Commutativity Conditions on Derivations and Lie Ideals in $\sigma$-prime Rings

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Abstract. Let $R$ be a 2-torsion free $\sigma$-prime ring, $U$ a nonzero square closed $\sigma$-Lie ideal of $R$ and let $d$ be a derivation of $R$. In this paper it is shown that:
1) If $d$ is centralizing on $U$, then $d = 0$ or $U \subseteq Z(R)$.
2) If either $d([x,y]) = 0$ for all $x, y \in U$, or $[d(x), d(y)] = 0$ for all $x, y \in U$ and $d$ commutes with $\sigma$ on $U$, then $d = 0$ or $U \subseteq Z(R)$.

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1. Introduction

Throughout this paper, $R$ will represent an associative ring with center $Z(R)$. Recall that $R$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. $R$ is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$ for all $a$ and $b$ in $R$. If $\sigma$ is an involution in $R$, then $R$ is said to be $\sigma$-prime if $aRb = aR\sigma(b) = 0$ implies that $a = 0$ or $b = 0$. It is obvious that every prime ring equipped with an involution $\sigma$ is also $\sigma$-prime, but the converse need not be true in general. An additive mapping $d : R \rightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y$ in $R$. A mapping $F : R \rightarrow R$ is said to be centralizing on a subset $S$ of $R$
if \([F(s), s] \in Z(R)\) for all \(s \in S\). In particular, if \([F(s), s] = 0\) for all \(s \in S\), then \(F\) is commuting on \(S\). In all that follows \(S\alpha_\sigma(R)\) will denote the set of symmetric and skew-symmetric elements of \(R\); i.e., \(S\alpha_\sigma(R) = \{x \in R/\sigma(x) = \pm x\}\). For any \(x, y \in R\), the commutator \(xy - yx\) will be denoted by \([x, y]\). An additive subgroup \(U\) of \(R\) is said to be a Lie ideal of \(R\) if \([u, r] \in U\) for all \(u \in U\) and \(r \in R\). A Lie ideal \(U\) which satisfies \(\sigma(U) \subseteq U\) is called a \(\sigma\)-Lie ideal. If \(U\) is a Lie (resp. \(\sigma\)-Lie) ideal of \(R\), then \(U\) is called a square closed Lie (resp. \(\sigma\)-Lie) ideal if \(u^2 \in U\) for all \(u \in U\). Since \((u + v)^2 \in U\) and \([u, v] \in U\), we see that \(2uv \in U\) for all \(u, v \in U\). Therefore, for all \(r \in R\) we get \(2r[u, v] = 2[u, rv] - 2[u, r]v \in U\) and \(2[u, v]r = 2[u, vr] - 2v[u, r] \in U\), so that \(2R[U, U] \subseteq U\) and \(2[U, U]R \subseteq U\). This remark will be freely used in the whole paper.

Many works concerning the relationship between commutativity of a ring and the behavior of derivations defined on this ring have been studied. The first important result in this subject is Posner’s theorem, which states that the existence of a nonzero centralizing derivation on a prime ring forces this ring to be commutative ([9]). This result has been generalized by many authors in several ways.

In [3], I. N. Herstein proved that if \(R\) is a prime ring of characteristic not 2 which has a nonzero derivation \(d\) such that \([d(x), d(y)] = 0\) for all \(x, y \in R\), then \(R\) is commutative. Motivated by this result, H. E. Bell, in [1], studied derivations \(d\) satisfying \(d(x, y)) = 0\) for all \(x, y \in R\). In [4] and [7], L. Oukhtite and S. Salhi generalized these results to \(\sigma\)-prime rings. In particular, they proved that if \(R\) is a 2-torsion free \(\sigma\)-prime ring equipped with a nonzero derivation which is centralizing on \(R\), then \(R\) is necessarily commutative.

Our purpose in this paper is to extend these results to square closed \(\sigma\)-Lie ideals.

2. Preliminaries and results

In order to prove our main theorems, we shall need the following lemmas.

**Lemma 1.** ([8], Lemma 4) If \(U \not\subseteq Z(R)\) is a \(\sigma\)-Lie ideal of a 2-torsion free \(\sigma\)-prime ring \(R\) and \(a, b \in R\) such that \(aUb = \sigma(a)Ub = 0\) or \(aUb = aU\sigma(b) = 0\), then \(a = 0\) or \(b = 0\).

**Lemma 2.** ([5], Lemma 2.3) Let \(0 \neq U\) be a \(\sigma\)-Lie ideal of a 2-torsion free \(\sigma\)-prime ring \(R\). If \([U, U] = 0\), then \(U \subseteq Z(R)\).

**Lemma 3.** ([6], Lemma 2.2) Let \(R\) be a 2-torsion free \(\sigma\)-prime ring and \(U\) a nonzero \(\sigma\)-Lie ideal of \(R\). If \(d\) is a derivation of \(R\) which commutes with \(\sigma\) and satisfies \(d(U) = 0\), then either \(d = 0\) or \(U \subseteq Z(R)\).

**Remark.** One can easily verify that Lemma 3 is still valid if the condition that \(d\) commutes with \(\sigma\) is replaced by \(d \circ \sigma = -\sigma \circ d\).

**Theorem 1.** Let \(R\) be a 2-torsion free \(\sigma\)-prime ring and \(U\) a square closed \(\sigma\)-Lie ideal of \(R\). If \(d\) is a derivation of \(R\) satisfying \([d(u), u] \in Z(R)\) for all \(u \in U\), then \(U \subseteq Z(R)\) or \(d = 0\).
Proof. Suppose that \( U \not\subseteq Z(R) \). As \( [d(x), x] \in Z(R) \) for all \( x \in U \), by linearization \([d(x), y] + [d(y), x] \in Z(R) \) for all \( x, y \in U \). Since \( \text{char} R \neq 2 \), the fact that \([d(x), x^2] + [d(x^2), x] \in Z(R) \) yields \( x[d(x), x] \in Z(R) \) for all \( x \in U \); hence
\[
[r, x][d(x), x] = 0 \quad \text{for all} \quad x \in U, r \in R,
\]
and therefore \([d(x), x]^2 = 0 \) for all \( x \in U \). Since \([d(x), x] \in Z(R) \),
\[
[d(x), x]R[d(x), x]\sigma([d(x), x]) = 0 \quad \text{for all} \quad x \in U
\]
and the \( \sigma \)-primeness of \( R \) yields \([d(x), x] = 0 \) or \([d(x), x]\sigma([d(x), x]) = 0 \). If
\([d(x), x]\sigma([d(x), x]) = 0 \), then \([d(x), x]R\sigma([d(x), x]) = 0 \); and the fact that \([d(x), x]^2 = 0 \) gives
\[
[d(x), x]R\sigma([d(x), x]) = [d(x), x]R[d(x), x] = 0.
\]
Since \( R \) is \( \sigma \)-prime, we obtain
\[
[d(x), x] = 0 \quad \text{for all} \quad x \in U.
\]
Let us consider the map \( \delta : R \rightarrow R \) defined by \( \delta(x) = d(x) + \sigma \circ d \circ \sigma(x) \). One can easily verify that \( \delta \) is a derivation of \( R \) which commutes with \( \sigma \) and satisfies
\[
[\delta(x), x] = 0 \quad \text{for all} \quad x \in U.
\]
Linearizing this equality, we obtain
\[
[\delta(x), y] + [\delta(y), x] = 0 \quad \text{for all} \quad x, y \in U.
\]
Writing \( 2xz \) instead of \( y \) and using \( \text{char} R \neq 2 \), we find that
\[
\delta(x)[x, z] = 0 \quad \text{for all} \quad x, z \in U.
\]
Replacing \( z \) by \( 2zy \) in this equality, we conclude that \( \delta(x)z[x, y] = 0 \), so that
\[
\delta(x)U[x, y] = 0 \quad \text{for all} \quad x, y \in U. \tag{1}
\]
By virtue of Lemma 1, it then follows that
\[
\delta(x) = 0 \quad \text{or} \quad [x, U] = 0, \quad \text{for all} \quad x \in U \cap S_{\sigma}(R).
\]
Let \( u \in U \). Since \( u - \sigma(u) \in U \cap S_{\sigma}(R) \), it follows that
\[
\delta(u - \sigma(u)) = 0 \quad \text{or} \quad [u - \sigma(u), U] = 0.
\]
If \( \delta(u - \sigma(u)) = 0 \), then \( \delta(u) \in S_{\sigma}(R) \) and (1) yields \( \delta(u) = 0 \); or \( [u, U] = 0 \). If
\([u - \sigma(u), U] = 0 \), then \([u, y] = [\sigma(u), y] \) for all \( y \in U \) and (1) assures that
\[
\delta(u)U[y, u] = 0 = \delta(u)U\sigma([u, y]), \quad \text{for all} \quad y \in U.
\]
Applying Lemma 1, we find that \( \delta(u) = 0 \) or \([u, U] = 0 \). Hence, \( U \) is a union of two additive subgroups \( G_1 \) and \( G_2 \), where
\[
G_1 = \{u \in U \text{ such that } \delta(u) = 0\} \quad \text{and} \quad G_2 = \{u \in U \text{ such that } [u, U] = 0\}.
\]
Since a group cannot be a union of two of its proper subgroups, we are forced to $U = G_1$ or $U = G_2$. Since $U \not\subseteq Z(R)$, Lemma 2 assures that $U = G_1$ and therefore $\delta(U) = 0$. Now applying Lemma 3, we get $\delta = 0$ and therefore $d \circ \sigma = -\sigma \circ d$. As $[d(x), x] = 0$ for all $x \in U$, in view of the above Remark, similar reasoning leads to $d = 0$. 

**Corollary 1.** ([7], Theorem 1.1) Let $R$ be a 2-torsion free $\sigma$-prime ring and $d$ a nonzero derivation of $R$. If $d$ is centralizing on $R$, then $R$ is commutative.

**Theorem 2.** Let $U$ be a square closed $\sigma$-Lie ideal of a 2-torsion free $\sigma$-prime ring $R$ and $d$ a derivation of $R$ which commutes with $\sigma$ on $U$. If $[d(x), d(y)] = d([y, x])$ for all $x, y \in U$, then $d = 0$ or $U \subseteq Z(R)$.

**Proof.** Suppose that $U \not\subseteq Z(R)$. We have

$$[d(x), d(y)] = d([y, x]) \quad \text{for all} \quad x, y \in U. \quad (2)$$

Substituting $2xy$ for $y$ in (2) and using $\text{char} R \neq 2$, we get

$$d(x)[y, x] = [d(x), x]d(y) + d(x)[d(x), y] \quad \text{for all} \quad x, y \in U. \quad (3)$$

Replacing $y$ by $2[y, z]x$ and using (3), we find that

$$[d(x), x][y, z] + d(x)[y, z][d(x), x] = 0 \quad \text{for all} \quad x, y, z \in U. \quad (4)$$

Replace $y$ by $2[y, z]d(x)$ in (3) to get

$$d(x)[y, z][d(x), x] - [d(x), x][y, z]d^2(x) = 0 \quad \text{for all} \quad x, y, z \in U. \quad (5)$$

From (4) and (5) we obtain

$$[d(x), x][y, z][d(x) + d^2(x)] = 0 \quad \text{for all} \quad x, y, z \in U. \quad (6)$$

Writing $2[u, v](d(x) + d^2(x))y$ instead of $y$ in (6), where $u, v \in U$, we obtain

$$[d(x), x][u, v]z(d(x) + d^2(x))y(d(x) + d^2(x)) = 0,$$

so that

$$[d(x), x][u, v]z(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \quad \text{for all} \quad x, u, v, z \in U. \quad (7)$$

If $x \in U \cap S_{\sigma}(R)$, then Lemma 1 together with (7) assures that

$$d(x) + d^2(x) = 0 \quad \text{or} \quad [d(x), x][u, v]z(d(x) + d^2(x)) = 0 \quad \text{for all} \quad u, v, z \in U.$$

Suppose that $[d(x), x][u, v]z(d(x) + d^2(x)) = 0$. Then

$$[d(x), x][u, v]U(d(x) + d^2(x)) = 0. \quad (8)$$

Since $d$ commutes with $\sigma$ and $x \in S_{\sigma}(R)$, in view of (8), Lemma 1 gives

$$d(x) + d^2(x) = 0 \quad \text{or} \quad [d(x), x][u, v] = 0 \quad \text{for all} \quad u, v \in U. \quad (9)$$
If \([d(x), x][u, v] = 0\), then replacing \(u\) by \(2uw\) in (9) where \(w \in U\), we obtain
\[
[d(x), x]U[u, v] = 0.
\] (10)

As \(\sigma(U) = U\) and \([U, U] \neq 0\), by (10), Lemma 2 yields that \([d(x), x] = 0\). Thus, in any event,
\[
either [d(x), x] = 0 \text{ or } d(x) + d^2(x) = 0 \text{ for all } x \in U \cap S_{a_\sigma}(R).
\]

Let \(x \in U\). Since \(x + \sigma(x) \in U \cap S_{a_\sigma}(R)\), either \(d(x + \sigma(x)) + d^2(x + \sigma(x)) = 0\) or \([d(x + \sigma(x)), x + \sigma(x)] = 0\).

If \(d(x + \sigma(x)) + d^2(x + \sigma(x)) = 0\), then \(d(x) + d^2(x) \in S_{a_\sigma}(R)\) and (7) yields that \(d(x) + d^2(x) = 0 \text{ or } [d(x), x][u, v]U(d(x) + d^2(x)) = 0\).

If \([d(x), x][u, v]U(d(x) + d^2(x)) = 0\), once again using \(d(x) + d^2(x) \in S_{a_\sigma}(R)\), we find that \(d(x) + d^2(x) = 0\), or \([d(x), x][u, v]\) for all \(u, v \in U\), in which case \([d(x), x] = 0\).

Now suppose that \([d(x + \sigma(x)), x + \sigma(x)] = 0\). As \(x - \sigma(x) \in U \cap S_{a_\sigma}(R)\) we have to distinguish two cases:

1) If \(d(x - \sigma(x)) + d^2(x - \sigma(x)) = 0\), then \(d(x) + d^2(x) \in S_{a_\sigma}(R)\). Reasoning as above we get \(d(x) + d^2(x) = 0\) or \([d(x), x] = 0\).

2) If \([d(x - \sigma(x)), x - \sigma(x)] = 0\), then \([d(x), x] \in S_{a_\sigma}(R)\). Replace \(u\) by \(2yu\) in (7), with \(y \in U\), to get \([d(x), x][y][u, v]z(d(x) + d^2(x))U(d(x) + d^2(x)) = 0\), so that
\[
[d(x), x][u, v]z(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \text{ for all } x, u, v, z \in U.
\] (11)

Since \([d(x), x] \in S_{a_\sigma}(R)\), from (11) it follows that
\[
[d(x), x] = 0 \text{ or } [u, v]U(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \text{ for all } u, v \in U.
\]

Suppose \([u, v]U(d(x) + d^2(x))U(d(x) + d^2(x)) = 0\). As \(\sigma(U) = U\) and \([U, U] \neq 0\), then
\[
(d(x) + d^2(x))U(d(x) + d^2(x)) = 0.
\] (12)

In (6), write \(2[u, v](d(x) + d^2(x))r\) instead of \(z\), where \(u, v \in U\) and \(r \in R\), to obtain
\[
[d(x), x][u, v]y(d(x) + d^2(x))r(d(x) + d^2(x)) = 0, \text{ for all } u, v, y \in U, r \in R.
\] (13)

Replacing \(r\) by \(r\sigma(d(x) + d^2(x))z\) in (13), where \(z \in U\), we find that
\[
[d(x), x][u, v]y(d(x) + d^2(x))r\sigma(d(x) + d^2(x))z(d(x) + d^2(x)) = 0,
\]
which leads us to
\[
[d(x), x][u, v]y(d(x) + d^2(x))U\sigma(d(x) + d^2(x))U(d(x) + d^2(x)) = 0.
\] (14)

Since \(\sigma(d(x) + d^2(x))U(d(x) + d^2(x))\) is invariant under \(\sigma\), by virtue of (14), Lemma 1 yields
\[
\sigma(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \text{ or } [d(x), x][u, v]y(d(x) + d^2(x)) = 0.
\]
If \( \sigma(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \), then (12) implies that \( d(x) + d^2(x) = 0 \).

Now assume that
\[
[d(x), x][u, v]y(d(x) + d^2(x)) = 0 \quad \text{for all } u, v, y \in U.
\] (15)

Replace \( v \) by \( 2wv \) in (15), where \( w \in U \), and use (15) to get
\[
[d(x), x]w[u, v]y(d(x) + d^2(x)) = 0,
\]
so that
\[
[d(x), x]U[u, v]y(d(x) + d^2(x)) = 0 \quad \text{for all } u, v, y \in U.
\] (16)

As \( [d(x), x] \in \text{Sa}_\sigma(R) \), (16) yields \( [u, v]U(d(x) + d^2(x)) = 0 \), in which case \( d(x) + d^2(x) = 0 \), or \( [d(x), x] = 0 \).

In conclusion, for all \( x \in U \) we have either \([d(x), x] = 0 \) or \( d(x) + d^2(x) = 0 \).

Now let \( x \in U \) such that \( d(x) + d^2(x) = 0 \). In (2), put \( y = 2[y, z]d(x) \) to get
\[
d([y, z])[d(x), x] = [[y, z], x]d(x) + [d(x), [y, z]]d(x) = [y, z][d(x), x].
\] (17)

If in (2) we put \( y = 2[y, z]x \), we get
\[
[[y, z], x]d(x) = [d(x), [y, z]]d(x) + d([y, z])[d(x), x] = 0.
\] (18)

From (17) and (18) it then follows that
\[
[y, z][d(x), x] = 0 \quad \text{for all } y, z \in U,
\]
hence \([y, z]U[d(x), x] = 0 \) for all \( y, z \in U \). Applying Lemma 1, this leads to
\[
[d(x), x] = 0, \quad \text{for all } x \in U.
\]

By virtue of Theorem 1, this yields that \( d = 0 \). \qed

Note that if \( d \) is a derivation of \( R \) which acts as an anti-homomorphism on \( U \), then \( d \) satisfies the condition \([d(x), d(y)] = d([y, x]) \) for all \( x, y \in U \). Thus we have the following corollary.

**Corollary 2.** ([6], Theorem 1.1) Let \( d \) be a derivation of a 2-torsion free \( \sigma \)-prime ring \( R \) which acts as an anti-homomorphism on a nonzero square closed \( \sigma \)-Lie ideal \( U \) of \( R \). If \( d \) commutes with \( \sigma \), then either \( d = 0 \) or \( U \subseteq Z(R) \).

**Theorem 3.** Let \( U \) be a square closed \( \sigma \)-Lie ideal of a 2-torsion free \( \sigma \)-prime ring \( R \) and \( d \) a derivation of \( R \). If either \( d([x, y]) = 0 \) for all \( x, y \in U \), or \([d(x), d(y)] = 0 \) for all \( x, y \in U \) and \( d \) commutes with \( \sigma \) on \( U \), then \( d = 0 \) or \( U \subseteq Z(R) \).
Proof. Suppose that \( U \not\subseteq Z(R) \). Assume that \( d([x, y]) = 0 \); for all \( x, y \in U \). Let \( \delta \) be the derivation of \( R \) defined by \( \delta(x) = d(x) + \sigma \circ d \circ \sigma(x) \).

Clearly, \( \delta \) commutes with \( \sigma \) and \( \delta([x, y]) = 0 \) for all \( x, y \in U \), so that
\[
[\delta(x), y] = [\delta(y), x] \quad \text{for all} \quad x, y \in U. \quad (19)
\]

Writing \([x, y]\) instead of \(y\) in (19), we find that
\[
[\delta(x), [x, y]] = 0 \quad \text{for all} \quad x, y \in U. \quad (20)
\]

Replacing \(x\) by \(x^2\) in (19), we conclude that
\[
\delta(x)[x, y] + [x, y]\delta(x) = 0 \quad \text{for all} \quad x, y \in U. \quad (21)
\]

As \( \text{char} R \neq 2 \), from (20) and (21) it follows that
\[
\delta(x)[x, y] = 0 \quad \text{for all} \quad x, y \in U. \quad (22)
\]

Replacing \(y\) by \(2zy\) in (22), we get \(\delta(x)z[x, y] = 0\), so that
\[
\delta(x)U[x, y] = 0 \quad \text{for all} \quad x, y \in U.
\]

From the proof of Theorem 1, we conclude that \( \delta = 0 \) and thus \( d \circ \sigma = -\sigma \circ d \).

Since \( d \) satisfies \( d([x, y]) = 0 \) for all \( x, y \in U \), by similar reasoning, we are forced to \( d = 0 \).

Now assume that \( d \) commutes with \( \sigma \) and satisfies \([d(x), d(y)] = 0\) for all \( x, y \in U \).

The fact that \([d(x), d(2xy)] = 0\) implies that
\[
d(x)[d(x), y] + [d(x), x]d(y) = 0 \quad \text{for all} \quad x, y \in U. \quad (23)
\]

Replace \(y\) by \(2[y, z]d(u)\) in (23), where \(z, u \in U\), to find that
\[
[d(x), x][y, z]d^2(u) = 0 \quad \text{for all} \quad x, y, u \in U. \quad (24)
\]

Write \(2[s, t]d^2(w)y\) instead of \(y\) in (24), where \(s, t, w \in U\), thereby concluding that
\[
[d(x), x][s, t]d^2(w)yv^2(u) = 0. \quad \text{Accordingly,}
\]
\[
[d(x), x][s, t]d^2(w)Ud^2(u) = 0 \quad \text{for all} \quad s, t, u, w, x \in U. \quad (25)
\]

Since \( d \) commutes with \( \sigma \) and \( \sigma(U) = U \), using (25) we find that
\[
d^2(U) = 0 \quad \text{or} \quad [d(x), x]U[s, t]d^2(w) = 0.
\]

Suppose that
\[
[d(x), x]U[s, t]d^2(w) = 0 \quad \text{for all} \quad s, t, w, x \in U. \quad (26)
\]

Replacing \(t\) by \(2tv\) in (26), where \(v \in U\), we are forced to
\[
[d(x), x][s, t]vd^2(w) = 0
\]
and hence
\[ [d(x), x][s, t]Ud^2(w) = 0 \text{ for all } s, t, w, x \in U. \quad (27) \]

Since \( \sigma(U) = U \) and \( d \) commutes with \( \sigma \), then (27) implies that either \( d^2(U) = 0 \), or \( [d(x), x][s, t] = 0 \) for all \( s, t, x \in U \), in which case \( [d(x), x] = 0 \) for all \( x \in U \).

Thus, in any event, we find that
\[ d^2(U) = 0 \text{ or } [d(x), x] = 0 \text{ for all } x \in U. \]

If \( d^2(U) = 0 \), then [5], Theorem 1.1 assures that \( d = 0 \).

If \( [d(x), x] = 0 \) for all \( x \in U \), then Theorem 1 yields \( d = 0 \). \( \square \)

**Corollary 3.** ([4], Theorem 3.3) Let \( d \) be a nonzero derivation of a 2-torsion free \( \sigma \)-prime ring \( R \). If \( d([x, y]) = 0 \) for all \( x, y \in R \), then \( R \) is commutative.

**References**


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