DIHEDRAL COVERS OF THE COMPLETE GRAPH $K_5$

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Abstract. A regular cover of a connected graph is called dihedral if its transformation group is dihedral. In this paper, the author classifies all dihedral coverings of the complete graph $K_5$ whose fibre-preserving automorphism subgroups act arc-transitively.

1. Introduction

Throughout this paper, we consider finite connected graphs without loops or multiple edges. For a graph $X$, every edge of $X$ gives rise to a pair of opposite arcs. By $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$, we denote the vertex set, the edge set, the arc set and the automorphism group of the graph $X$, respectively. The neighborhood of a vertex $v \in V(X)$ denoted by $N(v)$ is the set of vertices adjacent to $v$ in $X$. Let a group $G$ act on a set $\Omega$ and let $\alpha \in \Omega$. We denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing $\alpha$. The group $G$ is said to be semiregular if $G_\alpha = 1$ for each $\alpha \in \Omega$, and regular if $G$ is semiregular and transitive on $\Omega$. A graph $\tilde{X}$ is called a covering of a graph $X$ with projection $p: \tilde{X} \to X$ if there is a surjection $p: V(\tilde{X}) \to V(X)$ such that $p|_{N(\tilde{v})}: N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\tilde{X}$ is called the covering graph and $X$ is the base graph. A covering $\tilde{X}$ of $X$ with a projection $p$ is said to be regular (or $K$-covering) if there is a semiregular subgroup $K$ of the automorphism group $\text{Aut}(\tilde{X})$ such that graph $X$ is isomorphic to the quotient graph $\tilde{X}/K$, say by $h$, and the

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quotient map $\tilde{X} \to \tilde{X}/K$ is the composition $ph$ of $p$ and $h$ (for the purpose of this paper, all functions are composed from left to right). If $K$ is cyclic, elementary abelian or dihedral then $\tilde{X}$ is called a cyclic, elementary abelian or dihedral covering of $X$, respectively. If $\tilde{X}$ is connected, $K$ is the covering transformation group. The fibre of an edge or a vertex is its preimage under $p$. An automorphism of $\tilde{X}$ is said to be fibre-preserving if it maps a fibre to a fibre while an element of the covering transformation group fixes each fibre setwise. All of fibre-preserving automorphisms form a group called the fibre-preserving group.

An $s$-arc in a graph $X$ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words, a directed walk of length $s$ which never includes a backtracking. A graph $X$ is said to be $s$-arc-transitive if $\text{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. In particular, $0$-arc-transitive means vertex-transitive, and $1$-arc-transitive means arc-transitive or symmetric. An $s$-arc-transitive graph is said to be $s$-transitive if it is not $(s+1)$-arc-transitive. In particular, a subgroup of the automorphism group of a graph $X$ is said to be $s$-regular if it acts regularly on the set of $s$-arcs of $X$. Also if the subgroup is the full automorphism group $\text{Aut}(X)$ of $X$, then $X$ is said to be $s$-regular. Thus, if a graph $X$ is $s$-regular, then $\text{Aut}(X)$ is transitive on the set of $s$-arcs and the only automorphism fixing an $s$-arc is the identity automorphism of $X$.

Regular coverings of a graph have received considerable attention. For example, for a graph $X$ which is the complete graph $K_4$, the complete bipartite graph $K_{3,3}$, hypercube $Q_3$ or Petersen graph $O_3$, the $s$-regular cyclic or elementary abelian coverings of $X$, whose fibre-preserving groups are arc-transitive, classified for each $1 \leq s \leq 5$ [3, 4, 6, 7]. As an application of these classifications, all $s$-regular cubic graphs of order $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, and $10p^2$ constructed for each $1 \leq s \leq 5$ and each prime $p$ [3, 4, 6]. In [14], it was shown that all cubic graphs admitting a solvable edge-transitive group of automorphisms arise as regular covers of one of the following basic graphs: the complete graph $K_4$, the dipole Dip3 with two vertices and three parallel edges,
the complete bipartite graph $K_{3,3}$, the Pappus graph of order 18, and the Gray graph of order 54. Also all dihedral coverings of the complete graph $K_4$ and cubic symmetric graphs of order $2p$ were classified in [5, 8]. But apart from the octahedron graph [11], graphs of higher valencies have not received much attention. For more results see [1, 2, 13, 15]. In a series of reductions of this kind, the final, irreducible graph is often a complete graph. Thus studying $K_5$ is the obvious next choice in order to establish a base of examples for further investigation. All pairwise non-isomorphic connected arc-transitive $p$-elementary abelian covers of the complete graph $K_5$ are constructed in [10]. In this paper all dihedral coverings of the complete graph $K_5$ whose fibre-preserving automorphism subgroups act arc-transitively are determined. Also we give a family of 2-arc-transitive graphs.

Let $n$ be a non-negative integer. Let $Z_n$ denote the cyclic group of order $n$ and $D_{2n}$ the dihedral group of order $2n$. Set

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$ 

By $\{0, 1, 2, 3, 4\}$ denote the vertex set of $K_5$. For $n \geq 3$, the graph $DK(2n)$ is defined to have vertex set

$$V(DK(2n)) = \{0, 1, 2, 3, 4\} \times D_{2n}$$

and edge set

$$E(DK(2n)) = \{(0, c)(3, c), (1, c)(3, c), (1, c)(4, c), (2, c)(4, c), (0, c)(1, bc), (0, c)(2, a^{-1}bc), (0, c)(4, ac), (1, c)(2, bc), (2, c)(3, ac), (3, c)(4, a^{-2}bc), (4, c)(0, a^{-1}c) \mid c \in D_{2n} \}.$$ 

Note that the first four edges in the edge set $E(DK(2n))$ correspond with the tree edges in the spanning tree $T$ as depicted by the dashed lines in Fig. 1 and these four edges have the common $c$ as the second coordinates. In fact, the graph $DK(2n)$ is the covering graph derived from a
T-reduced voltage assignment $\phi: A(K_5) \to D_{2n}$ which assigns the six values $b, a^{-1}b, a, b, a^{-2}b, a^{-1}$ to the six cotree edges in $K_5$.

The following theorem is the main result of this paper.

**Theorem 1.1.** Let $\tilde{X}$ be a connected $D_{2n}$-covering ($n \geq 3$) of the complete graph $K_5$ whose fibre-preserving subgroup is arc-transitive. Then $\tilde{X}$ is arc-transitive if and only if $\tilde{X}$ is isomorphic to $DK(2n)$ for $n \geq 3$.

2. Preliminaries related to coverings

Let $X$ be a graph and $K$ a finite group. By $a^{-1}$, we mean the reverse arc to an arc $a$. A voltage assignment (or $K$-voltage assignment) of $X$ is a function $\phi: A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of $\phi$ are called voltages and $K$ is the voltage group. The graph $X \times_\phi K$ derived from a voltage assignment $\phi: A(X) \to K$ has a vertex set $V(X) \times K$ and an edge set $E(X) \times K$, so that an edge $(e, g)$ of $X \times_\phi K$ joins a vertex $(u, g)$ to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of $X$ with the first coordinate projection $p: X \times_\phi K \to X$ which is called the natural projection. By defining $(u, g')g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, $K$ becomes a subgroup of Aut$(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a $K$-covering. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{u, g \mid g \in K\}$ is the fibre of $u$ and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of $uv$, where $a = (u, v)$. Conversely, each regular covering $\tilde{X}$ of $X$ with a covering transformation group $K$ can be derived from a $K$-voltage assignment. Given a spanning tree $T$ of the graph $X$, a voltage assignment $\phi$ is said to be $T$-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [9] showed that every regular covering $\tilde{X}$ of a graph $X$ can be derived from a $T$-reduced voltage assignment $\phi$ with respect to an arbitrary fixed spanning tree $T$ of $X$. It is
clear that if $\phi$ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group $K$.

Let $\tilde{X}$ be a $K$-covering of $X$ with a projection $p$. If $\alpha \in \text{Aut}(X)$ and $\tilde{\alpha} \in \text{Aut}(\tilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a lift of $\alpha$, and $\alpha$ the projection of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\text{Aut}(X)$ and the projection of a subgroup of $\text{Aut}(\tilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\text{Aut}(\tilde{X})$ and $\text{Aut}(X)$, respectively. In particular, if the covering graph $\tilde{X}$ is connected, then the covering transformation group $K$ is the lift of the trivial group, that is $K = \{\tilde{\alpha} \in \text{Aut}(\tilde{X}) : p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of $\alpha$, then $K\tilde{\alpha}$ consists of all the lifts of $\alpha$.

**Figure 1.** A choice of the six cotree arcs in $K_5$. 
Let $X \times_{\phi} K \rightarrow X$ be a connected $K$-covering derived from a $T$-reduced voltage assignment $\phi$. The problem whether an automorphism $\alpha$ of $X$ lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \text{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group $K$ by

$$(\phi(C))^\bar{\alpha} = \phi(C^\alpha),$$

where $C$ ranges over all fundamental closed walks at $v$, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on $C$ and $C^\alpha$, respectively. Note that if $K$ is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at $v$ can be substituted by the fundamental cycles generated by the cotree arcs of $X$.

The next proposition is a special case of [12, Theorem 3.5].

**Proposition 2.1.** Let $X \times_{\phi} K \rightarrow X$ be a connected $K$-covering derived from a $T$-reduced voltage assignment $\phi$. Then, an automorphism $\alpha$ of $X$ lifts if and only if $\bar{\alpha}$ extends to an automorphism of $K$.

Two coverings $\tilde{X}_1$ and $\tilde{X}_2$ of $X$ with projections $p_1$ and $p_2$, respectively, are said to be equivalent if there exists a graph isomorphism $\tilde{\alpha}: \tilde{X}_1 \rightarrow \tilde{X}_2$ such that $\tilde{\alpha}p_2 = p_1$. We quote the following proposition.

**Proposition 2.2 ([16]).** Two connected regular coverings $X \times_{\phi} K$ and $X \times_{\psi} K$, where $\phi$ and $\psi$ are $T$-reduced, are equivalent if and only if there exists an automorphism $\sigma \in \text{Aut}(K)$ such that $\phi(u, v)^\sigma = \psi(u, v)$ for any cotree arc $(u, v)$ of $X$. 
3. Proof of Theorem 1.1

Suppose that $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. If $n = 2$, then $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now since elementary abelian coverings of the complete graph $K_5$ were classified by Kuzman [10], we only consider $n \geq 3$.

By $K_5$, we denote the complete graph with vertex set \{0, 1, 2, 3, 4\}. Let $T$ be a spanning tree of $K_5$ as shown by dashed lines in Figure 2. Let $\phi$ be such a voltage assignment defined by $\phi = 1$ on $T$ and $\phi = a_0, a_1, a_2, a_3, a_4$, and $b_0$ on the cotree arcs (0, 1), (1, 2), (2, 3), (3, 4), (4, 0), and (0, 2), respectively. Let $\rho = (01234), \tau = (0132)$ and $\sigma = (024)$. Then $\rho, \tau, \sigma$ are automorphisms of $K_5$.

By $i_1i_2\ldots i_s$ denote a directed cycle which has vertices $i_1, i_2, \ldots, i_s$ in a consecutive order. There are six fundamental cycles 130, 124, 1423, 134, 1403, and 13024 in $K_5$ which are generated by the six cotree arcs (0, 1), (1, 2), (2, 3), (3, 4), (4, 0) and (0, 2), respectively. Each cycle is mapped to a cycle of the same length under the actions of $\rho, \tau, \sigma$. We list all these cycles and their voltages in Table 1 in which $C$ denotes a fundamental cycle of $K_5$ and $\phi(C)$ denotes the voltage of $C$.

Let $\tilde{X} = K_5 \times \phi D_{2n}$ be a covering graph of the graph $K_5$ satisfying the hypotheses in the theorem, where $\phi = 1$ on the spanning tree $T$ which is depicted by the dashed lines in Figure 2. Note that the vertices of $K_5$ are labeled by 0, 1, 2, 3, and 4. By the hypotheses, the fibre-preserving group, say $\tilde{L}$, of the covering graph $K_5 \times \phi D_{2n}$ acts arc-transitively on $K_5 \times \phi D_{2n}$. Hence, the projection of $\tilde{L}$, say $L$, is arc-transitive on the base graph $K_5$. Thus $L$ is isomorphic to $AGL(1, 5) = \langle \rho, \tau \rangle$, $A_5 = \langle \rho, \sigma \rangle$, or $S_5 = \langle \rho, \sigma, \tau \rangle$. Consider the mapping $\bar{\rho}$ from the set \{a_0, a_1, a_2, a_3, a_4, b_0\} of the voltages of the six fundamental cycles of $K_5$ to the group $D_{2n}$, defined by $(\phi(C))^{\bar{\rho}} = \phi(C^\rho)$, where $C$ ranges over the six fundamental cycles. From Table 1, one can see that $a_0^{\bar{\rho}} = a_1, a_1^{\bar{\rho}} = a_2b_0, a_2^{\bar{\rho}} = b_0^{-1}a_3, a_3^{\bar{\rho}} = a_4b_0, a_4^{\bar{\rho}} = b_0^{-1}a_0$ and $b_0^{\bar{\rho}} = b_0$. Similarly, we can define $\bar{\sigma}$ and $\bar{\tau}$. 
Here we make the following general assumption.

(I) Let \( \tilde{X} \) be a connected \( D_{2n} \)-covering \((n \geq 3)\) of the complete graph \( K_5 \) whose fibre-preserving subgroup is arc-transitive.

For the three following lemmas we suppose that \( n \) is an odd number.

**Lemma 3.1.** Suppose that the subgroup of \( \text{Aut}(\tilde{X}) \) generated by \( \rho \) and \( \sigma \), say \( L \), lifts. Under the assumption (I), \( \tilde{X} \) is arc-transitive if and only if \( \tilde{X} \) is isomorphic to \( DK(6) \).

**Proof.** Since \( \rho, \sigma \in L \), Proposition 2.1 implies that \( \bar{\rho} \) and \( \bar{\sigma} \) can be extended to automorphisms of \( D_{2n} \). We denote by \( \rho^* \) and \( \sigma^* \) these extended automorphisms, respectively. In this case \( o(a_0) = o(a_1) = o(a_3) \). Now we consider the following two subcases:

### Table 1. Fundamental cycles and their images with corresponding voltages.

<table>
<thead>
<tr>
<th>( C )</th>
<th>( \phi(C) )</th>
<th>( C^\rho )</th>
<th>( \phi(C^\rho) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>130</td>
<td>( a_0 )</td>
<td>241</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>124</td>
<td>( a_1 )</td>
<td>230</td>
<td>( a_2 b_0 )</td>
</tr>
<tr>
<td>1423</td>
<td>( a_2 )</td>
<td>2034</td>
<td>( b_0^{-1} a_3 )</td>
</tr>
<tr>
<td>134</td>
<td>( a_3 )</td>
<td>240</td>
<td>( a_4 b_0 )</td>
</tr>
<tr>
<td>1403</td>
<td>( a_4 )</td>
<td>2014</td>
<td>( b_0^{-1} a_0 )</td>
</tr>
<tr>
<td>13024</td>
<td>( b_0 )</td>
<td>24130</td>
<td>( b_0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( C^\sigma )</th>
<th>( \phi(C^\sigma) )</th>
<th>( C^\tau )</th>
<th>( \phi(C^\tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>132</td>
<td>( a_2^{-1} a_1^{-1} )</td>
<td>321</td>
<td>( a_2^{-1} a_1^{-1} )</td>
</tr>
<tr>
<td>140</td>
<td>( a_4 a_0 )</td>
<td>304</td>
<td>( a_4^{-1} a_3^{-1} )</td>
</tr>
<tr>
<td>1043</td>
<td>( a_0^{-1} a_4^{-1} a_3^{-1} )</td>
<td>3402</td>
<td>( a_3 a_4 b_0 a_2 )</td>
</tr>
<tr>
<td>130</td>
<td>( a_0 )</td>
<td>324</td>
<td>( a_2^{-1} a_3^{-1} )</td>
</tr>
<tr>
<td>1023</td>
<td>( a_0^{-1} b_0 a_2 )</td>
<td>3412</td>
<td>( a_3 a_1 a_2 )</td>
</tr>
<tr>
<td>13240</td>
<td>( a_2^{-1} a_4 a_0 )</td>
<td>32104</td>
<td>( a_2^{-1} a_1^{-1} a_0^{-1} a_4^{-1} a_3^{-1} )</td>
</tr>
</tbody>
</table>
Subcase I. $o(a_0) = o(a_1) = o(a_3) = 2$.

By considering $a_1^* = a_4a_0$, we have $o(a_4a_0) = 2$. It follows that $o(a_4) \neq 2$. Since $a_4^* = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$. So $o(b_0^{-1}) = 2$, and hence $o(a_2) \neq 2$, by $a_2^* = b_0^{-1}a_3$. Now we may assume that $a_0 = a^ib, a_1 = a^jb, a_3 = a^kb, a_2 = a^r, a_4 = a^s$ and $b_0 = a^t$, where $0 \leq i, j, k, l \leq n - 1$ and $0 < r, s \leq n - 1$. Since Aut$(D_{2n})$ acts transitively on involutions, by Proposition 2.2 we may assume that $a_0 = b, a_1 = a^ib, a_3 = a^kb, a_2 = a^r, a_4 = a^s$ and $b_0 = a^t$, where $0 \leq i, j, k, l \leq n - 1$ and $0 < r, s \leq n - 1$. Also since $K_5 \times \phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{ i, j, k, r, s \}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get the same results. First suppose that $(i, n) = 1$. Since $\sigma: a \mapsto a^i, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b, a_1 = ab, a_3 = a^ib, a_2 = a^r, a_4 = a^s$, and $b_0 = a^j$, where $0 \leq i, j \leq n - 1$ and $0 < r, s \leq n - 1$. From Table 1, one can see that $a_0^* = b_0^* = ab, a_1^* = (ab)^* = a^rb^* = a^rj$. Thus $a^r = a^{r+j-1}$. By considering the image of $a_2 = a^r, a_4 = a^s$ and $b_0 = a^j$ under $\rho^*$, we conclude that $a^{r+j-1} = a^r, a^{s(r+j-1)} = a^j$ and $a^{j(r+j-1)}ab = a^j$. Also $a_0^* = b_0^* = a^r+jb$ and $a_1^* = (ab)^* = a^srb^* = a^sb$. Thus $a^s = a^{n+r+1}.

Now by considering the image of $a_2 = a^r, a_4 = a^s$ and $b_0 = a^j$ under $\sigma^*$, we conclude that $a^{r+s-1} = a^{s-i}, a^{s(r+s-1)} = a^{s^{-}r+b}$ and $a^{j(s+r-1)}a^{-r+1}b = a^{s-r}b$.

Therefore, we have the following:

\begin{align*}
(1) \quad r(r+j-1) &= j-i, \\
(2) \quad s(r+j-1) &= j, \\
(3) \quad j(r+j-1) &= j-1, \\
(4) \quad j(s+r-1) &= s-1, \\
(5) \quad r(s+r-1) &= s-i, \\
(6) \quad s(s+r-1) &= -j+r.
\end{align*}

By (1) and (3), $rj(r+j-1) = j^2 - ij$ and $rj(r+j-1) = rj - r$. Thus $j^2 - ji = rj - r$. Also by (4) and (5), $rj(s+r-1) = sr - r$ and $rj(s+r-1) = sj - ij$. Thus $sj - ij = sr - r$. So $j^2 - rj = sj - sr$, and hence $(j - r)(j - s) = 0$. Also by (2) and (3), $sj(r+j-1) = j^2$ and $sj(r+j-1) = sj - s$. Thus $j^2 = sj - s$. By $(j - r)(j - s) = 0$, we have $j = r$ or $j = s$. If $j = r$,
then \( s^2 + sr - s = 0 \), by (6). Thus \( s = 0 \) or \( s = -r + 1 \). If \( s = 0 \), then \( j = 0 \) by (2). Thus \( r = 0 \), a contradiction. If \( s = -r + 1 \), then \( s = 1 \) by \( j(s + r - 1) = s - 1 \). So \( r = 0 \), a contradiction. If \( j = s \), then by \( j^2 = sj - s \), we have \( s = 0 \), a contradiction.

Now suppose that \((r,n) = 1\). Since \( \sigma : a \mapsto a^r, b \mapsto b \) is an automorphism of \( D_{2n} \), by Proposition 2.2, we may assume that \( a_0 = b \), \( a_1 = aib \), \( a_3 = a^{j}b \), \( a_2 = a \), \( a_4 = a^r \) and \( b_0 = a^kb \), where \( 0 \leq i, j, k \leq n - 1 \) and \( 0 < r \leq n - 1 \). From Table 1, one can see that \( a_0^\sigma = b^{\sigma^*} = a^ib \), \( a_2^\sigma = (a)^{\rho^*} = a^{k-j} \). By considering the image of \( a_1 = a^ib \), \( a_3 = a^j, a_4 = a^r \) and \( b_0 = a^kb \) under \( \rho^* \), we conclude that \( a_{i(k-j)}a^ib = a^{k+1}b \), \( a_{i(k-j)}a^{j}b = a^{r+k}b \), \( a_{r(k-j)} = a^k \) and \( a_{(k-j)}a^ib = a^kb \). Also \( a_{0}^\sigma = b^{\sigma^*} = a^{i-1}b \) and \( a_{2}^\sigma = (a)^{\sigma^*} = a^{r-j} \). Now by considering the image of \( a_1 = a^ib \), \( a_3 = a^j \), \( a_4 = a^r \) and \( b_0 = a^kb \) under \( \sigma^* \), we conclude that \( a_{i(r-j)}a^{j}b = a^rb \), \( a_{j(r-j)}a^{i-1}b = b \), \( a_{i(r-j)}a^{i-1}b = a^{r-1}b \).

Therefore, we have the following:

1. \( i(k-j) + i = k + 1 \),
2. \( j(k-j) + i = r + k \),
3. \( r(k-j) = k \),
4. \( k(k-j) + i = k \),
5. \( i(r-j) + i - 1 = r \),
6. \( j(r-j) + i - 1 = 0 \),
7. \( r(r-j) = -k + 1 \),
8. \( k(r-j) = r - i \).

By (2) and (3), \( rj(k-j) = r^2 + rk - ir \) and \( rj(k-j) = kj \). Thus \( r^2 + rk - ir = kj \). Also by (7) and (8), \( rk(r-j) = -k^2 + k \) and \( rk(r-j) = r^2 - ir \). Thus \( -k^2 + k = r^2 - ir \). So \( kj - rk = -k^2 + k \), and hence \( k(j - r + k - 1) = 0 \). Thus \( k = 0 \) or \( j = r - k + 1 \). If \( k = 0 \), then \( i = 0 \) by (4). Thus by \(-k^2 + k = r^2 - ir \), we have \( r = 0 \), a contradiction. If \( j = r - k + 1 \), then \((k - 1)(r + 1) = 0 \) by (7). Hence \( k = 1 \) or \( r = -1 \). If \( k = 1 \), then \( j = r \). Now by (6), \( i = 1 \), and so by (8), we have \( r = 1 \). So by (5), \( 1 = 0 \), a contradiction. If \( r = -1 \), then \( j = -k \). Also by (5), \( i(r - j + 1) = 0 \), and so \( i = 0 \) or \( r = j - 1 \). If \( i = 0 \), then by (1), \( k = -1 \). Thus \( j = 1 \), and so by (3), \( 2 = -1 \). Therefore, \( n = 3 \) and

\[
\begin{align*}
a_0 &= b, & a_1 &= b, & a_3 &= ab, & a_2 &= a, & a_4 &= a^{-1}, & b_0 &= a^{-1}b.
\end{align*}
\]
From Table 1, it is easy to check that $\tilde{\rho}$, $\tilde{\sigma}$ and $\tilde{\tau}$ can be extended to automorphisms of $D_{2n}$. Thus by Proposition 2.1, $\rho$, $\sigma$ and $\tau$ lift. Since $S_5 = \langle \rho, \sigma, \tau \rangle$ is 2-arc-transitive, it follows that $\text{Aut}(\tilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle \rho, \sigma, \tau \rangle$. Therefore, $\tilde{X}$ is 2-arc-transitive.

Finally, if $r = j - 1$, then by $r = -1$, we have $j = 0$. So by (6), $i = 1$. Also by (7), $k = 0$. Now by (2), $1 = -1$, and so $n = 2$, a contradiction.

Subcase II. $o(a_0) = o(a_1) = o(a_3) \neq 2$.

By considering $a_0^* = a_4a_0$, we have $o(a_4a_0) \neq 2$. It follows that $o(a_4) \neq 2$. Since $a_4^* = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$. So $o(b_0^{-1}) \neq 2$, and hence $o(a_2) \neq 2$ by $a_2^* = b_0^{-1}a_3$. Now we may assume that $a_0 = a^i$, $a_1 = a^j$, $a_2 = a^k$, $a_3 = a^l$, $a_4 = a^m$ and $b_0 = a^n$, where $0 \leq i, j, k, l, m, n \leq n - 1$. Since $K_5 \times \phi D_{2n}$ is connected, we have a contradiction. \hfill $\Box$

Lemma 3.2. Suppose that the subgroup of $\text{Aut}(\tilde{X})$ generated by $\rho$ and $\tau$, say $L$, lifts. Under the assumption (I), $\tilde{X}$ is arc-transitive if and only if $\tilde{X}$ is isomorphic to $DK(2n)$ for $n \geq 3$.

Proof. Since $\rho, \tau \in L$, Proposition 2.1 implies that $\tilde{\rho}$ and $\tilde{\tau}$ can be extended to automorphisms of $D_{2n}$. We denote these extended automorphisms by $\rho^*$ and $\tau^*$, respectively. In this case $o(a_0) = o(a_1)$. Now we consider the following two subcases:

Subcase I. $o(a_0) = o(a_1) = 2$.

By considering $a_0^* = a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1}) = 2$. It follows that $o(a_2) \neq 2$. Since $a_2^* = b_0^{-1}a_3$, we have either $o(b_0) = o(a_3) = 2$ or $o(b_0) \neq 2$ and $o(a_3) \neq 2$. First suppose that $o(b_0) \neq 2$ and $o(a_3) \neq 2$. Since $a_3^* = a_4b_0$, we have $o(a_4) \neq 2$. Also since $a_4^* = b_0^{-1}a_0$, it follows that $o(a_0) \neq 2$, a contradiction.

Now suppose that $o(b_0) = o(a_3) = 2$. Since $a_3^* = a_4b_0$, it implies that $o(a_4) \neq 2$. Now we may assume that $a_0 = a^ib$, $a_1 = a^jb$, $a_3 = a^kb$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^lb$, where $0 \leq i, j, k, l \leq n - 1$ and $0 < r, s \leq n - 1$. Since $\text{Aut}(D_{2n})$ acts transitively on involutions, we may
assume that \(a_0 = b, a_1 = a^ib, a_3 = a^jb, a_2 = a^r, a_4 = a^s\) and \(b_0 = a^kb\), where \(0 \leq i, j, k \leq n - 1\) and \(0 < r, s \leq n - 1\). Since \(K_5 \times_b D_{2n}\) is assumed to be connected, \(D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle\). Thus we may assume that \((t, n) = 1\), where \(t \in \{i, j, k, r, s\}\). Without loss of generality, we may assume that \((i, n) = 1\) or \((r, n) = 1\). In fact, with the same arguments as in other cases we get the same results. First suppose that \((i, n) = 1\). Since \(\sigma: a \mapsto a^i, b \mapsto b\) is an automorphism of \(D_{2n}\), by Proposition 2.2, we may assume that \(a_0 = b, a_1 = ab, a_3 = a^jb, a_2 = a^r, a_4 = a^s\) and \(b_0 = a^j b\), where \(0 \leq i, j \leq n - 1\) and \(0 < r, s \leq n - 1\). From Table 1, one can see that \(a_0^ρ^* = bρ^* = ab, a_1^ρ^* = (ab)^ρ^* = a^ρ^* b^ρ^* = a^{r+j}b\). Thus \(a^ρ^* = a^{r+j-1}\). By considering the image of \(a_2 = a^r, a_3 = a^ib\) and \(b_0 = a^j b\) under \(\rho^*\), we conclude that \(a^{r(r+j-1)} = a^{j-i}, a^{i(r+j-1)}ab = a^{s+j}b\) and \(a^{j(r+j-1)}ab = a^j b\). Also \(a_0^{τ^*} = b^{τ^*} = a^{-r+1}b, a_1^{τ^*} = (ab)^{τ^*} = a^{r+b}τ^* = a^{i-s}b\). Thus \(a^{τ^*} = a^{i-s+r-1}\). By considering the image of \(a_2 = a^r\) and \(b_0 = a^j b\) under \(τ^*\), we conclude that \(a^{r(i-s+r-1)} = a^{i-s-j+r}\) and \(a^{j(i-s+r-1)}a^{-r+1} = a^{r+1-s+i}\).

Therefore, we have the following:

\[
\begin{align*}
(1) & \quad r(r+j-1) = j - i, \\
(2) & \quad i(r+j-1) + 1 = s + j, \\
(3) & \quad j(r+j-1) + 1 = j, \\
(4) & \quad r(i-s+r-1) = i-s-j+r, \\
(5) & \quad j(i-s+r-1) = i-s.
\end{align*}
\]

By (4) and (5), \((j-r)(i-s+r-2) = 0\). Thus \(j = r\) or \(i - s + r = 2\). If \(i - s + r = 2\), then by (4) \(j = i - s\). Now by (1), \(r(i-s+1) = -s\). So by considering (4) \(i + r = j\). Thus \(r = -s\) by \(j = i - s\). So \(i = 2s + 2\), and hence \(j = s + 2\). Now by (2), \(1 = 0\), a contradiction. If \(j = r\), then \(r(2r-1) = r - i\) by (1). Also by (3), \(r(2r-1) = r - 1\). So \(i = 1\), and hence by (2), \(s = r\). Now by (5), \(s = r = j = 1\). Thus by (1), \(1 = 0\), a contradiction.

Now suppose that \((r, n) = 1\). Since \(\sigma: a \mapsto a^r, b \mapsto b\) is an automorphism of \(D_{2n}\), by Proposition 2.2, we may assume that \(a_0 = b, a_1 = a^ib, a_3 = a^jb, a_2 = a, a_4 = a^r\) and \(b_0 = a^kb\), where \(0 \leq i, j, k \leq n - 1\) and \(0 < r \leq n - 1\). From Table 1, one can see that \(a_0^ρ^* = bρ^* = a^ib, a_2^ρ^* = (a)^ρ^* = a^{k-j}\). By considering the image of \(a_1 = a^ib, a_3 = a^jb, a_4 = a^r\) and \(b_0 = a^kb\) under
ρ*, we conclude that $a^{i(k-j)}a^ib = a^{k+1}b$, $a^{j(k-j)}a^ib = a^{k+r}b$, $a^{r(k-j)} = a^k$ and $a^{k(k-j)}a^ib = a^{k}b$. Also $a_0^{\tau^*} = b^{\tau^*} = a_1^{i-1}b$, $a_2^{\tau^*} = a^{i-r-k+1}$. By considering the image of $a_1 = a^ib$, $a_3 = a^jb$ and $b_0 = a^{k}b$ under $\tau^*$, we conclude that $a^{i(j-r-k+1)}a^{i-1}b = a^{j-r}b$, $a^{j(r-k+1)}a^{i-1}b = a^{j-1}b$ and $a^{k(j-r-k+1)}a^{i-1}b = a^{i-1-r+j}b$.

Therefore, we have the following:

(1) $\ i k - i j + i = k + 1$, \quad (2) \ $j k - j^2 + i = r + k$,
(3) \ $r k - r j = k$, \quad (4) \ $k^2 - k j + i = k$,
(5) \ $i(j - r - k + 1) + i - 1 = -r + j$, \quad (6) \ $j(j - r - k + 1) = j - i$,
(7) \ $k(j - r - k + 1) = j - r$.

By (6), $j^2 - j r + j k + i = 0$. Also by (6) and (7), we have $k j(j - r - k + 1) = k j - k i$ and $k j(j - r - k + 1) = j^2 - r j$. Thus $j^2 - j r = k j - k i$. Thus $i(k - 1) = 0$, and so $i = 0$ or $k = 1$. If $i = 0$, then by (1), we have $k = -1$. Also by (4), $j = -2$. Now by (2), $r = -1$. Therefore,

$a_0 = b$, \quad $a_1 = b$, \quad $a_3 = a^{-2}b$, \quad $a_2 = a$, \quad $a_4 = a^{-1}$, \quad $b_0 = a^{-1}b$.

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$ and $\tau$ lift. Clearly, $AGL(1, 5) = \langle \rho, \tau \rangle$ is 1-regular. Thus $Aut(X)$ contains a 1-regular subgroup lifted by $\langle \rho, \tau \rangle$.

Now if $k = 1$, then by (3) and (4), $r - r j = 1$ and $i - j = 0$. Since $i = j$, it follows that $i(i - r) = -r + 1$ by (5). So $i^2 - i r = -r + 1 = -1 - r j + 1$. Thus $i = j = 0$, and so $r = 1$. Now by (2), $2 = 0$, a contradiction.

**Subcase II.** $o(a_0) = o(a_1) \neq 2$.

By considering $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1}) \neq 2$. It follows that $o(a_2) \neq 2$. Since $a_2^{\tau^*} = b_0^{-1}a_3$, we have either $o(b_0) = o(a_3) = 2$ or $o(b_0) \neq 2$ and $o(a_3) \neq 2$. First suppose that $o(b_0) = o(a_3) = 2$. Since $a_3^{\tau^*} = a_4b_0$, it follows that $o(a_4) \neq 2$. Now by considering $a_4^{\tau^*} = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$ a contradiction.
Now suppose that $o(b_0) \neq 2$ and $o(a_3) \neq 2$. Since $a_3^\ast = a_4b_0$, we have $o(a_4) \neq 2$. Therefore, $K_5 \times \phi D_{2n}$ is not connected, a contradiction.

□

**Lemma 3.3.** Suppose that the subgroup of $\text{Aut}(\tilde{X})$ generated by $\rho$, $\sigma$ and $\tau$, say $L$, lifts. Under the assumption (I), $\tilde{X}$ is arc-transitive if and only if $\tilde{X}$ is isomorphic to $DK(2n)$ for $n \geq 3$.

*Proof.* $\rho$ and $\sigma$ lift. With the same arguments as in Cubcase I, we have $n = 3$ and

\[
a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.
\]

From Table 1, it is easy to check that $\tilde{\rho} \tilde{\sigma}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$, $\sigma$ and $\tau$ lift. Also $S_5 = \langle \rho, \sigma, \tau \rangle$ is 2-arc-transitive. Thus $\text{Aut}(\tilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle \rho, \sigma, \tau \rangle$. Thus $\tilde{X}$ is 2-arc-transitive. Moreover, $\rho$ and $\tau$ lift. With the same arguments as in Subcase II, we have

\[
a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.
\]

From Table 1, it is easy to check that $\tilde{\sigma}$ can be extended to automorphisms of $D_{2n}$ whenever $n = 3$. Now if $n = 3$, then by Proposition 2.1, $\sigma$ lift. Now with the same arguments as above, $\tilde{X}$ is 2-arc-transitive. □

Now suppose that $n$ is even.

**Lemma 3.4.** Suppose that the subgroup of $\text{Aut}(\tilde{X})$ generated by $\rho$ and $\sigma$, say $L$, lifts. Then there is no connected regular covering of the complete graph $K_5$ whose fibre-preserving group is arc-transitive.

*Proof.* Since $\rho, \sigma \in L$, Proposition 2.1 implies that $\tilde{\rho}$ and $\tilde{\sigma}$ can be extended to automorphisms of $D_{2n}$. We denote these extended automorphisms by $\rho^\ast$ and $\sigma^\ast$, respectively. In this case $o(a_0) = o(a_1) = o(a_3)$. Now we consider the following two subcases:
Subcase I. \( o(a_0) = o(a_1) = o(a_3) = 2. \)

Since \( o(a_0) = 2 \), we may assume that \( a_0 = a^{n/2} \) or \( a_0 \neq a^{n/2} \) and \( a_0 = a^i b \) \((0 \leq i < n)\). If \( a_0 = a^{n/2} \), then \( a_1 = a_3 = a^{n/2} \). By Table 1, \( a_1^\ast = a_4 a_0 \) and \( a_3^\ast = a_0 \). Thus \( a_4 = 1 \) and so by 
\[ a_4^\ast = b_0^{-1} a_0, \]
we have \( b_0 = a^{n/2} \). Also by \( a_2^\ast = b_0^{-1} a_3 \), we have \( a_2 = 1 \). Therefore \( K_5 \times_\phi D_{2n} \) is not connected, a contradiction.

Thus we may assume that \( a_0 \neq a^{n/2} \). So \( a_1 \neq a^{n/2} \) and \( a_3 \neq a^{n/2} \). Thus we may assume that \( a_0 = a^i b, a_1 = a^j b \) and \( a_3 = a^k b \), where \( 0 \leq i, j, k < n \). By considering \( a_1^\ast = a_2 b_0 \), we have one of the following cases:

i) \( a_2 = a^l b, b_0 = a^t \) \((0 \leq l < n, 0 < t < n)\);

ii) \( a_2 = a^l, b_0 = a^t b \) \((0 < l < n, 0 \leq t < n)\).

First suppose that \( a_2 = a^l b, b_0 = a^t \) \((0 \leq l < n, 0 < t < n)\). Since \( a_4^\ast = b_0^{-1} a_0 \), we may suppose that \( a_4 = a^s b \), where \( 0 \leq s < n \). Now since \( b_0^\ast = a_2^{-1} a_4 a_0 \), we have a contradiction. Now suppose that \( a_2 = a^l, b_0 = a^t b \) \((0 < l < n, 0 \leq t < n)\). Since \( a_4^\ast = b_0^{-1} a_0 \), we have \( o(a_4) \neq 2 \) or \( a_4 = a^{n/2} \). First suppose that \( o(a_4) \neq 2 \). Now by Proposition 2.2, we may assume that \( a_0 = a^i b, a_1 = a^j b, a_3 = a^k b, a_2 = a^l, a_4 = a^k \) and \( b_0 = a^t b \), where \( 0 \leq i, j, k, t \leq n - 1 \) and \( 0 < l, k \leq n - 1 \). Now with the same arguments as in Subcase I, when \( n \) is odd, we have

\[ a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b. \]

From Table 1, it is easy to check that \( \bar{\rho} \) and \( \bar{\sigma} \) can be extended to automorphisms of \( D_{2n} \) when \( n = 3 \), a contradiction.

Now suppose that \( a_4 = a^{n/2} \). Now we may assume that \( a_0 = a^i b, a_1 = a^j b, a_3 = a^k b, a_2 = a^r, a_4 = a^{n/2} \), and \( b_0 = a^t b \), where \( 0 \leq i, j, k, l \leq n - 1 \) and \( 0 < r \leq n - 1 \). Since \( \text{Aut}(D_{2n}) \) acts transitively on involutions, by Proposition 2.2, we may assume that \( a_0 = b, a_1 = a^i b, a_3 = a^j b, a_2 = a^r, a_4 = a^{n/2} \) and \( b_0 = a^k b \), where \( 0 \leq i, j, k \leq n - 1 \) and \( 0 < r \leq n - 1 \). Since \( K_5 \times_\phi D_{2n} \) is assumed to be connected, \( D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle \). Thus we may assume that \( (t, n) = 1 \), where
$t \in \{i, j, k, r\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get same results. First suppose that $(i, n) = 1$. Since $\sigma: a \mapsto a^i, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b, a_1 = ab, a_3 = a^ib, a_2 = a^r, a_4 = a^{(n/2)}$ and $b_0 = a^rb, where 0 \leq i, j \leq n - 1 and 0 < r \leq n - 1$. Now with the same arguments as in Subcase I, when $n$ is odd (by replacing $s$ with $(n/2)$), we have a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma: a \mapsto a^r, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b, a_1 = a^ib, a_3 = a^jb, a_2 = a, a_4 = a^{(n/2)}$ and $b_0 = a^kb, where 0 \leq i, j, k \leq n - 1$. Now by replacing $r$ with $(n/2)$ in Case I, when $n$ is odd, we have $(n/2)(k - j) = k$ and $(n/2)((n/2) - j) = -k + 1$ (see Equations (3) and (7) in Subcase I). So $n = 2$, a contradiction.

Subcase II. $o(a_0) = o(a_1) = o(a_3) \neq 2$.

By considering $a_1^\ast = a_4a_0$, we have $o(a_4a_0) \neq 2$. So we have $o(a_4) \neq 2$ or $o(a_4) = 2$ and $a_4 = a^{n/2}$. If $o(a_4) \neq 2$, then $o(b_0^{-1}a_0) \neq 2$ by $a_4^\ast = b_0^{-1}a_0$. Now we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. If $b_0 = a^{n/2}$, then $o(a_2) \neq 2$ by $a_2^\ast = b_0^{-1}a_3$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. If $o(b_0) \neq 2$, then by $a_2^\ast = b_0^{-1}a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. Finally, if $a_4 = a^{n/2}$, then by considering $a_3^\ast = a_4b_0$, we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. Clearly, $b_0 \neq a^{n/2}$ by $a_3^\ast = a_4b_0$. Thus $o(b_0) \neq 2$, and so by $a_2^\ast = b_0^{-1}a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction.

Lemma 3.5. Suppose that the subgroup of Aut($\tilde{X}$) generated by $\rho$ and $\tau$, say $L$, lifts. Under the assumption (I), $\tilde{X}$ is arc-transitive if and only if $\tilde{X}$ is isomorphic to $DK(2n)$ for $n > 3$. 

Proof. Since \( \rho, \tau \in L \), Proposition 2.1 implies that \( \bar{\rho} \) and \( \bar{\tau} \) can be extended to automorphisms of \( D_{2n} \). We denote these extended automorphisms by \( \rho^* \) and \( \tau^* \), respectively. In this case \( o(a_0) = o(a_1) \). Now we consider the following two subcases:

**Subcase I.** \( o(a_0) = o(a_1) = 2 \).

Since \( o(a_0) = 2 \), we may assume that \( a_0 = a^{n/2} \) or \( a_0 \neq a^{n/2} \) and \( a_0 = a^ib \) (0 ≤ \( i < n \)). If \( a_0 = a^{n/2} \), then \( a_1 = a^{n/2} \). By Table 1, we have \( a_0^* = a_2^{-1}a_1^{-1} \) and \( a_1^* = a_2b_0 \). Therefore, \( a_2 = 1 \) and \( b_0 = a^{n/2} \). Also by \( a_2^* = b_0^{-1}a_3 \), we have \( a_3 = a^{n/2} \). Now by \( a_3^* = a_4b_0 \), we have \( a_4 = 1 \). Thus \( \bar{X} \) is not connected, a contradiction. Thus we may assume that \( a_0 \neq a^{n/2} \) and \( a_0 = a^ib \). So \( a_1 \neq a^{n/2} \) and so we may assume that \( a_0 = a^ib, a_1 = a^jb, \) where 0 ≤ \( i, j < n \). By considering \( a_0^* = a_2^{-1}a_1^{-1} \), we have \( o(a_2) \neq 2 \) or \( a_2 = a^{n/2} \). First assume that \( o(a_2) \neq 2 \). Thus \( b_0 = a^kb \) (0 ≤ \( k < n \)) by \( a_1^* = a_2b_0 \). Also since \( a_2^* = b_0^{-1}a_3 \), we have \( o(a_3) = 2 \) and \( a_3 = a^lb \) (0 ≤ \( l < n \)). Finally, since \( a_4^* = b_0^{-1}a_0 \), we have \( o(a_4) \neq 2 \) or \( a_4 = a^{n/2} \). First suppose that \( a_4 = a^{n/2} \). We have \( a_0 = a^ib, a_1 = a^jb, a_3 = a^kb, a_2 = a^r, a_4 = a^{n/2} \) and \( b_0 = a^l \), where 0 ≤ \( i, j, k, l < n - 1 \) and 0 < \( r \leq n - 1 \). Since \( \text{Aut}(D_{2n}) \) acts transitively on involutions, by Proposition 2.2, we may assume that \( a_0 = b, a_1 = a^ib, a_3 = a^jb, a_2 = a^r, a_4 = a^{n/2} \) and \( b_0 = a^k \), where 0 ≤ \( i, j, k, l \leq n - 1 \) and 0 < \( r \leq n - 1 \). Since \( a_4^* = b_0^{-1}a_0 \), we have \( k = n/2 \). Now \( a_4a_0 = b_0 \), and so \( (a_4a_0)^* = b_0^* \). Thus \( a_0 = a_1 \), and so \( i = 0 \). We have \( a_0^* = a_1^* \). So \( a_1 = a_2b_0 \), and hence \( r = n/2 \). Now \( a_2 = a_4 \), and so \( a_2^* = a_4^* \). Therefore, \( a_0 = a_3 \), and hence \( a_3 = b \). Now \( K_5 \times \phi D_{2n} \) is not connected a contradiction.

Now suppose that \( o(a_4) \neq 2 \). With the same arguments as in Subcase II, when \( n \) is odd, we have

\[
a_0 = b, a_1 = b, a_3 = a^{-2}b, a_2 = a, a_4 = a^{-1}, b_0 = a^{-1}b.
\]
From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$ and $\tau$ lift. Also $AGL(1, 5) = \langle \rho, \tau \rangle$ is 1-regular. Thus $\text{Aut}(\bar{X})$ contains a 1-regular subgroup lifted by $\langle \rho, \tau \rangle$.

Now assume that $a_2 = a^{n/2}$. Thus $b_0 = a^k b$ ($0 \leq k < n$) by $a_1^{\rho^*} = a_2 b_0$. Also since $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_3) = 2$ and $a_3 = a^l b$ ($0 \leq l < n$). Finally, since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $a_4 = a^{n/2}$. We have $a_0 = a^i b, a_1 = a^j b, a_3 = a^k b$, $a_2 = a_4 = a^{n/2}$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n - 1$. Since $a_4^{-1} = a_3 a_1 a_2$, we have $k = j$. Also since $a_2^{-1} = a_3 a_4 b_0 a_2$, we have $l = k = j$. Since $\text{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b, a_1 = a^i b, a_3 = a^i b, a_2 = a_4 = a^{n/2}$, and $b_0 = a^i b$, where $0 \leq i, j, k \leq n - 1$. Since $a_4^{-1} = b_0^{-1} a_0$, we have $i = n/2$, a contradiction.

Now suppose that $a_0 = a^i b, a_1 = a^j b, a_3 = a^k b, a_2 = a^{n/2}, a_4 = a^s$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n - 1$ and $0 < s \leq n - 1$. Since $\text{Aut}(D_{2n})$ acts transitively on involutions, we may assume that $a_0 = b, a_1 = a^i b, a_3 = a^j b, a_2 = a^{n/2}, a_4 = a^s$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n - 1$ and $0 < s \leq n - 1$. Since $K_5 \times \phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, s\}$. Without loss of generality we may assume that $(i, n) = 1$ or $(s, n) = 1$. In fact, with the same arguments the in other cases we get the same results. First suppose that $(i, n) = 1$. Therefore, we may assume that $a_0 = b, a_1 = a b, a_3 = a^i b, a_2 = a^{(n/2)}$, $a_4 = a^s$, and $b_0 = a^l b$, where $0 \leq i, j \leq n - 1$ and $0 < s \leq n - 1$. Now with the same arguments as in Case II, when $n$ is odd we get a contradiction. Now suppose that $(s, n) = 1$. Therefore, we may assume that $a_0 = b, a_1 = a^i b, a_3 = a^j b, a_2 = a^{n/2}, a_4 = a^s$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n - 1$. From Table 1, one can see that $a_0^{\rho^*} = b_0^{\rho^*} = a_1^{\rho^*}, a_4^{\rho^*} = (a)^{\rho^*} = a^k$. By considering the image of $a_1 = a^i b, a_3 = a^j b$ and $a_2 = a^{n/2}$ under $\rho^*$, we conclude that $a^{ik+i} b = a^{(n/2) + k} b, a^{jk+i} b = a^{k+1} b$ and $a^{(n/2) k} = a^{k-j}$. Thus, we have $ik + i = n/2 + k, jk + i = k + 1$ and $(n/2) k = k - j$. By $(n/2) k = k - j$, we have $nk = 2k - 2j$. It follows that $2j = 2k$. Also $a^{\tau^*} = a^j b a^i b a^{(n/2)} = a^j - i + (n/2)$. 


Thus $a^*_2 = a^{n/2(j-i+(n/2))} = a^j b a a^n ba^{(n/2)} = a^{j-1-k+(n/2)}$. So, $2j - 2k - 2 = 0$ and so $2 = 0$, a contradiction.

Subcase II. $o(a_0) = o(a_1) \neq 2$.

By considering $a^*_0 = a^{-1}_2 a_1^{-1}$, we have $o(a^{-1}_2 a_1^{-1}) \neq 2$. Thus $o(a_2) \neq 2$ or $a_2 = a^{n/2}$. First suppose that $o(a_2) \neq 2$. By considering $a^*_2 = b^{-1}_0 a_3$, we have one of the following cases:

i) $a_3 = a^i b$, $b_0 = a^j b$ ($0 \leq i, j < n$);

ii) $a_3 = a^i$, $b_0 = a^{n/2}$ ($0 < i < n$);

iii) $a_3 = a^{n/2}$, $b_0 = a^i$ ($0 < i < n$).

By $a^*_1 = a_2 b_0$, we have a contradiction in the first case. Now consider the second case. Since $a^*_3 = a_4 b_0$, we have $o(a_4) \neq 2$. Now $K_5 \times \phi D_{2n}$ is not connected, a contradiction. Now consider the last case. Since $a^*_3 = a_4 b_0$, we have $o(a_4) \neq 2$. Thus $K_5 \times \phi D_{2n}$ is not connected, a contradiction.

Now suppose that $a_2 = a^{n/2}$. By $a^*_1 = a_2 b_0$, we have $o(b_0) \neq 2$. Also since $a^*_2 = b^{-1}_0 a_3$, we have $o(a_3) \neq 2$. Finally, since $a^*_3 = a_4 b_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. Thus $K_5 \times \phi D_{2n}$ is not connected, a contradiction.

□

Lemma 3.6. Suppose that the subgroup of Aut($\tilde{X}$) generated by $\rho, \sigma$ and $\tau$, say $L$, lifts. Then there is no connected regular covering of the complete graph $K_5$ whose fibre-preserving group is arc-transitive.

Proof. $\rho$ and $\sigma$ lift. With the same arguments as in Case I, we have a contradiction. Also $\rho$ and $\tau$ lift. With the same arguments as in Subcase II, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2} b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b.$$ 

From Table 1, it is easy to check that $\tilde{\sigma}$ can be extended to automorphisms of $D_{2n}$ whenever $n = 3$, a contradiction. □
**Proof of Theorem 1.1.** This follows from Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6.

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