TANGENT BUNDLE OF ORDER TWO 
AND BIHARMONICITY

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ABSTRACT. The problem studied in this paper is related to the biharmonicity of a section from a Riemannian manifold \((M, g)\) to its tangent bundle \(T^2M\) of order two equipped with the diagonal metric \(g^D\). We show that a section on a compact manifold is biharmonic if and only if it is harmonic. We also investigate the curvature of \((T^2M, g^D)\) and the biharmonicity of section of \(M\) as a map from \((M, g)\) to \((T^2M, g^D)\).

1. Introduction

Harmonic (resp., biharmonic) maps are critical points of energy (resp., bienergy) functional defined on the space of smooth maps between Riemannian manifolds introduced by Eells and Sampson [4] (resp., Jiang [6]). In this paper, we present some properties for biharmonic section between a Riemannian manifold and its second tangent bundle which generalize the results of Ishihara [5], Konderak [7], Oproiu [9] and Djaa-Ouakkas [3].

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Consider a smooth map \( \phi: (M^n, g) \to (N^n, h) \) between two Riemannian manifolds, then the energy functional is defined by

\[
E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g
\]

(or over any compact subset \( K \subset M \)).

A map is called harmonic if it is a critical point of the energy functional \( E \) (or \( E(K) \) for all compact subsets \( K \subset M \)). For any smooth variation \( \{\phi_t\}_{t \in I} \) of \( \phi \) with \( \phi_0 = \phi \) and \( V = \frac{d\phi_t}{dt} |_{t=0} \), we have

\[
\frac{d}{dt} E(\phi_t)|_{t=0} = -\frac{1}{2} \int_M h(\tau(\phi), V) dv_g,
\]

where

\[
\tau(\phi) = \text{tr}_g \nabla d\phi
\]

is the tension field of \( \phi \). Then we have the following theorem.

**Theorem 1.1.** A smooth map \( \phi: (M^m, g) \to (N^n, h) \) is harmonic if and only if

\[
\tau(\phi) = 0.
\]

If \((x^i)_{1 \leq i \leq m}\) and \((y^\alpha)_{1 \leq \alpha \leq n}\) denote local coordinates on \( M \) and \( N \), respectively, then equation (4) takes the form

\[
\tau(\phi)^\alpha = \left( \Delta \phi^\alpha + g^{ij} \Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \right) = 0,
\]

where \( \Delta \phi^\alpha = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial \phi^\alpha}{\partial x^j}) \) is the Laplace operator on \((M^m, g)\) and \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols on \( N \).
Definition 1.2. A map \( \phi: (M, g) \rightarrow (N, h) \) between Riemannian manifolds is called biharmonic if it is a critical point of bienergy functional

\[
E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv^g.
\]

The Euler-Lagrange equation attached to bienergy is given by vanishing of the bitension field

\[
\tau_2(\phi) = -J_\phi(\tau(\phi)) = -(\Delta^\phi \tau(\phi) + \text{tr}_g R^N(\tau(\phi), d\phi)d\phi),
\]

where \( J_\phi \) is the Jacobi operator defined by

\[
J_\phi: \Gamma(\phi^{-1}(TN)) \rightarrow \Gamma(\phi^{-1}(TN))
\]

\[
V \mapsto \Delta^\phi V + \text{tr}_g R^N(V, d\phi)d\phi.
\]

Theorem 1.3. A smooth map \( \phi: (M^m, g) \rightarrow (N^n, h) \) is biharmonic if and only if

\[
\tau_2(\phi) = 0.
\]

From Theorem 1.1 and formula (7), we have the following corollary.

Corollary 1.4. If \( \phi: (M^m, g) \rightarrow (N^n, h) \) is harmonic, then \( \phi \) is biharmonic.

(For more details see [6]).

2. Preliminary Notes

2.1. Horizontal and vertical lifts on \( TM \)

Let \( (M, g) \) be an \( n \)-dimensional Riemannian manifold and \( (TM, \pi, M) \) be its tangent bundle. A local chart \( (U, x^i)_{i=1...n} \) on \( M \) induces a local chart \( (\pi^{-1}(U), x^i, y^j)_{i,j=1,...,n} \) on \( TM \). Denote the Christoffel symbols of \( g \) by \( \Gamma^k_{ij} \) and the Levi-Civita connection of \( g \) by \( \nabla \).
We have two complementary distributions on $TM$, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$ defined by

$$\mathcal{V}(x,u) = \text{Ker}(d\pi(x,u))$$

$$= \left\{ a^i \frac{\partial}{\partial y^i}(x,u) ; \ a^i \in \mathbb{R} \right\},$$

$$\mathcal{H}(x,u) = \left\{ a^i \frac{\partial}{\partial x^i}(x,u) - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}(x,u) ; \ a^i \in \mathbb{R} \right\},$$

where $(x, u) \in TM$, such that $T(x,u)TM = \mathcal{H}(x,u) \oplus \mathcal{V}(x,u)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on $M$. The vertical and the horizontal lifts of $X$ are defined by

(10) $$X^V = X^i \frac{\partial}{\partial y^i}$$

(11) $$X^H = X^i \delta^i_{x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\}.$$

For consequences, we have:

1. $\left( \frac{\partial}{\partial x^i} \right)^H = \delta^i_{x^i}$ and $\left( \frac{\partial}{\partial x^i} \right)^V = \frac{\partial}{\partial y^i}$.

2. $\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right)_{i,j=1,...,n}$ is a local frame on $TM$.

3. If $u = u^i \frac{\partial}{\partial x^i} \in T_x M$, then $u^H = u^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\}$ and $u^V = u^i \frac{\partial}{\partial y^i}$.
**Definition 2.1.** Let \((M, g)\) be a Riemannian manifold and \(F : TM \to TM\) be a smooth bundle endomorphism of \(TM\). Then we define a vertical and horizontal vector fields \(VF, HF\) on \(TM\) by

\[
VF : TM \to TTM \quad (x, u) \mapsto (F(u))^V,
\]

\[
HF : TM \to TTM \quad (x, u) \mapsto (F(u))^H.
\]

Locally we have

\[
VF = y^i F^j_i \frac{\partial}{\partial y^j} = y^i \left( F(\frac{\partial}{\partial x^i}) \right)^V,
\]

\[
HF = y^i F^j_i \frac{\partial}{\partial x^j} - y^i y^k F^j_i \Gamma^s_{jk} \frac{\partial}{\partial y^s} = y^i \left( F(\frac{\partial}{\partial x^i}) \right)^H.
\]

**Proposition 2.2 ([1]).** Let \((M, g)\) be a Riemannian manifold and \(\nabla\) be the Levi-Civita connection of the tangent bundle \((TM, g^s)\) equipped with the Sasaki metric. If \(F\) is a tensor field of type \((1, 1)\) on \(M\), then

\[
(\nabla_X V F)(x, u) = (F(X))^V_{(x, u)},
\]

\[
(\nabla_X H F)(x, u) = (F(X))^H_{(x, u)} + \frac{1}{2} (R_x(u, X_x) F(u))^H,
\]

\[
(\nabla_X H V F)(x, u) = V(\nabla_X F)(x, u) + \frac{1}{2} (R_x(u, F_x(u)) X_x)^H,
\]

\[
(\nabla_X V H F)(x, u) = H(\nabla_X F)(x, u) - \frac{1}{2} (R_x(X_x, F_x(u)) u)^V.
\]
where \((x,u) \in TM\) and \(X \in \Gamma(TM)\).

### 2.2. Second Tangent Bundle

Let \(M\) be an \(n\)-dimensional smooth differentiable manifold and \((U_{\alpha}, \psi_{\alpha})_{\alpha \in I}\) a corresponding atlas. For each \(x \in M\), we define an equivalence relation on

\[ C_x = \{ \gamma: (-\varepsilon, \varepsilon) \to M; \gamma \text{ is smooth and } \gamma(0) = x, \varepsilon > 0 \} \]

by

\[ \gamma \approx_x h \iff \gamma'(0) = h'(0) \quad \text{and} \quad \gamma''(0) = h''(0), \]

where \(\gamma'\) and \(\gamma''\) denote the first and the second derivation of \(\gamma\), respectively,

\[ \gamma': (-\varepsilon, \varepsilon) \to TM; \quad t \mapsto [d\gamma(t)](1) \]

\[ \gamma'': (-\varepsilon, \varepsilon) \to T(TM); \quad t \mapsto [d\gamma'(t)](1). \]

**Definition 2.3.** We define the second tangent space of \(M\) at the point \(x\) to be the quotient \(T^2_xM = C_x / \approx_x\) and the second tangent bundle of \(M\) the union of all second tangent space, \(T^2M = \bigcup_{x \in M} T^2_xM\). We denote the equivalence class of \(\gamma\) by \(j^2_x\gamma\) with respect to \(\approx_x\), and by \(j^2\gamma\) an element of \(T^2M\).

In the general case, the structure of higher tangent bundle \(T^rM\) is considered in [8, Chapters 1–2] and [2].

**Proposition 2.4 ([3]).** Let \(M\) be an \(n\)-dimensional manifold, then \(TM\) is sub-bundle of \(T^2M\) and the map

\[ i: TM \to T^2M \]

\[ j^1_x f = j^2_x \tilde{f} \]
is an injective homomorphism of natural bundles (not of vector bundles), where
\[
\tilde{f}^i = \int_0^t f^i(s) ds - tf^i(0) + f^i(0) \quad i = 1 \ldots n.
\]

**Theorem 2.5.** Let \((M, g)\) be a Riemannian manifold and \(\nabla\) be the Levi-Civita connection. If \(TM \oplus TM\) denotes the Whitney sum, then
\[
S: T^2M \to TM \oplus TM
\]
\[
j^2\gamma(0) \mapsto (\dot{\gamma}(0), (\nabla_{\dot{\gamma}(0)}\dot{\gamma})(0))
\]
is a diffeomorphism of natural bundles. In the induced coordinate we have
\[
(x^i; y^i; z^i) \mapsto (x^i, y^i, z^i + y^j y^k \Gamma^i_{jk}).
\]

**Remark 2.6.** The diffeomorphism \(S\) determines a vector bundle structure on \(T^2M\) by
\[
\alpha.\Psi_1 + \beta.\Psi_2 = S^{-1}(\alpha S(\Psi_1) + \beta S(\Psi_2)),
\]
where \(\Psi_1, \Psi_2 \in T^2M\) and \(\alpha, \beta \in \mathbb{R}\), for which \(S\) is a linear isomorphism of vector bundles and \(i: TM \to T^2M\) is an injective linear homomorphism of vector bundles (for more details see [2]).

**Definition 2.7 ([3]).** Let \((M, g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism \(S\). For any section \(\sigma \in \Gamma(T^2M)\), we define two vector fields on \(M\) by
\[
X_\sigma = P_1 \circ S \circ \sigma,
\]
\[
Y_\sigma = P_2 \circ S \circ \sigma,
\]
where \(P_1\) and \(P_2\) denote the first and the second projections from \(TM \oplus TM\) on \(TM\).

From Remark 2.6 and Definition 2.7, we deduce the following.
**Proposition 2.8.** For all sections \( \sigma, \varpi \in \Gamma(T^2M) \) and \( \alpha \in \mathbb{R} \), we have
\[
X_{\alpha \sigma + \varpi} = \alpha X_\sigma + X_\varpi,
\]
\[
Y_{\alpha \sigma + \varpi} = \alpha Y_\sigma + Y_\varpi,
\]
where \( \alpha \sigma + \varpi = S^{-1}(\alpha S(\sigma) + S(\varpi)) \).

**Definition 2.9 ([3]).** Let \((M, g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism \(S\). We define a connection \(\hat{\nabla}\) on \(\Gamma(T^2M)\) by
\[
\hat{\nabla}: \Gamma(TM) \times \Gamma(T^2M) \to \Gamma(T^2M)
\]
\[(z, \sigma) \mapsto \hat{\nabla}_Z \sigma = S^{-1}(\nabla_Z X_\sigma, \nabla_Z Y_\sigma)\]
where \(\nabla\) is the Levi-Civita connection on \(M\).

**Proposition 2.10.** If \((U, x^i)\) is a chart on \(M\) and \((\sigma^i, \bar{\sigma}^i)\) are the components of section \(\sigma \in \Gamma(T^2M)\), then
\[
X_\sigma = \sigma^i \frac{\partial}{\partial x^i}
\]
(21)
\[
Y_\sigma = (\bar{\sigma}^k + \sigma^i \sigma^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k}.
\]
(22)

**Proposition 2.11.** Let \((M, g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two, then
\[
J: \Gamma(TM) \to \Gamma(T^2M)
\]
\[Z \mapsto S^{-1}(Z, 0)\]
(23)
is an injective homomorphism of vector bundles.
Locally if \((U; x^i)\) is a chart on \(M\) and \((U; x^i; y^i)\) and \((U; x^i; y^i; z^i)\) are the induced charts on \(TM\) and \(T^2M\). respectively, then we have

\[
J: (x^i, y^i) \mapsto (x^i, y^i, -y^j y^k \Gamma^i_{jk}).
\]

**Definition 2.12.** Let \((M, g)\) be a Riemannian manifold and \(X \in \Gamma(TM)\) be a vector field on \(M\). For \(\lambda = 0, 1, 2\), the \(\lambda\)-lift of \(X\) to \(T^2M\) is defined by

\[
\begin{align*}
X^0 &= S^{-1}_*(X^H, X^H) \\
X^1 &= S^{-1}_*(X^V, 0) \\
X^2 &= S^{-1}_*(0, X^V).
\end{align*}
\]

**Theorem 2.13** ([2]). Let \((M, g)\) be a Riemannian manifold and \(R\) its tensor curvature, then for all vector fields \(X, Y \in \Gamma(TM)\) and \(p \in T^2M\), we have:

1. \([X^0, Y^0]_p = [X, Y]_p^0 - (R(X, Y)u)^1 - (R(X, Y)w)^2\),
2. \([X^0, Y^i] = (\nabla_X Y)^i\),
3. \([X^i, Y^j] = 0\),

where \((u, w) = S(p)\) and \(i, j = 1, 2\).

**Definition 2.14.** Let \((M, g)\) be a Riemannian manifold. For any section \(\sigma \in \Gamma(T^2M)\), we define the vertical lift of \(\sigma\) to \(T^2M\) by

\[
\sigma^V = S^{-1}_*(X^V, Y^V) \in \Gamma(T(T^2M)).
\]
Remark 2.15. From Definition 2.7 and the formulae (14), (23) and (28), we obtain

\[\sigma^V = X^1_\sigma + Y^2_\sigma,\]

\[(\tilde{\nabla}_Z \sigma)^V = (\nabla_Z X_\sigma)^1 + (\nabla_Z Y_\sigma)^2,\]

\[Z^1 = J(Z)^V,\]

\[Z^2 = i(Z)^V\]

for all \(\sigma \in \Gamma(T^2M)\) and \(Z \in \Gamma(TM)\).

2.3. Diagonal metric

Theorem 2.16 ([3]). Let \((M, g)\) be a Riemannian manifold and \(TM\) its tangent bundle equipped with the Sasakian metric \(g^s\), then

\[g^D = S_{-1}^{-1}(\tilde{g}, \tilde{g})\]

is the only metric that satisfies the following formulae

(29)

\[g^D(X^i, Y^j) = \delta_{ij} \cdot g(X, Y) \circ \pi_2\]

for all vector fields \(X, Y \in \Gamma(TM)\) and \(i, j = 0, \ldots, 2\), where \(\tilde{g}\) is the metric defined by

\[\tilde{g}(X^H, Y^H) = \frac{1}{2} g^s(X^H, Y^H),\]

\[\tilde{g}(X^H, Y^V) = g^s(X^H, Y^V),\]

\[\tilde{g}(X^V, Y^V) = g^s(X^V, Y^V).\]

\(g^D\) is called the diagonal lift of \(g\) to \(T^2M\).

Proposition 2.17. Let \((M, g)\) be a Riemannian manifold and \(\tilde{\nabla}\) be the Levi-Civita connection of the tangent bundle of order two equipped with the diagonal metric \(g^D\). Then:
1. \( (\tilde{\nabla}_X^0 Y^0)_p = (\nabla_X Y)^0 - \frac{1}{2}(R(X, Y)u)^1 - \frac{1}{2}(R(X, Y)w)^2, \)
2. \( (\tilde{\nabla}_X^1 Y^1)_p = (\nabla_X Y)^1 + \frac{1}{2}(R(u, Y)X)^0, \)
3. \( (\tilde{\nabla}_X^2 Y^2)_p = (\nabla_X Y)^2 + \frac{1}{2}(R(w, Y)X)^0, \)
4. \( (\tilde{\nabla}_X^1 Y^0)_p = \frac{1}{2}(R_x(u, X)Y)^0, \)
5. \( (\tilde{\nabla}_X^2 Y^0)_p = \frac{1}{2}(R_x(w, X)Y)^0, \)
6. \( (\tilde{\nabla}_X Y^j)_p = 0 \)

for all vector fields \( X, Y \in \Gamma(TM) \) and \( p \in \Gamma(T^2M) \), where \( i, j = 1, 2 \) and \((u, w) = S(p)\).

3. Biharmonicity Of section

3.1. The Curvature Tensor

**Definition 3.1.** Let \((M, g)\) be a Riemannian manifold and \( F : TM \to TM \) be a smooth bundle endomorphism of \( TM \). For \( \lambda = 0, 1, 2 \), the \( \lambda \)-lift of \( F \) to \( T^2M \) is defined by

\[
F^0 = S_*^{-1}(HF, HF),
F^1 = S_*^{-1}(VF, 0),
F^2 = S_*^{-1}(0, VF).
\]

From Proposition 2.17, we obtain the following lemma.
Lemma 3.2. Let \( F : TM \to TM \) be a smooth bundle endomorphism of \( TM \), then we have

\[
\begin{align*}
(\tilde{\nabla}_X F^0)_p &= F(X)_p^0 + \frac{1}{2}(R(u, X)F(u))_p^0, \\
(\tilde{\nabla}_X F^0)_p &= F(X)_p^0 + \frac{1}{2}(R(w, X)F(w))_p^0, \\
(\tilde{\nabla}_X F^i)_p &= F(X)_p^i, \quad i, j = 1, 2, \\
(\tilde{\nabla}_X F^1)_p &= (\nabla_X F)_p^1 + \frac{1}{2}(R(u, F_x(u))X)_p^1, \\
(\tilde{\nabla}_X F^2)_p &= (\nabla_X F)_p^2 + \frac{1}{2}(R(w, F_x(w))X)_p^2, \\
(\tilde{\nabla}_X F^0)_p &= (\nabla_X F)_p^0 - \frac{1}{2}(R(X_x, F_x(u))u)_p^1 - \frac{1}{2}(R(X_x, F_x(w))w)_p^2
\end{align*}
\]

for any \( p \in T^2 M, \quad i, j = 1, 2 \) and \( X \in \Gamma(TM) \).

Using the formula of curvature and Lemma 3.2, we have the following.

Proposition 3.3. Let \( R \) be a curvature tensor of \((M, g)\), and \( \tilde{R} \) be curvature tensor of \((T^2 M, g^D)\) equipped with the diagonal lift of \( g \). Then we have the following

1. \[
\tilde{R}(X^0, Y^0)Z^0 &= \left( R(X, Y)Z + \frac{1}{4}R(u, R(Z, Y)u)X + \frac{1}{4}R(w, R(Z, Y)w)X \right)_p^0 \\
&\quad + \left( \frac{1}{4}R(u, R(X, Z)u)Y + \frac{1}{4}R(w, R(X, Z)w)Y \right)_p^0 \\
&\quad + \left( \frac{1}{2}R(u, R(X, Y)u)Z + \frac{1}{2}R(w, R(X, Y)w)Z \right)_p^0 \\
&\quad + \frac{1}{2} \left( \nabla_Z R)(X, Y)u \right)_p^1 + \frac{1}{2} \left( \nabla_Z R)(X, Y)w \right)_p^2,
\]
2. \( \tilde{R}(X^0, Y^0)Z^i = \left( R(X, Y)Z + \frac{1}{4}R(R(u, Z)Y, X)u + \frac{1}{4}R(R(w, Z)Y, X)w \\
- \frac{1}{4}R(R(u, Z)X, Y)u - \frac{1}{4}R(R(w, Z)X, Y)w \right)^i \\
+ \frac{1}{2}\left( (\nabla_X R)(u, Z)Y + (\nabla_X R)(w, Z)Y - (\nabla_Y R)(u, Z)X \\
- (\nabla_Y R)(w, Z)X \right)^0, \)

3. \( \tilde{R}(X^1, Y^1)Z^0 = \left( R(X, Y)Z + \frac{1}{4}R(u, X)R(u, Y)Z + \frac{1}{4}R(w, X)R(w, Y)Z \\
- \frac{1}{4}R(u, Y)R(u, X)Z - \frac{1}{4}R(w, Y)R(w, X)Z \right)^0, \)

4. \( \tilde{R}(X^i, Y^2)Z^0 = \left( R(X, Y)Z + \frac{1}{4}R(u, X)R(u, Y)Z + \frac{1}{4}R(w, X)R(w, Y)Z \\
- \frac{1}{4}R(u, Y)R(u, X)Z - \frac{1}{4}R(w, Y)R(w, X)Z \right)^0, \)

5. \( \tilde{R}(X^i, Y^0)Z^0 = -\left( \frac{1}{4}R(u, Y)Z, X)u + \frac{1}{4}R(w, Y)Z, X)w + \frac{1}{2}R(X, Z)Y \right)^i \\
+ \frac{1}{2}\left( (\nabla_X R)(u, Y)Z + (\nabla_X R)(w, Y)Z \right)^0, \)

6. \( \tilde{R}(X^i, Y^0)Z^j = \left( \frac{1}{2}R(Y, Z)X + \frac{1}{4}R(u, Y)R(u, X)Z + \frac{1}{4}R(w, Y)R(w, X)Z \right)^0 \)

7. \( \tilde{R}(X^1, Y^1)Z^i = \tilde{R}(X^2, Y^2)Z^i = 0 \)

for any \( \xi = (p, u, w) \in T^2M, \quad i, j = 1, 2 \quad \text{and} \quad X, Y, Z \in \Gamma(TM). \)
Lemma 3.4. Let \((M, g)\) be a Riemannian manifold and \(T^2M\) be the tangent bundle equipped with the diagonal metric. If \(Z \in \Gamma(TM)\) and \(\sigma \in \Gamma(T^2M)\), then
\[
d_x \sigma(Z_x) = Z_p^0 + (\nabla_{Z} \sigma)_p^V,
\]
where \(p = \sigma(x)\).

Proposition 3.5 ([3]). Let \((M, g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two equipped with the diagonal metric. Then the tension field associated with \(\sigma \in \Gamma(T^2M)\) is
\[
\tau(\sigma) = (\text{trace}_g \nabla^2 X_{\sigma})^1 + (\text{trace}_g \nabla^2 Y_{\sigma})^2 \\
+ \left( \text{trace}_g (R(X_{\sigma}, \nabla_X X_{\sigma}) + R(Y_{\sigma}, \nabla_Y Y_{\sigma})\right)^0 \\
= (\text{trace}_g \hat{\nabla}^2 \sigma)^V + \left( \text{trace}_g (R(X_{\sigma}, \nabla_X X_{\sigma}) + R(Y_{\sigma}, \nabla_Y Y_{\sigma})\right)^0,
\]
where \(-\text{trace}_g \nabla^2\) (resp., \(-\text{trace}_g \hat{\nabla}^2\)) denotes the Laplacian attached to \(\nabla\) (resp., \(\hat{\nabla}\)).

4. Biharmonicity of Section \(\sigma : (M, g) \to (T^2M, g^D)\)

For a section \(\sigma \in \Gamma(T^2M)\), we denote
\[
\tau^0(\sigma) = \tau^0(X_{\sigma}) + \tau^0(Y_{\sigma}),
\]
\[
\tau^V(\sigma) = \tau^1(X_{\sigma}) + \tau^2(Y_{\sigma}),
\]
\[
\overline{\tau}^0(\sigma) = \left( \tau^0(X_{\sigma}) + \tau^0(Y_{\sigma}) \right)^0,
\]
\[
\overline{\tau}^V(\sigma) = \left( \tau^1(X_{\sigma}) \right)^1 + \left( \tau^2(Y_{\sigma}) \right)^2.
\]
where
\[ \tau^0(X_\sigma) = \text{trace}_g(R(X_\sigma, \nabla X_\sigma)\ast), \]
\[ \tau^0(Y_\sigma) = \text{trace}_g(R(Y_\sigma, \nabla Y_\sigma)\ast), \]
\[ \tau^1(X_\sigma) = \text{trace}_g \nabla^2 X_\sigma, \]
\[ \tau^2(Y_\sigma) = \text{trace}_g \nabla^2 Y_\sigma. \]

From these notations, we have
\[ (36) \quad \tau(\sigma) = \tau^V + \tau^0. \]

**Theorem 4.1.** Let \((M, g)\) be a Riemannian compact manifold and \((T^2 M, g^D)\) be its tangent bundle of order two equipped with the diagonal metric and a vector bundle structure via the diffeomorphism \(S\) between \(T^2\) and \(TM \oplus TM\). Then \(\sigma: M \to T^2 M\) is a biharmonic section if and only if \(\sigma\) is harmonic.

**Proof.** First, if \(\sigma\) is harmonic, then from Corollary 1.4, we deduce that \(\sigma\) is biharmonic.

Conversely, assuming that \(\sigma\) is biharmonic. Let \(\sigma_t\) be a compactly supported variation of \(\sigma\) defined by \(\sigma_t = (1 + t)\sigma\). Using Proposition 2.8, we have
\[ (37) \quad X_{\sigma_t} = (1 + t)X_\sigma \quad \text{and} \quad Y_{\sigma_t} = (1 + t)Y_\sigma. \]

Substituting (37) in (32) to (35), we obtain
\[ (38) \quad \tau^0(\sigma_t) = (1 + t)^2 \tau^0(\sigma) \quad \text{and} \quad \tau^V(\sigma_t) = (1 + t)\tau^V(\sigma) \]
\[ (39) \quad \tau^0(\sigma_t) = (1 + t)^2 \tau^0(\sigma) \quad \text{and} \quad \tau^V(\sigma_t) = (1 + t)\tau^V(\sigma). \]

Then
\[ E_2(\sigma_t) = \frac{1}{2} \int |\tau(\sigma_t)|^2_{g_D} v_g = \frac{1}{2} \int |\bar{\tau}^0(\sigma_t)|^2_{g_D} v_g + \frac{1}{2} \int |\bar{\tau}^V(\sigma_t)|^2_{g_D} v_g \]
\[ = \frac{(1 + t)^4}{2} \int |\bar{\tau}^0(\sigma)|^2_{g_D} v_g + \frac{(1 + t)^2}{2} \int |\bar{\tau}^V(\sigma)|^2_{g_D} v_g. \]

Since the section \( \sigma \) is biharmonic, then for the variation \( \sigma_t \), we have

\[ 0 = \frac{d}{dt} E_2(\sigma_t)|_{t=0} = 2 \int |\bar{\tau}^0(\sigma)|^2_{g_D} v_g + \int |\bar{\tau}^V(\sigma)|^2_{g_D} v_g. \]

Hence

\[ \bar{\tau}^0(\sigma) = 0 \quad \text{and} \quad \bar{\tau}^V(\sigma) = 0, \quad \text{then} \quad \tau(\sigma) = 0. \]

\[ \square \]

In the case where \( M \) is not compact, the characterization of biharmonic sections requires the following two lemmas.

**Lemma 4.2.** Let \((M, g)\) be a Riemannian manifold and \((T^2M, g^D)\) be its tangent bundle of order two equipped with the diagonal metric. If \( \sigma \in \Gamma(T^2M) \) is a smooth section, then the Jacobi tensor \( J_\sigma(\tau^V(\sigma)) \) is given by

\[ J_\sigma(\bar{\tau}^V(\sigma)) = \left\{ \begin{array}{l} \text{trace}_g \nabla^2(\tau^V(\sigma)) \\ + \left\{ \text{trace}_g \left( R(u, \nabla_* \tau^1(X_\sigma)) * + R(w, \nabla_* \tau^2(Y_\sigma)) * + R(\tau^V(\sigma), \nabla_* \sigma) * \\ + \frac{1}{2} R(u, \tau^1(X_\sigma)) R(u, \nabla_* X_\sigma) * + \frac{1}{2} R(w, \tau^2(Y_\sigma)) R(w, \nabla_* Y_\sigma) * \right\} \end{array} \right\}^0. \]
Proof. Let $p \in T^2M$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on $M$ such that $(\nabla e_i e_i)_x = 0$. If we denote $F_i(x, u, w) = \frac{1}{2} R(u, \tau^1(X_\sigma))e_i + \frac{1}{2} R(w, \tau^2(Y_\sigma))e_i$, then we have

$$\tilde{\nabla}_e^\sigma \tilde{\nabla}^V(\sigma)_p = (\tilde{\nabla}_e^\sigma + (\nabla_e X_\sigma)^1 + (\nabla e Y_\sigma)^2)(\tau^1(X_\sigma))^1 + (\tau^2(Y_\sigma))^2)_p$$

$$= (\nabla e_i(\tau^V(\sigma))_p + \frac{1}{2}(R(u, \tau^1(X_\sigma))e_i + R(w, \tau^2(Y_\sigma))e_i)^0 = (\nabla e_i(\tau^V(\sigma))_p + (F_i(x, u, w))^0,$$

hence

$$(\text{trace}_g \tilde{\nabla}^2 \tilde{\nabla}^V(\sigma))_p = \sum_{i=1}^m \left\{ \tilde{\nabla}_e^\sigma \tilde{\nabla}_e^\sigma (\tilde{\nabla}^V(\sigma)) \right\}_p = \sum_{i=1}^m \left\{ \tilde{\nabla}_e^\sigma_0 + (\nabla e X_\sigma)^1 + (\nabla e Y_\sigma)^2)((\nabla e_i(\tau^V(\sigma))^1 + (F_i)^0) \right\}_p$$

$$= \sum_{i=1}^m \left\{ \tilde{\nabla}_e^\sigma(\nabla e_i^1(X_\sigma))^1 + \tilde{\nabla}_e^\sigma(\nabla e_i^1(Y_\sigma))^2 + \tilde{\nabla}_e^0 F_i^0 + \tilde{\nabla}(\nabla e_i X_\sigma)^1 F_i^0 + \tilde{\nabla}(\nabla e_i Y_\sigma)^2 F_i^0 \right\}_p.$$ 

Using Proposition 2.17, we obtain

$$(\text{trace}_g \tilde{\nabla}^2 \tilde{\nabla}^V(\sigma))_p = \sum_{i=1}^m \left\{ (\nabla e_i \nabla e_i^1(X_\sigma)) - \frac{1}{4} R(e_i, R(u, \tau^1(X_\sigma))e_i)u \right\}_p$$

$$+ \sum_{i=1}^m \left\{ (\nabla e_i \nabla e_i^1(Y_\sigma)) - \frac{1}{4} R(e_i, R(w, \tau^2(Y_\sigma))e_i)w \right\}_p + \sum_{i=1}^m \left\{ \frac{1}{2} R(u, \nabla e_i^1(X_\sigma))e_i \right.$$

$$+ \frac{1}{2} R(w, \nabla e_i^1(Y_\sigma))e_i + \frac{1}{2} (\nabla e_i R(u, \tau^1(X_\sigma))e_i) + \frac{1}{2} (\nabla e_i R(w, \tau^2(Y_\sigma))e_i)$$

$$+ \frac{1}{2} R(\tau^1(X_\sigma), \nabla e_i u)e_i + \frac{1}{2} R(\tau^2(Y_\sigma), \nabla e_i w)e_i + \frac{1}{4} R(u, \nabla e_i X_\sigma)R(u, \tau^1(X_\sigma))e_i$$

$$+ \frac{1}{4} R(w, \nabla e_i X_\sigma)R(w, \tau^2(Y_\sigma))e_i + \frac{1}{2} R(\nabla e_i X_\sigma, \tau^1(X_\sigma))e_i + \frac{1}{2} R(\nabla e_i X_\sigma, \tau^2(Y_\sigma))e_i \right\}_p.$$
From proposition 3.3, we have

\[
\text{trace}_g(\tilde{R}(\nabla^V(\sigma), d\sigma)d\sigma) = \sum_{i=1}^{m} \left\{ \tilde{R}(\nabla^1(\tau_i(\tau_1(\sigma)))^1, e_i^0e_i^0 + \tilde{R}(\nabla^1(\tau_1(\sigma)))^1, (\nabla_{e_i}X_\sigma)^1)e_i^0 \\
+ \tilde{R}(\nabla^1(\tau_1(\sigma)))^1, (\nabla_{e_i}Y_\sigma)^2)e_i^0 + \tilde{R}(\nabla^1(\tau_1(\sigma)))^1, (\nabla_{e_i}X_\sigma)^1 + \tilde{R}(\nabla^1(\tau_1(\sigma)))^1, (\nabla_{e_i}Y_\sigma)^2 \\
+ \tilde{R}(\nabla^2(Y_\sigma))^2, e_i^0e_i^0 + \tilde{R}(\nabla^2(Y_\sigma))^2, (\nabla_{e_i}X_\sigma)^1)e_i^0 + \tilde{R}(\nabla^2(Y_\sigma))^2, (\nabla_{e_i}Y_\sigma)^2)e_i^0 \\
+ \tilde{R}(\nabla^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i}X_\sigma)^1 + \tilde{R}(\nabla^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i}Y_\sigma)^2 \right\}.
\]

By calculating at point \( p \in T^2M \), we obtain

\[
\text{trace}_g(\tilde{R}(\nabla^V(\sigma), d\sigma)d\sigma)_p = \sum_{i=1}^{m} \left\{ -\frac{1}{4}R(u, \tau^1(X_\sigma)e_i, e_i)u \right\}^1 - \left\{ \frac{1}{4}R(R(u, \tau^2(Y_\sigma)e_i), e_i)w \right\}^2 \\
+ \sum_{i=1}^{m} \left\{ R(\tau^1(X_\sigma), \nabla_{e_i}X_\sigma)e_i + R(\tau^2(Y_\sigma), \nabla_{e_i}Y_\sigma)e_i \\
+ \frac{1}{4}R(u, \tau^1(X_\sigma))R(u, \nabla_{e_i}X_\sigma)e_i - \frac{1}{4}R(u, \nabla_{e_i}Y_\sigma)R(w, \tau^2(Y_\sigma))e_i \\
+ \frac{1}{4}R(u, \tau^2(Y_\sigma))R(u, \nabla_{e_i}Y_\sigma)e_i - \frac{1}{4}R(u, \nabla_{e_i}X_\sigma)R(u, \tau^1(X_\sigma))e_i \\
+ \frac{1}{2}R(\tau^1(X_\sigma), \nabla_{e_i}X_\sigma)e_i + \frac{1}{2}R(\tau^2(Y_\sigma), \nabla_{e_i}Y_\sigma)e_i \\
+ \frac{1}{4}R(u, \tau^1(X_\sigma))R(u, \nabla_{e_i}X_\sigma)e_i + \frac{1}{4}R(u, \tau^2(Y_\sigma))R(w, \nabla_{e_i}Y_\sigma)e_i \\
- \frac{1}{2}(\nabla_{e_i}R(u, \tau^1(X_\sigma)e_i - \frac{1}{2}(\nabla_{e_i}R(u, \tau^2(Y_\sigma)e_i))^0. \right\}
\]
Considering the formula (8), we deduce

$$J_\sigma(\tau^V(\sigma)) = \left\{ \begin{array}{l} \text{trace}_g \nabla^2(\tau^V(\sigma)) \end{array} \right\}^V + \left\{ \begin{array}{l} \text{trace}_g(R(u, \nabla_* \tau^1(X_\sigma)^*) \\
+ R(w, \nabla_* \tau^2(Y_\sigma))^* + R(\tau^V(\sigma), \nabla_* \sigma)^* \\
+ \frac{1}{2} R(u, \tau^1(X_\sigma)) R(u, \nabla_* X_\sigma)^* + \frac{1}{2} R(w, \tau^2(Y_\sigma)) R(w, \nabla_* Y_\sigma)^*) \end{array} \right\}^0. \quad \square$$

**Lemma 4.3.** Let $(M, g)$ be a Riemannian manifold and $(T^2M, g^D)$ be its tangent bundle of order two equipped with the diagonal metric. If $\sigma \in \Gamma(T^2M)$ is a smooth section, then the Jacobi tensor $J_\sigma(\tau^0(\sigma))$ is given by

$$J_\sigma(\tau^0(\sigma))_p = \text{trace}_g \left\{ \begin{array}{l} 2R(\tau^0(X_\sigma), *) \nabla_* X_\sigma - R(*, \nabla_* \tau^0(X_\sigma)) u + \frac{1}{2} R(u, \nabla_* X_\sigma)^*, \tau^0(X_\sigma) \end{array} \right\}^1 + \text{trace}_g \left\{ \begin{array}{l} 2R(\tau^0(Y_\sigma), *) \nabla_* Y_\sigma - R(*, \nabla_* \tau^0(Y_\sigma)) w + \frac{1}{2} R(w, \nabla_* Y_\sigma)^*, \tau^0(Y_\sigma) \end{array} \right\}^2 + \text{trace}_g \left\{ \begin{array}{l} \nabla_* \nabla_* \tau^0(\sigma) + R(u, \nabla_* X_\sigma) \nabla_* \tau^0(X_\sigma) + R(w, \nabla_* Y_\sigma) \nabla_* \tau^0(Y_\sigma) \\
+ \frac{1}{2} R(u, \nabla_* \nabla_* X_\sigma) \tau^0(X_\sigma) + \frac{1}{2} R(w, \nabla_* \nabla_* Y_\sigma) \tau^0(Y_\sigma) + R(u, R(\tau^0(X_\sigma), *)) u^* \\
+ R(w, R(\tau^0(Y_\sigma), *) w)^* + R(\tau^0(\sigma), *) + (\nabla_{\tau^0(X_\sigma)} R)(u, \nabla_* X_\sigma)^* \\
+ (\nabla_{\tau^0(Y_\sigma)} R)(w, \nabla_* Y_\sigma)^* \end{array} \right\}^0_p$$

for all $p = (x, u, w) \in T^2M$. 
Proof. Let \( p = (x, u, w) \in T^2M \) and \( \{e_i\}_{i=1}^m \) be a local orthonormal frame on \( M \) such that \( (\nabla e_i e_i)_x = 0 \), denoted by

\[
F_i = F_iX + F_iY = \frac{1}{2} R(e_i, \tau^0(X_\sigma)) + \frac{1}{2} R(e_i, \tau^0(Y_\sigma))
\]

(40)

\[
G = G_X + G_Y = \frac{1}{2} R(\ast, \nabla_\ast X_\sigma) \tau^0(X_\sigma) + \frac{1}{2} R(\ast, \nabla_\ast Y_\sigma) \tau^0(Y_\sigma).
\]

(41)

First, using Lemma 3.4 and Proposition 2.17, we calculate

\[
\text{trace}_g \tilde{\nabla}^2(\tau^0(\sigma))_p = \sum_{i=1}^m \left\{ \tilde{\nabla}_e_i \tilde{\nabla}_e_i (\tau^0(\sigma))^0 \right\} = \sum_{i=1}^m \left\{ (\tilde{\nabla}_e_i - (\nabla e_i X_\sigma) + (\nabla e_i Y_\sigma)^2 (\nabla e_i \tau(\sigma))^0 - F_i^1 X - F_i^2 Y + G_i^0) \right\}_p.
\]

(42)

From Proposition 2.17, we have

\[
\text{trace}_g \tilde{\nabla}^2(\tau^0(\sigma))_p = \sum_{i=1}^m \left\{ (\nabla e_i \nabla e_i \tau^0(\sigma))^0 + \left( \frac{1}{2} R(u, \nabla e_i X_\sigma) \nabla e_i \tau^0(X_\sigma) \right) \right\}
\]

\[
+ \frac{1}{2} R(w, \nabla e_i Y_\sigma) \nabla e_i \tau^0(Y_\sigma) - (\nabla e_i F_i X)^1 - (\nabla e_i F_i Y)^2 - \left( \frac{1}{2} R(e_i, \nabla e_i \tau^0(X_\sigma)) u \right)^1
\]

\[
- (F_i X (\nabla e_i)_X \sigma)^1 - (F_i Y (\nabla e_i)_Y \sigma)^2 + (\nabla e_i G)^0 - \left( \frac{1}{2} R(e_i, G_X(u)) u \right)^1 - \left( \frac{1}{2} R(e_i, G_Y(u)) w \right)^2
\]

\[
+ (G_X(\nabla e_i X_\sigma))^0 + (G_Y(\nabla e_i Y_\sigma))^0 + \left( \frac{1}{2} R(u, \nabla e_i X_\sigma) G_X(u) \right)^0 + \left( \frac{1}{2} R(w, \nabla e_i Y_\sigma) G_Y(u) \right)^0 \right\}_p.
\]
Substituting (40) and (41) in (42), we arrive at

\[
\text{trace}_g \tilde{\nabla}^2(\tau^0(\sigma))_p = \sum_{i=1}^{m} \left\{ (\nabla e_i \nabla e_i \tau^0(\sigma) + R(u, \nabla e_i X_{\sigma}) \nabla e_i \tau^0(X_{\sigma}) + R(w, \nabla e_i Y_{\sigma} \nabla e_i Y_{\sigma}) \tau^0(Y_{\sigma}) + \frac{1}{2} R(u, \nabla e_i \nabla e_i X_{\sigma}) \tau^0(X_{\sigma})
\right.
\]

\[
+ \frac{1}{2} R(w, \nabla e_i \nabla e_i Y_{\sigma}) \tau^0(Y_{\sigma}) + \frac{1}{2} (\nabla e_i R)(u, \nabla e_i X_{\sigma}) \tau^0(X_{\sigma})
\]

\[
+ \frac{1}{2} (\nabla e_i R)(w, \nabla e_i Y_{\sigma}) \tau^0(Y_{\sigma}) + \frac{1}{4} R(u, \nabla e_i X_{\sigma}) R(u, \nabla e_i X_{\sigma}) \tau^0(X_{\sigma})
\]

\[
+ \frac{1}{4} R(w, \nabla e_i Y_{\sigma}) R(w, \nabla e_i Y_{\sigma}) \tau^0(Y_{\sigma}) - \frac{1}{4} R(u, R(e_i, \tau^0(X_{\sigma})) u) e_i - \sum_{i=1}^{m} \left\{ \frac{1}{2} R(e_i, \nabla e_i X_{\sigma}) \nabla e_i X_{\sigma}
\right.
\]

\[
+ R(e_i, \nabla e_i \tau^0(X_{\sigma})) u + \frac{1}{2} (\nabla e_i R)(e_i, \tau^0(X_{\sigma})) u
\]

\[
+ \frac{1}{4} R(e_i, R(u, \nabla e_i X_{\sigma}) \tau^0(X_{\sigma})) u \right\}_p
\]

\[
- \sum_{i=1}^{m} \left\{ \frac{1}{2} R(e_i, \tau^0(Y_{\sigma})) \nabla e_i Y_{\sigma} + R(e_i, \nabla e_i \tau^0(Y_{\sigma})) w
\right.
\]

\[
+ \frac{1}{2} (\nabla e_i R)(e_i, \tau^0(Y_{\sigma})) w + \frac{1}{4} R(e_i, R(w, \nabla e_i Y_{\sigma}) \tau^0(Y_{\sigma})) w \right\}_p.
\]
On the other hand, we have

$$\text{trace}_g \left\{ \widetilde{R}(\tau^0(\sigma), d\sigma) d\sigma \right\}_p$$

$$= \sum_{i=1}^{m} \left\{ R(\tau^0(\sigma), e_i) e_i + \frac{3}{4} R(u, R(\tau^0(X_{\sigma}), e_i) u) e_i ight. \\
+ \frac{3}{4} R(w, R(\tau^0(Y_{\sigma}), e_i) w) e_i + \nabla_{\tau^0}(X_{\sigma} R)(u, \nabla_{e_i} X_{\sigma}) e_i \\
+ \nabla_{\tau^0}(Y_{\sigma} R)(w, \nabla_{e_i} Y_{\sigma}) e_i - \frac{1}{2} (\nabla_{e_i} R)(u, \nabla_{e_i} X_{\sigma}) \tau^0(X_{\sigma}) \\
- \frac{1}{2} (\nabla_{e_i} R)(w, \nabla_{e_i} Y_{\sigma}) \tau^0(Y_{\sigma}) - \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) R(u, \nabla_{e_i} X_{\sigma}) \tau^0(X_{\sigma}) \\
- \frac{1}{4} R(w, \nabla_{e_i} Y_{\sigma}) R(w, \nabla_{e_i} Y_{\sigma}) \tau^0(Y_{\sigma}) \right\}^0 \\
+ \sum_{i=1}^{m} \left\{ \frac{1}{2} (\nabla_{e_i} R) \tau^0(X_{\sigma}, e_i) u + \frac{1}{2} R(R(u, \nabla_{e_i} X_{\sigma}) e_i, \tau^0(X_{\sigma})) u ight. \\
+ \frac{3}{2} R(\tau^0(X_{\sigma}), e_i) \nabla_{e_i} X_{\sigma} - \frac{1}{4} R(R(u, \nabla_{e_i} X_{\sigma}) \tau^0(X_{\sigma}), e_i) u \right\}^1 \\
+ \sum_{i=1}^{m} \left\{ \frac{1}{2} (\nabla_{e_i} R) \tau^0(Y_{\sigma}, e_i) w + \frac{1}{2} R(R(w, \nabla_{e_i} Y_{\sigma}) e_i, \tau^0(Y_{\sigma})) w ight. \\
+ \frac{3}{2} R(\tau^0(Y_{\sigma}), e_i) \nabla_{e_i} Y_{\sigma} - \frac{1}{4} R(R(w, \nabla_{e_i} Y_{\sigma}) \tau^0(Y_{\sigma}), e_i) w \right\}^2.$$  \hspace{1cm} (44)

By summing (43) and (44), the proof of Lemma 4.3 is completed. \hfill \Box
From Lemma 4.2 and 4.3, we deduce the following theorems

**Theorem 4.4.** Let \((M, g)\) be a Riemannian manifold and \(\left(T^2 M, g^D\right)\) be its tangent bundle of order two equipped with the diagonal metric. If \(\sigma: M \to T^2 M\) is a smooth section, then the bitension field of \(\sigma\) is given by

\[
\tau_2(\sigma)_p = \text{trace}_g \left\{ \nabla^2 \tau^1(X_\sigma) + 2R(\tau^0(X_\sigma), *)\nabla_* X_\sigma - R(\ast, \nabla_* \tau^0(X_\sigma)) u \\
+ \frac{1}{2} R(R(u, \nabla_* \ast, \tau^0(X_\sigma))) u \right\}
\]

\[
+ \text{trace}_g \left\{ \nabla^2 \tau^2(Y_\sigma) + 2R(\tau^0(Y_\sigma), *)\nabla_* Y_\sigma \\
- R(\ast, \nabla_* \tau^0(Y_\sigma)) w + \frac{1}{2} R(R(w, \nabla_* \ast, \tau^0(Y_\sigma))) w \right\}
\]

\[
+ \text{trace}_g \left\{ R(u, \nabla_* \tau^1(X_\sigma)) \ast + R(w, \nabla_* \tau^2(Y_\sigma)) \ast + R(\tau^1(X_\sigma), \nabla_* X_\sigma) \ast \\
+ R(\tau^2(Y_\sigma), \nabla_* Y_\sigma) \ast + \frac{1}{2} R(u, \tau^1(X_\sigma)) R(u, \nabla_* X_\sigma) \ast \\
+ \frac{1}{2} R(w, \tau^2(Y_\sigma)) R(w, \nabla_* Y_\sigma) \ast + \nabla_* \nabla_* \tau^0(\sigma) + R(u, \nabla_* X_\sigma) \nabla_* \tau^0(X_\sigma) \\
+ R(w, \nabla_* Y_\sigma) \nabla_* \tau^0(Y_\sigma) + R(\tau^0(\sigma), \ast) \ast + \frac{1}{2} R(u, \nabla_* \nabla_* X_\sigma) \tau^0(X_\sigma) \\
+ \frac{1}{2} R(w, \nabla_* \nabla_* Y_\sigma) \tau^0(Y_\sigma) + R(u, R(\tau^0(X_\sigma), \ast) u) \ast \\
+ R(w, R(\tau^0(Y_\sigma), \ast) w) \ast + (\nabla_{\tau^0(X_\sigma)} R)(u, \nabla_* X_\sigma) \ast \\
+ (\nabla_{\tau^0(Y_\sigma)} R)(w, \nabla_* Y_\sigma) \ast \right\}_p
\]
for all \( p \in T^2M \).

**Theorem 4.5.** Let \((M, g)\) be a Riemannian manifold and \((T^2M, g^D)\) be its tangent bundle of order two equipped with the diagonal metric. A section \( \sigma: M \to T^2M \) is biharmonic if and only if the following conditions are verified:

1) \[ 0 = \text{trace}_g \left\{ \nabla^2 \tau^1(X_\sigma) + 2R(\tau^0(X_\sigma), *)\nabla_\sigma X_\sigma - R(*, \nabla_\sigma \tau^0(X_\sigma))u \\
+ \frac{1}{2}R(u, \nabla_\sigma *)_*, \tau^0(X_\sigma))u \right\}_p , \]

2) \[ 0 = \text{trace}_g \left\{ \nabla^2 \tau^2(Y_\sigma) + 2R(\tau^0(Y_\sigma), *)\nabla_\sigma Y_\sigma - R(*, \nabla_\sigma \tau^0(Y_\sigma))w \\
+ \frac{1}{2}R(w, \nabla_\sigma *)_*, \tau^0(Y_\sigma))w \right\}_p , \]

3) \[ 0 = \text{trace}_g \left\{ R(u, \nabla_\sigma \tau^1(X_\sigma)) + R(w, \nabla_\sigma \tau^2(Y_\sigma)) \\
+ R(\tau^1(X_\sigma), \nabla_\sigma X_\sigma) + R(\tau^2(Y_\sigma), \nabla_\sigma Y_\sigma) \\
+ \frac{1}{2}R(u, \nabla^2(X_\sigma))R(u, \nabla_\sigma X_\sigma) + \frac{1}{2}R(w, \nabla^2(Y_\sigma))R(w, \nabla_\sigma Y_\sigma) \\
+ \nabla_\sigma \nabla_\sigma \tau^0(\sigma) + R(u, \nabla_\sigma X_\sigma)\nabla_\sigma \tau^0(X_\sigma) + R(w, \nabla_\sigma Y_\sigma)\nabla_\sigma \tau^0(Y_\sigma) \\
+ \frac{1}{2}R(u, \nabla_\sigma \nabla_\sigma X_\sigma)\tau^0(X_\sigma) + \frac{1}{2}R(w, \nabla_\sigma \nabla_\sigma Y_\sigma)\tau^0(Y_\sigma) \\
+ R(u, R(\tau^0(X_\sigma), *)u) + R(w, R(\tau^0(Y_\sigma), *)w) \\
+ R(\tau^0(\sigma), *) + (\nabla_{\tau^0(\sigma)}R)(u, \nabla_\sigma X_\sigma) \\
+ (\nabla_{\tau^0(Y_\sigma)}R)(w, \nabla_\sigma Y_\sigma) \right\}_p \]

for all \( p = S^{-1}(x, u, w) \in T^2M \).
**Corollary 4.6.** Let \((M,g)\) be a Riemannian manifold and \((T^2M,g^D)\) be its tangent bundle of order two equipped with the diagonal metric. If \(\sigma: M \to T^2M\) is a section such that \(X_\sigma\) and \(Y_\sigma\) are biharmonic vector fields, then \(\sigma\) is biharmonic.

(For biharmonic vector see [1]).