On the Geometry of Symplectic Involutions

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Abstract. Let $V$ be a $2n$-dimensional vector space over a field $F$ and $\Omega$ be a non-degenerate symplectic form on $V$. Denote by $\mathfrak{H}_k(\Omega)$ the set of all $2k$-dimensional subspaces $U \subset V$ such that the restriction $\Omega|_U$ is non-degenerate. Our main result (Theorem 1) says that if $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets is induced by a semi-symplectic automorphism of $V$. For the case when $n \neq 2k$ this fails, but we have a weak version of this result (Theorem 2). If the characteristic of $F$ is not equal to 2 then there is a one-to-one correspondence between elements of $\mathfrak{H}_k(\Omega)$ and symplectic $(2k, 2n-2k)$-involutions and Theorem 1 can be formulated as follows: for the case when $n \neq 2k$ and $\max(k, n-k) \geq 5$ any commutativity preserving bijective transformation of the set of symplectic $(2k, 2n-2k)$-involutions can be extended to an automorphism of the symplectic group.

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1. Introduction

Let $W$ be an $n$-dimensional vector space over a division ring $R$ and $n \geq 3$. We put $\mathcal{G}_k(W)$ for the Grassmannian of $k$-dimensional subspaces of $W$. The projective space associated with $W$ will be denoted by $\mathcal{P}(W)$.
Let us consider the set $\mathcal{G}_k(W)$ of all pairs
$$(S, U) \in \mathcal{G}_k(W) \times \mathcal{G}_{n-k}(W),$$
where $S + U = W$. If $B$ is a base for $\mathcal{P}(W)$ then the base subset of $\mathcal{G}_k(W)$ associated with the base $B$ consists of all $(S, U)$ such that $S$ and $U$ are spanned by elements of $B$. If $n \neq 2k$ then any bijective transformation of $\mathcal{G}_k(W)$ preserving the class of base subsets is induced by a semi-linear isomorphism of $W$ to itself or to the dual space $W^*$ (for $n = 2k$ this fails, but some weak version of this result holds true). Using Mackey’s ideas [7] J. Dieudonné [2] and C. E. Rickart [9] have proved this statement for $k = 1$, $n - 1$. For the case when $1 < k < n - 1$ it was established by author [8]. Note that adjacency preserving transformations of $\mathcal{G}_k(W)$ were studied in [6].

Now suppose that the characteristic of $R$ is not equal to 2 and consider an involution $u \in \text{GL}(W)$. There exist two subspaces $S_+(u)$ and $S_-(u)$ such that
$$u(x) = x \text{ if } x \in S_+(u), \quad u(x) = -x \text{ if } x \in S_-(u)$$
and
$$W = S_+(u) + S_-(u).$$
We say that $u$ is a $(k, n - k)$-involution if the dimensions of $S_+(u)$ and $S_-(u)$ are equal to $k$ and $n - k$, respectively. The set of $(k, n - k)$-involutions will be denoted by $\mathcal{I}_k(W)$. There is the natural one-to-one correspondence between elements of $\mathcal{I}_k(W)$ and $\mathcal{G}_k(W)$ such that each base subset of $\mathcal{G}_k(W)$ corresponds to a maximal set of mutually permutable $(k, n - k)$-involutions. Thus any commutativity preserving transformation of $\mathcal{I}_k(W)$ can be considered as a transformation of $\mathcal{G}_k(W)$ preserving the class of base subsets, and our statement shows that if $n \neq 2k$ then any commutativity preserving bijective transformation of $\mathcal{I}_k(W)$ can be extended to an automorphism of $\text{GL}(W)$.

In the present paper we give symplectic analogues of these results.

2. Results

2.1.

Let $V$ be a $2n$-dimensional vector space over a field $F$ and $\Omega : V \times V \to F$ be a non-degenerate symplectic form. The form $\Omega$ defines on the set of subspaces of $V$ the orthogonal relation which will be denoted by $\perp$. For any subspace $S \subset V$ we put $S^\perp$ for the orthogonal complement to $S$. A subspace $S \subset V$ is said to be non-degenerate if the restriction $\Omega|_S$ is non-degenerate; for this case $S$ is even-dimensional and $S + S^\perp = V$. We put $\mathfrak{H}_k(\Omega)$ for the set of non-degenerate $2k$-dimensional subspaces. Any element of $\mathfrak{H}_k(\Omega)$ can be presented as the sum of $k$ mutually orthogonal elements of $\mathfrak{H}_1(\Omega)$.

Let us consider the projective space $\mathcal{P}(V)$ associated with $V$. The points of this space are 1-dimensional subspaces of $V$, and each line consists of all 1-dimensional subspaces contained in a certain 2-dimensional subspace.
A line of $\mathcal{P}(V)$ is called \textit{hyperbolic} if the corresponding 2-dimensional subspace belongs to $\mathcal{H}_1(\Omega)$; otherwise, the line is said to be \textit{isotropic}.

Points of $\mathcal{P}(V)$ together with the family of isotropic lines form the well-known \textit{polar space}.

Some results related with the hyperbolic symplectic geometry (spanned by points of $\mathcal{P}(V)$ and hyperbolic lines) can be found in [1], [4], [5].

A base $B = \{P_1, \ldots, P_{2n}\} \in \mathcal{P}(V)$ is called \textit{symplectic} if for any $i \in \{1, \ldots, 2n\}$ there is unique $\sigma(i) \in \{1, \ldots, 2n\}$ such that $P_i \not\perp P_{\sigma(i)}$. Then the set $S_i := P_i + P_{\sigma(i)}$ is said to be the \textit{base subset} of $\mathcal{H}_1(\Omega)$ associated with the base $B$.

For any $k \in \{2, \ldots, n-1\}$ the set $S_k$ consisting of all $S_{i_1} + \cdots + S_{i_k}$ ($S_{i_1}, \ldots, S_{i_k}$ are different) will be called the \textit{base subset} of $\mathcal{H}_k(\Omega)$ associated with $S_1$ (or defined by $S_1$). Now suppose that the characteristic of $F$ is not equal to 2. An involution $u \in \text{GL}(V)$ is symplectic (belongs to the group $\text{Sp}(\Omega)$) if and only if $S_+^+(u)$ and $S_-^-(u)$ are non-degenerate and $S_-(u) = (S_+(u))^\perp$. We denote by $I_k(\Omega)$ the set of symplectic $(2k, 2n-2k)$-involutions. There is the natural bijection

\[ i_k : I_k(\Omega) \to \mathcal{H}_k(\Omega), \quad u \to S_+(u). \]

We say that $X \subset I_k(\Omega)$ is an \textit{MC-subset} if any two elements of $X$ are commutative and for any $u \in I_k(\Omega) \setminus X$ there exists $s \in X$ such that $su \neq us$ (in other words, $X$ is a maximal set of mutually permutable elements of $I_k(\Omega)$).

Fact 1. [2], [3] $X$ is a \textit{MC-subset} of $I_k(\Omega)$ if and only if $i_k(X)$ is a base subset of $\mathcal{H}_k(\Omega)$. For any two commutative elements of $I_k(\Omega)$ there is a \textit{MC-subset} containing them.

Fact 1 shows that a bijective transformation $f$ of $\mathcal{H}_k(\Omega)$ preserves the class of base subsets if and only if $i_k^{-1}fi_k$ is commutativity preserving.

2.2.

If $l$ is an element of $\Gamma\text{Sp}(\Omega)$ (the group of semi-linear automorphisms which preserved $\Omega$ to within a non-zero scalar and an automorphism of $F$) then for each number $k \in \{1, \ldots, n-1\}$ we have the bijective transformation

\[ (l)_k : \mathcal{H}_k(\Omega) \to \mathcal{H}_k(\Omega), \quad U \to l(U) \]

which preserves the class of base subsets. The bijection

\[ p_k : \mathcal{H}_k(\Omega) \to \mathcal{H}_{n-k}(\Omega), \quad U \to U^\perp \]

sends base subsets to base subsets. We will need the following trivial fact.

Fact 2. Let $f$ be a bijective transformation of $\mathcal{H}_k(\Omega)$ preserving the class of base subsets. Then the same holds for the transformation $p_kfp_{n-k}$. Moreover, if $f = (l)_k$ for certain $l \in \Gamma\text{Sp}(\Omega)$ then $p_kfp_{n-k} = (l)_{n-k}$. 

Two distinct elements of $\mathfrak{H}_1(\Omega)$ are orthogonal if and only if there exists a base subset containing them, thus for any bijective transformation $f$ of $\mathfrak{H}_1(\Omega)$ the following condition are equivalent:

- $f$ preserves the relation $\perp$,
- $f$ preserves the class of base subsets.

It is not difficult to prove (see [2], p. 26–27 or [9], p. 711–712) that if one of these conditions holds then $f$ is induced by an element of $\Gamma\text{Sp}(\Omega)$. Fact 2 guarantees that the same is fulfilled for bijective transformations of $\mathfrak{H}_{n-1}(\Omega)$ preserving the class of base subsets. This result was exploited by J. Dieudonné [2] and C. E. Rickart [9] to determine automorphisms of the group $\text{Sp}(\Omega)$.

**Theorem 1.** If $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets is induced by an element of $\Gamma\text{Sp}(\Omega)$.

**Corollary 1.** Suppose that the characteristic of $F$ is not equal to 2. If $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any commutativity preserving bijective transformation $f$ of $\mathfrak{I}_k(\Omega)$ can be extended to an automorphism of $\text{Sp}(\Omega)$.

**Proof of Corollary.** By Fact 1, $i_k f i_k^{-1}$ preserves the class of base subsets. Theorem 1 implies that $i_k f i_k^{-1}$ is induced by $l \in \Gamma\text{Sp}(\Omega)$. The automorphism $u \to lut^{-1}$ is as required. \hfill $\square$

2.3.

For the case when $n = 2k$ Theorem 1 fails.

**Example 1.** Suppose that $n = 2k$ and $\mathfrak{X}$ is a subset of $\mathfrak{H}_k(\Omega)$ such that for any $U \in \mathfrak{X}$ we have $U^\perp \in \mathfrak{X}$. Consider the transformation of $\mathfrak{H}_k(\Omega)$ which sends each $U \in \mathfrak{X}$ to $U^\perp$ and leaves fixed all other elements. This transformation preserves the class of base subsets (any base subset of $\mathfrak{H}_k(\Omega)$ contains $U$ together with $U^\perp$), but it is not induced by a semilinear automorphism if $\mathfrak{X} \neq \emptyset$. If $n = 2k$ then we denote by $\overline{\mathfrak{H}}_k(\Omega)$ the set of all subsets $\{U, U^\perp\} \subset \mathfrak{H}_k(\Omega)$. Then every $l \in \Gamma\text{Sp}(\Omega)$ induces the bijection

$$(l)_k' : \overline{\mathfrak{H}}_k(\Omega) \to \overline{\mathfrak{H}}_k(\Omega), \quad \{U, U^\perp\} \to \{l(U), l(U^\perp) = l(U)^\perp\}.$$ 

The transformation from Example 1 gives the identical transformation of $\overline{\mathfrak{H}}_k(\Omega)$.

**Theorem 2.** Let $n = 2k \geq 14$ and $f$ be a bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets. Then $f$ preserves the relation $\perp$ and induces a bijective transformation of $\overline{\mathfrak{H}}_k(\Omega)$. The latter mapping is induced by an element of $\Gamma\text{Sp}(\Omega)$.

**Corollary 2.** Let $n = 2k \geq 14$ and $f$ be a commutativity preserving bijective transformation of $\mathfrak{I}_k(\Omega)$. Suppose also that the characteristic of $F$ is not equal to 2. Then there exists an automorphism $g$ of the group $\text{Sp}(\Omega)$ such that $f(u) = \pm g(u)$ for any $u \in \mathfrak{I}_k(\Omega)$. 

3. Inexact subsets

In this section we suppose that \( n \geq 4 \) and \( 1 < k < n - 1 \).

3.1. Inexact subsets of \( G_k(W) \)

Let \( B = \{ P_1, \ldots, P_n \} \) be a base of \( P(W) \). For any \( m \in \{ 1, \ldots, n - 1 \} \) we denote by \( B_m \) the base subset of \( G_m(W) \) associated with \( B \) (the definition was given in Section 1).

If \( \alpha = (M, N) \in B_m \) then we put \( B_k(\alpha) \) for the set of all \( (S, U) \in B_k \) where \( S \) is incident to \( M \) or \( N \) (then \( U \) is incident to \( N \) or \( M \), respectively), the set of all \( (S, U) \in B_k \) such that \( S \) is incident to \( M \) will be denoted by \( B_k^+(\alpha) \).

A subset \( X \subset B_k \) is called exact if there is only one base subset of \( G_k(W) \) containing \( X \); otherwise, \( X \) is said to be inexact.

If \( \alpha \in B_2 \) then \( B_k(\alpha) \) is a maximal inexact subset of \( B_k \) (Example 1 in [8]). Conversely, we have the following:

Lemma 1. (Lemma 2 of [8]) If \( X \) is a maximal inexact subset of \( B_k \) then there exists \( \alpha \in B_2 \) such that \( X = B_k(\alpha) \).

Lemma 2. (Lemmas 5 and 8 of [8]) Let \( g \) be a bijective transformation of \( B_k \) preserving the class of maximal inexact subsets. Then for any \( \alpha \in B_{k-1} \) there exists \( \beta \in B_{k-1} \) such that

\[
g(B_k(\alpha)) = B_k(\beta);
\]

moreover, we have

\[
g(B_k^+(\alpha)) = B_k^+(\beta)
\]

if \( n \neq 2k \).

3.2. Inexact subsets of \( H_k(\Omega) \)

Let \( G_1 = \{ S_1, \ldots, S_n \} \) be a base subset of \( H_1(\Omega) \). For each number \( m \in \{ 2, \ldots, n - 1 \} \) we denote by \( G_m \) the base subset of \( H_m(\Omega) \) associated with \( G_1 \).

Let \( M \in G_m \). Then \( M^\perp \in G_{n-m} \). We put \( G_k(M) \) for the set of all elements of \( G_k \) incident to \( M \) or \( M^\perp \). The set of all elements of \( G_k \) incident to \( M \) will be denoted by \( G_k^+(M) \).

Let \( X \) be a subset of \( G_k \). We say that \( X \) is exact if it is contained only in one base subset of \( H_k(\Omega) \); otherwise, \( X \) will be called inexact. For any \( i \in \{ 1, \ldots, n \} \) we denote by \( X_i \) the set of all elements of \( X \) containing \( S_i \). If \( X_i \) is not empty then we define

\[
U_i(X) := \bigcap_{U \in X_i} U,
\]

and \( U_i(X) := \emptyset \) if \( X_i \) is empty. It is trivial that our subset is exact if \( U_i(X) = S_i \) for each \( i \).

Lemma 3. \( X \) is exact if \( U_i(X) \neq S_i \) only for one \( i \).
Proof. Let $\mathcal{S}_1'$ be a base subset of $\mathcal{H}_1(\Omega)$ which defines a base subset of $\mathcal{H}_k(\Omega)$ containing $\mathcal{X}$. If $j \neq i$ then $U_j(\mathcal{X}) = S_j$ implies that $S_j$ belongs to $\mathcal{S}_1'$. Let us take $S' \in \mathcal{S}_1'$ which does not coincide with any $S_j, j \neq i$. Since $S'$ is orthogonal to all such $S_j$, we have $S' = S_i$ and $\mathcal{S}_1' = \mathcal{S}_1$.

Example 2. Let $M \in \mathcal{S}_2$. Then $M = S_i + S_j$ for some $i, j$. We choose orthogonal $S_i', S_j' \in \mathcal{H}_1(\Omega)$ such that $S_i' + S_j' = M$ and $\{S_i, S_j\} \neq \{S_i', S_j'\}$. Then

$$(\mathcal{S}_1 \setminus \{S_i, S_j\}) \cup \{S_i', S_j'\}$$

is a base subset of $\mathcal{H}_1(\Omega)$ which defines another base subset of $\mathcal{H}_k(\Omega)$ containing $\mathcal{S}_k(M)$. Therefore, $\mathcal{S}_k(M)$ is inexact. Any $U \in \mathcal{S}_k \setminus \mathcal{S}_k(M)$ intersects $M$ by $S_i$ or $S_j$ and

$$U_p(\mathcal{S}_k(M) \cup \{U\}) = S_p$$

if $p = i$ or $j$; the same holds for all $p \neq i, j$. By Lemma 3, $\mathcal{S}_k(M) \cup \{U\}$ is exact for any $U \in \mathcal{S}_k \setminus \mathcal{S}_k(M)$. Thus the inexact subset $\mathcal{S}_k(M)$ is maximal.

Lemma 4. Let $\mathcal{X}$ be a maximal inexact subset of $\mathcal{S}_k$. Then $\mathcal{X} = \mathcal{S}_k(M)$ for certain $M \in \mathcal{S}_2$.

Proof. By the definition, there exists another base subset of $\mathcal{H}_k(\Omega)$ containing $\mathcal{X}$; the associated base subset of $\mathcal{H}_1(\Omega)$ will be denoted by $\mathcal{S}_1'$. Since our inexact subset is maximal, we need to prove the existence of $M \in \mathcal{S}_2$ such that $\mathcal{X} \subset \mathcal{S}_k(M)$.

Let us consider $i \in \{1, \ldots, n\}$ such that $U_i$ is not empty (from this moment we write $U_i$ in place of $U_i(\mathcal{X})$). We say that the number $i$ is of first type if the inclusion $U_j \subset U_i, j \neq i$ implies that $U_j = \emptyset$ or $U_j = U_i$. If $i$ is not of first type and the inclusion $U_j \subset U_i, j \neq i$ holds only for the case when $U_j = \emptyset$ or $j$ is of first type then $i$ is said to be of second type. Similarly, other types of numbers can be defined.

Suppose that there exists a number $j$ of first type such that $\dim U_j \geq 4$. Then $U_j$ contains certain $M \in \mathcal{S}_2$. Since $j$ is of first type, for any $U \in \mathcal{X}$ one of the following possibilities is realized:

- $U \in \mathcal{X}_j$ then $M \subset U_j \subset U$,
- $U \in \mathcal{X} \setminus \mathcal{X}_j$ then $U \subset U_j^\perp \subset M^\perp$.

This means that $M$ is as required.

Now suppose that $U_j = S_j$ for all $j$ of first type, so $S_j \in \mathcal{S}_1'$ if $j$ is of first type. Consider any number $i$ of second type. If $U_i \in \mathcal{S}_m$ then $m \geq 2$ and there are exactly $m - 1$ distinct $j$ of first type such that $S_j = U_j$ is contained in $U_i$; since all such $S_j$ belong to $\mathcal{S}_1'$ and $U_i$ is spanned by elements of $\mathcal{S}_1'$, we have $S_i \in \mathcal{S}_1'$. Step by step we establish the same for other types. Thus $S_i \in \mathcal{S}_1'$ if $U_i$ is not empty. Since $\mathcal{X}$ is inexact, Lemma 3 implies the existence of two distinct numbers $i$ and $j$ such that $U_i = U_j = \emptyset$. We define $M := S_i + S_j$. Then any element of $\mathcal{X}$ is contained in $M^\perp$ and we get the claim.\qed
Let $\mathcal{G}_1'$ be another base subset of $\mathcal{H}_1(\Omega)$ and $\mathcal{G}_m'$, $m \in \{2, \ldots, n - 1\}$, be the base subset of $\mathcal{H}_m(\Omega)$ defined by $\mathcal{G}_1'$.  

**Lemma 5.** Let $h$ be a bijection of $\mathcal{G}_k$ to $\mathcal{G}_k'$ such that $h$ and $h^{-1}$ send maximal inexact subsets to maximal inexact subsets. Then for any $M \in \mathcal{G}_{k-1}$ there exists $M' \in \mathcal{G}'_{k-1}$ such that 

$$h(\mathcal{G}_k(M)) = \mathcal{G}_k'(M');$$ 

moreover, we have 

$$h(\mathcal{G}_k^+(M)) = \mathcal{G}_k'^+(M')$$ 

if $n \neq 2k$.

**Proof.** Let $\mathcal{B}_m$, $m \in \{1, \ldots, n - 1\}$, be as in subsection 3.1. For each $m$ there is the natural bijection $b_m : \mathcal{B}_m \rightarrow \mathcal{G}_m$ sending $(S, U) \in \mathcal{B}_m$, $S = P_{i_1} + \cdots + P_{i_m}$ to $S_{i_1} + \cdots + S_{i_m}$. For any $M \in \mathcal{G}_m$ we have 

$$\mathcal{G}_k(M) = b_k(\mathcal{G}_k(b_m^{-1}(M))) \quad \text{and} \quad \mathcal{G}_k^+(M) = b_k(\mathcal{G}_k^+(b_m^{-1}(M))).$$

Let $b'_m$ be the similar bijection of $\mathcal{B}_m$ to $\mathcal{G}'_m$. Then $(b'_k)^{-1}hb_k$ is a bijective transformation of $\mathcal{B}_k$ preserving the class of base subsets and our statement follows from Lemma 2. \qed

4. **Proof of Theorems 1 and 2**

By Fact 2, we need to prove Theorem 1 only for $k < n - k$. Throughout the section we suppose that $1 < k \leq n - k$ and $n - k \geq 5$; for the case when $n = 2k$ we require that $n \geq 14$.

4.1.

Let $f$ be a bijective transformation of $\mathcal{H}_k(\Omega)$ preserving the class of base subsets. The restriction of $f$ to any base subset satisfies the condition of Lemma 5.

For any subspace $T \subset V$ we denote by $\mathcal{H}_k(T)$ the set of all elements of $\mathcal{H}_k(\Omega)$ incident to $T$ or $T^\perp$, the set of all elements of $\mathcal{H}_k(\Omega)$ incident to $T$ will be denoted by $\mathcal{H}_k^+(T)$.

In this subsection we show that Theorems 1 and 2 are simple consequences of the following lemma.

**Lemma 6.** There exists a bijective transformation $g$ of $\mathcal{H}_{k-1}(\Omega)$ such that 

$$g(\mathcal{H}_k^+(T)) = \mathcal{H}_k^+(g(T)) \quad \forall \ T \in \mathcal{H}_{k-1}(\Omega)$$

if $n \neq 2k$, and

$$g(\mathcal{H}_k(T)) = \mathcal{H}_k(g(T)) \quad \forall \ T \in \mathcal{H}_{k-1}(\Omega)$$

for the case when $n = 2k$. 

The proof of Lemma 6 will be given later.

Let \( \mathcal{S}_{k-1} \) be a base subset of \( \mathcal{H}_{k-1}(\Omega) \) and \( \mathcal{S}_k \) be the associated base subset of \( \mathcal{H}_k(\Omega) \) (these base subsets are defined by the same base subset of \( \mathcal{H}_1(\Omega) \)). By our hypothesis, \( f(\mathcal{S}_k) \) is a base subset of \( \mathcal{H}_k(\Omega) \); we denote by \( \mathcal{S}'_{k-1} \) the associated base subset of \( \mathcal{H}_{k-1}(\Omega) \). It is easy to see that \( g(\mathcal{S}_{k-1}) = \mathcal{S}'_{k-1} \), so \( g \) maps base subsets to base subsets. Since \( f^{-1} \) preserves the class of base subset, the same holds for \( g^{-1} \). Thus \( g \) preserves the class of base subsets.

Now suppose that \( g = (l)_{k-1} \) for certain \( l \in \Gamma\text{Sp}(\Omega) \). Let \( U \) be an element of \( \mathcal{H}_k(\Omega) \). We take \( M, N \in \mathcal{H}_{k-1}(\Omega) \) such that \( U = M + N \). If \( n \neq 2k \) then

\[
\{U\} = \mathcal{H}^+_k(M) \cap \mathcal{H}^+_k(N) \quad \text{and} \quad \{f(U)\} = \mathcal{H}^+_k(l(M)) \cap \mathcal{H}^+_k(l(N)),
\]

so \( f(U) = l(M) + l(N) = l(U) \), and we get \( f = (l)_k \). For the case when \( n = 2k \) we have

\[
\{U, U^\perp\} = \mathcal{H}_k(M) \cap \mathcal{H}_k(N) \quad \text{and} \quad \{f(U), f(U)^\perp\} = \mathcal{H}_k(l(M)) \cap \mathcal{H}_k(l(N));
\]

since \( l(M) + l(N) = l(U) \) and \( (l(M))^\perp \cap (l(N))^\perp = (l(M) + l(N))^\perp = l(U)^\perp \),

\[
\{f(U), f(U)^\perp\} = \{l(U), l(U)^\perp\};
\]

the latter means that \( f = (l)'_k \). Therefore, Theorem 1 can be proved by induction and Theorem 2 follows from Theorem 1.

To prove Lemma 6 we use the following:

**Lemma 7.** Let \( M \in \mathcal{H}_m(\Omega) \) and \( N \) be a subspace contained in \( M \). Then the following assertions are fulfilled:

1. If \( \dim N > m \) then \( N \) contains an element of \( \mathcal{H}_1(\Omega) \).
2. If \( \dim N > m + 2 \) then \( N \) contains two orthogonal elements of \( \mathcal{H}_1(\Omega) \).
3. If \( \dim N > m + 4 \) then \( N \) contains three distinct mutually orthogonal elements of \( \mathcal{H}_1(\Omega) \).

**Proof.** The form \( \Omega|_M \) is non-degenerate. If \( \dim N > m \) then the restriction of \( \Omega|_M \) to \( N \) is non-zero. This implies the existence of \( S \in \mathcal{H}_1(\Omega) \) contained in \( N \). We have

\[
\dim N \cap S^\perp \geq \dim N - 2,
\]

and for the case when \( \dim N > m + 2 \) there is an element of \( \mathcal{H}_1(\Omega) \) contained in \( N \cap S^\perp \). Similarly, (3) follows from (2).

\( \square \)

**4.2. Proof of Lemma 6 for \( k < n - k \)**

Let \( T \in \mathcal{H}_{k-1}(\Omega) \) and \( \mathcal{S}_1 = \{S_1, \ldots, S_n\} \) be a base subset of \( \mathcal{H}_1(\Omega) \) such that

\[
T^\perp = S_1 + \cdots + S_{n-k+1} \quad \text{and} \quad T = S_{n-k+2} + \cdots + S_n.
\]
We put \( \mathfrak{G}_k \) for the base subset of \( \mathfrak{H}_k(\Omega) \) associated with \( \mathfrak{G}_1 \). Then \( \mathfrak{G}_k^+(T) \) consists of all
\[
U_i := T + S_i,
\]
where \( i \in \{1, \ldots, n - k + 1\} \). By Lemma 5, there exists \( T' \in \mathfrak{H}_{k-1}(\Omega) \) such that
\[
f(\mathfrak{G}_k^+(T)) \subset \mathfrak{H}_k^+(T').
\]
We need to show that \( f(\mathfrak{H}_k^+(T)) \) coincides with \( \mathfrak{H}_k^+(T') \).

**Lemma 8.** Let \( U \in \mathfrak{H}_k^+(T) \). Suppose that there exist two distinct \( M, N \in \mathfrak{H}_k^+(T) \) such that \( f(M), f(N) \) belong to \( \mathfrak{H}_k^+(T') \) and there is a base subset of \( \mathfrak{H}_k(\Omega) \) containing \( M, N \) and \( U \). Then \( f(U) \) is an element of \( \mathfrak{H}_k^+(T') \).

**Proof.** If there exists a base subset of \( \mathfrak{H}_k(\Omega) \) containing \( M, N \) and \( U \) then \( T \) belongs to the associated base subset of \( \mathfrak{H}_{k-1}(\Omega) \) and Lemma 5 implies the existence of \( T'' \in \mathfrak{H}_{k-1}(\Omega) \) such that \( f(M), f(N) \) and \( f(U) \) belong to \( \mathfrak{H}_k^+(T'') \). On the other hand, \( f(M) \) and \( f(N) \) are different elements of \( \mathfrak{H}_k^+(T') \) and \( f(M) \cap f(N) \) coincides with \( T' \). Hence \( T'' = T' \).

For any \( U \in \mathfrak{H}_k^+(T) \) we denote by \( S(U) \) the intersection of \( U \) and \( T^\perp \), it is clear that \( S(U) \) is an element of \( \mathfrak{H}_1(\Omega) \).

If \( S(U) \) is contained in \( S_1 + \cdots + S_{n-k-1} \) then \( U \), \( S_{n-k} \), \( S_{n-k+1} \) are mutually orthogonal and there exists a base subset of \( \mathfrak{H}_k(\Omega) \) containing \( U, S_{n-k}, S_{n-k+1} \). All \( f(U_i) \) belong to \( \mathfrak{H}_k^+(T') \) and Lemma 8 shows that \( f(U) \in \mathfrak{H}_k^+(T') \).

Let \( U \) be an element of \( \mathfrak{H}_k^+(T) \) such that \( S(U) \) is contained in \( S_1 + \cdots + S_{n-k} \). We have
\[
\dim(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp \geq 2(n - k - 2) > n - k - 1
\]
(the latter inequality follows from the condition \( n - k \geq 5 \)) and Lemma 7 implies the existence of \( S' \in \mathfrak{H}_1(\Omega) \) contained in
\[
(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp.
\]
Then \( S(U), S', S_{n-k+1} \) are mutually orthogonal and there exists a base subset of \( \mathfrak{H}_k(\Omega) \) containing \( U, T + S', S_{n-k+1} \). It was proved above that \( f(T + S') \) belongs to \( \mathfrak{H}_k^+(T') \). Since \( f(U_i) \in \mathfrak{H}_k^+(T') \) for each \( i \), Lemma 8 guarantees that \( f(U) \) is an element of \( \mathfrak{H}_k^+(T') \).

Now suppose that \( S(U) \) is not contained in \( S_1 + \cdots + S_{n-k} \). Since \( n - k \geq 5 \),
\[
\dim(S_1 + \cdots + S_{n-k}) \cap S(U)^\perp \geq 2(n - k - 1) > n - k + 2.
\]
By Lemma 7, there exist two orthogonal \( S', S'' \in \mathfrak{H}_1(\Omega) \) contained in
\[
(S_1 + \cdots + S_{n-k}) \cap S(U)^\perp.
\]
Then \( S', S'' \), \( S(U) \) are mutually orthogonal and there exists a base subset of \( \mathfrak{H}_k(\Omega) \) containing \( S' + T, S'' + T \) and \( U \). We have shown above that \( f(S' + T), f(S'' + T) \) belong to \( \mathfrak{H}_k^+(T') \) and Lemma 8 shows that the same holds for \( f(U) \).

So \( f(\mathfrak{H}_k^+(T)) \subset \mathfrak{H}_k^+(T') \). Since \( f^{-1} \) preserves the class of base subsets, the inverse inclusion holds true. We define \( g : \mathfrak{H}_{k-1}(\Omega) \to \mathfrak{H}_{k-1}(\Omega) \) by \( g(T) := T' \).

This transformation is bijective (otherwise, \( f \) is not bijective).
4.3. Proof of Lemma 6 for \( n = 2k \)

We start with the following:

**Lemma 9.** If \( n = 2k \) then \( f(U^\perp) = f(U)^\perp \) for any \( U \in \mathcal{H}_k(\Omega) \).

**Proof.** We take a base subset \( \mathcal{S}_k \) containing \( U \). Then \( U^\perp \in \mathcal{S}_k \). Denote by \( \mathcal{S}_{k-1} \) the base subset of \( \mathcal{H}_{k-1}(\Omega) \) associated with \( \mathcal{S}_k \). Let \( \mathcal{S}_{k-1} \) be the base subset of \( \mathcal{H}_{k-1}(\Omega) \) associated with \( \mathcal{S}_k' := f(\mathcal{S}_k) \). We choose \( M, N \in \mathcal{S}_{k-1} \) such that \( U = M + N \). Then

\[
\{U, U^\perp\} = \mathcal{S}_k(M) \cap \mathcal{S}_k(N)
\]

and Lemma 5 guarantees that

\[
\{f(U), f(U^\perp)\} = \mathcal{S}_k'(M') \cap \mathcal{S}_k'(N')
\]

for some \( M', N' \in \mathcal{S}_{k-1}' \). The set \( \mathcal{S}_k'(M') \cap \mathcal{S}_k'(N') \) is not empty if one of the following possibilities is realized:

- \( M' + N' \) and \( M'^\perp \cap N'^\perp \) are elements of \( \mathcal{H}_{k-1}(\Omega) \) and \( \mathcal{S}_k'(M') \cap \mathcal{S}_k'(N') \) consists of these two elements.
- \( M' \subseteq N'^\perp \) and \( N' \subseteq M'^\perp \), then \( \mathcal{S}_k'(M') \cap \mathcal{S}_k'(N') \) consists of 4 elements.

Thus

\[
\{f(U), f(U^\perp)\} = \{M' + N', M'^\perp \cap N'^\perp\}.
\]

Since \( M' + N' \) and \( M'^\perp \cap N'^\perp \) are orthogonal, we get the claim. \( \square \)

Let \( T \in \mathcal{H}_{k-1}(\Omega) \). As in the previous subsection we consider a base subset \( \mathcal{S}_1 = \{S_1, \ldots, S_n\} \) of \( \mathcal{H}_1(\Omega) \) such that

\[
T^\perp = S_1 + \cdots + S_{n-k+1} \quad \text{and} \quad T = S_{n-k+2} + \cdots + S_n.
\]

We denote by \( \mathcal{S}_k \) the base subset of \( \mathcal{H}_k(\Omega) \) associated with \( \mathcal{S}_1 \). Then \( \mathcal{S}_k(T) \) consists of

\[
U_i := T + S_i, \quad i \in \{1, \ldots, n - k + 1\}
\]

and their orthogonal complements. Lemma 5 implies the existence of \( T' \in \mathcal{H}_{k-1}(\Omega) \) such that

\[
f(\mathcal{S}_k(T)) \subset \mathcal{H}_k(T').
\]

We show that \( f(U) \) belongs to \( \mathcal{H}_k(T') \) for any \( U \in \mathcal{H}_k(T) \).

We need to establish this fact only for the case when \( U \) is an element of \( \mathcal{H}_k^+(T) \). Indeed, if \( U \in \mathcal{H}_k^+(T^\perp) \) then \( U^\perp \) is an element of \( \mathcal{H}_k^+(T) \) and \( f(U^\perp) \in \mathcal{H}_k(T') \) implies that \( f(U) = f(U^\perp)^\perp \) belongs to \( \mathcal{H}_k(T') \).

**Lemma 10.** Let \( U \in \mathcal{H}_k^+(T) \). Suppose that there exist distinct \( M_i \in \mathcal{H}_k^+(T) \), \( i = 1, 2, 3 \) such that each \( f(M_i) \) belongs to \( \mathcal{H}_k(T') \) and there is a base subset of \( \mathcal{H}_k(\Omega) \) containing \( M_1, M_2, M_3 \) and \( U \). Then \( f(U) \in \mathcal{H}_k(T') \).
Proof. By Lemma 5, there exists $T'' \in \mathfrak{H}_{k-1}(\Omega)$ such that $f(U)$, all $f(M_i)$, and their orthogonal complements belong to $\mathfrak{H}_k(T'')$. For any $i = 1, 2, 3$ one of the subspaces $f(M_i)$ or $f(M_i) \perp$ is an element of $\mathfrak{H}_k(T'')$; we denote this subspace by $M'_i$. Then

$$T'' = \bigcap_{i=1}^{3} M'_i$$

and $T'' \perp = M'_i + M'_j, i \neq j$;

note also that the intersection of any $M'_i$ and $M'_j$ does not belong to $\mathfrak{H}_{k-1}(\Omega)$. Since all $M'_i$ and $M'_j$ belong to $\mathfrak{H}_k(T'')$, we have $T' = T''$.

As in the previous subsection for any $U \in \mathfrak{H}_k(T')$ we denote by $S(U)$ the intersection of $U$ and $T' \perp$, it is an element of $\mathfrak{H}_1(\Omega)$.

If $S(U)$ is contained in $S_1 + \cdots + S_{n-k-2}$ then $S(U), S_{n-k-1}, S_{n-k}, S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, U_{n-k-1}, U_{n-k}, U_{n-k+1}$. Since $f(U_i) \in \mathfrak{H}_k(T')$ for each $i$, Lemma 10 shows that $f(U)$ belongs to $\mathfrak{H}_k(T')$.

Suppose that $S(U)$ is contained in $S_1 + \cdots + S_{n-k-1}$. We have

$$\dim(S_1 + \cdots + S_{n-k-2}) \cap S(U) \perp \geq 2(n - k - 3) > n - k - 2$$

(since $k = n - k \geq 7$) and Lemma 7 implies the existence of $S' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k-2}) \cap S(U) \perp$$

Then $S(U), S', S_{n-k}, S_{n-k+1}$ are mutually orthogonal, so $U, T + S', U_{n-k}, U_{n-k+1}$ are contained in a certain base subset of $\mathfrak{H}_k(\Omega)$. It was shown above that $f(T + S')$ is an element of $\mathfrak{H}_k(T')$ and Lemma 10 guarantees that $f(U) \in \mathfrak{H}_k(T')$ (recall that all $f(U_i)$ belong to $\mathfrak{H}_k(T')$).

Consider the case when $S(U)$ is contained in $S_1 + \cdots + S_{n-k}$. We have

$$\dim(S_1 + \cdots + S_{n-k-1}) \cap S(U) \perp \geq 2(n - k - 2) > (n - k - 1) + 2$$

(recall that $k = n - k \geq 7$) and there exist two orthogonal $S', S'' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k-1}) \cap S(U) \perp$$

(Lemma 7). Then $S(U), S', S'', S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T + S', T + S'', U_{n-k+1}$. It follows from Lemma 10 that $f(U) \in \mathfrak{H}_k(T')$ (since $f(T + S'), f(T + S'')$ and any $f(U_i)$ belong to $\mathfrak{H}_k(T')$).

Let $U$ be an element of $\mathfrak{H}_k(T')$ such that $S(U)$ is not contained in $S_1 + \cdots + S_{n-k}$. Since $n = 2k \geq 14$,

$$\dim(S_1 + \cdots + S_{n-k}) \cap S(U) \perp \geq 2(n - k - 1) > n - k + 4$$

By Lemma 7, there exist mutually orthogonal $S', S'', S''' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k}) \cap S(U) \perp.$$
A base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T + S', T + S'', T + S'''$ exists. It was shown above that $f(T + S'), f(T + S'')$ and $f(T + S''')$ belong to $\mathfrak{H}_k(T')$ and Lemma 10 implies that the same holds for $f(U)$.

Thus $f(\mathfrak{H}_k(T)) \subset \mathfrak{H}_k(T')$. As in the previous subsection we have the inverse inclusion and define $g : \mathfrak{H}_{k-1}(\Omega) \to \mathfrak{H}_{k-1}(\Omega)$ by $g(T) := T'$.

References


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