$

has a symmetry algebra with 8 generators, and (1.1) are exactly its solutions which are invariant under the action of the Lie subalgebra generated by infinitesimal symmetries

$$X_1 = x\partial_x - u\partial_u, X_2 = y\partial_y - u\partial_u, \tag{1.10}$$

where we denoted by ∂_w the partial derivative w.r.t. the corresponding index variable w. However, the characterization of its general solution is an open problem.

Using the standard procedure from [7], one can determine the infinitesimal generators of the PDE system (1.8), as follows:

Theorem 3. *The symmetry Lie algebra of the system of PDEs (1.8) is generated by the vector fields*

$$X_1 = x\partial_x, X_2 = \partial_x, X_3 = y\partial_y, X_4 = \partial_y, X_5 = u\partial_u. \tag{1.11}$$

Proof. The vanishing of the second prolongation of the PDE relative to the symmetry infinitesimal generator $X = \xi\partial_x + \eta\partial_y + \varphi\partial_u$ and vanishing of the coefficients of independent monomials in the partials of u mod the first PDE leads to the determining PDEs:

$$\xi_y = 0, \xi_u = 0, \xi_{uu} = 0, \eta_x = 0, \eta_u = 0, \eta_{uu} = 0, \varphi_{xy} = 0 \tag{1.12}$$

and

$$u^2(\varphi_{uu} - \xi_{xu} - \eta_{yu}) - u\varphi_u + \varphi = 0, \tag{1.13}$$

$$\varphi_y = u(\varphi_{uy} - \xi_{xy}), \varphi_x = u(\varphi_{xu} - \eta_{xy}). \tag{1.14}$$

The relations (1.12) show that $\xi = a(x)$ and $\eta = b(y)$. Then (1.13) becomes an Euler equation of φ in u, whence $\varphi = uc(x,y) + d(x,y)u\log|u|$; the PDEs (1.14) lead to d constant. Then the algebra of symmetries of the first PDE is generated by the fields of the form

$$X = a(x)\partial_x + b(y)\partial_y + (uc(x,y) + d \cdot u\log|u|)\partial_u. \tag{1.15}$$

Similarly, the determining PDEs of the second PDE in (1.8) are

$$\begin{gathered}\xi_y = 0, \xi_{yy} = 0, \xi_u = 0, \xi_{yu} = 0, \xi_{uu} = 0, \xi_{xy} = 0,\\ \eta_x = 0, \eta_{xu} = 0, \eta_{xx} = 0, \eta_u = 0, \eta_{uu} = 0, \varphi_{xx} = 0, \varphi_{xy} = 0, \varphi_{yy} = 0\end{gathered}$$

and

$$\begin{gathered}\varphi_{uu} = 2\xi_{xu}, \varphi_{uu} = 2\eta_{yu}, \xi_{xx} = 2\varphi_{xu}, \eta_{yy} = 2\varphi_{yu},\\ \varphi_{uu} = \xi_{xu} + \eta_{yu}, \eta_{xy} = \varphi_{xu}, \xi_{xy} = \varphi_{yu}.\end{gathered}$$

These lead to the solutions

$$\xi = a_1 x + a_2, \quad \eta = a_3 y + a_4, \quad \varphi = a_5 u + a_6 x + a_7 y + a_8,$$

where $a_i \in \mathbb{R}, i = \overline{1,8}$. Then the symmetry vector field X from (1.15) can be as well linearly expressed in terms of the generator fields

$$\begin{aligned} &X_1 = x\partial_x, X_2 = \partial_x, X_3 = y\partial_y, X_4 = \partial_y, \\ &X_5 = u\partial_u, X_6 = x\partial_u, X_7 = y\partial_u, X_8 = \partial_u. \end{aligned} \tag{1.16}$$

Taking into account the alternate form of X in (1.15), it follows that the generators of the Lie algebra of symmetries for the PDE system (1.8) are the ones in (1.11). □

Remark 3. The algebra generated by the fields (1.11) contains the subalgebra generated by the ones in (1.10); the extra generators are responsible for the translations which build orbits of Tzitzeica surfaces.

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References

[1] V. Balan and S.M. Bibic, *Solutions entailed by symmetries for quadric-derived PDEs*, Sci. Bull.Univ. Politehnica of Bucharest, Ser. A, 2005, to appear.

[2] N. Bîlă, *Lie groups applications to minimal surfaces PDE*, Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, Ed: Gr. Tsagas, BSG Proceedings 3, Geometry Balkan Press 1999, 197-205.

[3] N. Bîlă and C. Udrişte, *Infinitesimal symmetries of Camassa-Holm equation*, Proceedings of the Workshop on Global Analysis, Diff. Geom. & Lie Algebras, Ed: Gr. Tsagas, BSG Proceedings 4, Geometry Balkan Press 1999, 149-160.

[4] G. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*, Springer Verlag, 1974.

[5] B.K. Harrison and F.B. Estabrook, *Geometric approach to invariance groups and solution of partial differential systems*, J. Math. Phys. 12, 4 (1971), 653-666.

[6] J.Y. Lefebre and P. Metzger, *Quelques exemples de groupes d'inva-riance d'équations aux dérivées partielles*, C. R. Acad. Sci., Paris, Sér. I, Math. 279 (1974), 165-168.

[7] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Math., 107, Springer Verlag, New York, 1986.

[8] G. Tzitzeica, *Sur une nouvelle classe de surfaces*, C. R. Acad. Sci., Paris, Sér. I, Math. 144 (1907), 1257-1259.

[9] C. Udrişte and N. Bîlă, *Symmetry Lie groups of the Monge-Ampère equation*, Balkan J. Geom. Appl. 3, 2 (1998), 121-133.

[10] C. Udrişte and N. Bîlă, *Symmetry group of Tzitzeica surfaces PDE*, Balkan J. Geom. Appl. 4, 2 (1999), 123-140.

Vladimir Balan and Ştefania-Alina Vîlcea
University Politehnica of Bucharest, Department of Mathematics
Splaiul Independenţei 313, RO-060042 Bucharest, Romania
email addresses: vbalan@mathem.pub.ro, mili_alina@yahoo.com