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# COMMON TERMS IN CERTAIN BINARY RECURRENCES

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**Abstract.** The purpose of this paper is to prove that the common terms of linear recurrences $M(2a,-1,0,b)$ and $N(2c,-1,0,d)$ have at most 2 common terms if $p=2$, and have at most three common terms if $p>2$ where $D$ and $p$ are fixed positive integers and $p$ is a prime, such that neither $D$ nor $D+p$ is perfect square, further $a,b,c,d$ are nonzero integers satisfying the equations $a^2-Db^2=1$ and $c^2-(D+p)d^2=1$.

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## 1. Introduction

Let $G = G(A,B,G_0,G_1) = \{G_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by the recursion

$$G_n = AG_{n-1} + BG_{n-2} \quad (n > 1),$$

where $A, B$ and the initial terms $G_0, G_1$ are fixed integers, $AB \neq 0$ and $G_0^2+G_1^2 \neq 0$.

Let $\alpha$ and $\beta$ be the roots of the characteristic polynomial $x^2 - Ax - B$ of the sequence $G$. Throughout this paper we assume that $|\alpha| \geq |\beta|$ and the sequence $G$ is nondegenerate, that is, $\frac{\alpha}{\beta}$ is not a root of unity.

It is well-known that the terms of $G$ can be written in the form

$$G_n = \frac{q\alpha^n - e\beta^n}{\alpha - \beta}, \tag{1}$$

where $q = G_1 - G_0\beta$ and $e = G_1 - G_0\alpha$.

It can be proved that in the case $A^2 + 4B > 0$

$$|G_n| > c|\alpha|^n,$$

while in the case $A^2 + 4B < 0$

$$(2) \qquad |G_n| > \frac{c_1|\alpha|^n}{n^{-c_0}}$$

holds by the results of C. L. Stewart [13], where $c, c_1, c_0, n_0$ are positive real constants depending on the parameters of $G$ and $n > n_0$.

Thus $|G_n| > x$ for all fixed real $x$, if $n$ is large enough, that is all elements can be equal to finitely many other elements of the sequence $G$.

A similar problem is to determine the common terms of distinct sequences.

G. Revuz [11] proved a general theorem for the common terms of different second order linear recurrences $G$ and $H$ defined by the same $A, B$ constants: The equation $G_x = H_y$ has finitely many solutions $(x, y)$; if $x > n_0$ then $G_x \neq H_y$.

A variety of classical algebraic and elementary estimations to the common terms of recursive sequences and similar problems can be found in the papers of M. D. Hirsch [3], P. Kiss [4], [5], M. Mignotte [9], F. Mátyás [8], H. P. Schlickewei, W. M. Schmidt [12] and others.

Using Shure's theorem K. Liptai [7] proved that certain recursive sequences have finitely many common elements.

J. Binz [2] proved that the sequences $G\,(6, -1, 0, 6)$ and $H\,(10, -1, 0, 10)$ have only one common term.

There is a connection between the number of solutions of a special type of Pell's equations and the number of common terms in certain recurrences, that is why we use the following result:

Michael A. Bennett [1] proved that if $a$ and $b$ are distinct nonzero integers then the simultaneous Pell's equations

$$x^2 - az^2 = 1, y^2 - bz^2 = 1$$

possess at most three solutions in positive integers $(x, y, z)$.

## 1. Results and proofs

Some special cases are the most interesting because the number of the common terms can be determined.

The aim of the next part is to give the common terms in certain binary recurrences and generalize the result of J. Binz. Our main result is the following.

**Theorem 1.** *Let $D$ and $p$ be fixed positive integers, where $p$ is a prime, such that neither $D$ nor $D + p$ is perfect square. Further let $a, b, c, d$ be non-zero integers satisfying the equations $a^2 - Db^2 = 1$ and $c^2 - (D + p)\,d^2 = 1$. Then the sequences $M\,(2a, -1, 0, b)$ and $N\,(2c, -1, 0, d)$, apart from the zero initial terms, have at most two common terms if $p = 2$.*

**Proof.** First we prove that $(x, y) = (x, M_n)$ is a solution of the equation

$$x^2 - Dy^2 = 1 \tag{3}$$

for all $M_n$. The number pairs $(x_n, y_n)$ are also solutions, where

$$x_n + y_n\sqrt{D} = \left(a + b\sqrt{D}\right)^n \; (n = 0, 1, 2, ...). \tag{4}$$

This follows from the condition $(x, y) = (a, b)$ and

$$\begin{aligned} x_n^2 - Dy_n^2 &= \left(x_n + y_n\sqrt{D}\right)\left(x_n - y_n\sqrt{D}\right) \\ &= \left(a + b\sqrt{D}\right)^n \left(a - b\sqrt{D}\right)^n = \left(a^2 - Db^2\right)^n = 1. \end{aligned}$$

From (4) we have

$$y_n = \frac{1}{2\sqrt{D}}\left[\left(a + b\sqrt{D}\right)^n - \left(a - b\sqrt{D}\right)^n\right].$$

The roots of the characteristic polynomial $x^2 - 2ax + 1$ of the sequence $M$ are:

$$\alpha = a + \sqrt{a^2 - 1} = a + b\sqrt{D},$$

$$\beta = a - b\sqrt{D},$$

so with $M_0 = 0, M_1 = b$, $\alpha - \beta = 2b\sqrt{D}$ and by (1) the equality $y_n = M_n$ holds. It is similarly true for all terms $N_k$ that $(z, y) = (z, N_k)$ is a solution of the equation

$$z^2 - (D + p)\, y^2 = 1.$$

If the sequences $M$ and $N$ have some common terms, then the number of integer solutions $(x, y, z)$ of the equation system

$$\begin{aligned} x^2 - Dy^2 &= 1, \\ z^2 - (D + p)\, y^2 &= 1 \end{aligned} \tag{5}$$

is the number of the different common terms. It is enough to prove that the equation system has at most two solutions if $y \neq 0$. Assume that $(x, y, z)$ is the solution of (5). In this case

$$x^2 - Dy^2 = z^2 - (D + p)\, y^2$$

so

$$x^2 + py^2 = z^2 \tag{6}$$

and $\gcd(x,y)=1$, $\gcd(z,y)=1$. The solution $(x,y,z)$ is a positive solution of equation (6). If $\gcd(x,z)>1$ then $p \mid x^2+y^2$ and $p \mid y$ contradict to what is mentioned before. Now $p=2$. Then (6) can be written in form

$$x^2+2y^2=z^2. \tag{7}$$

The primitive solutions of (7) are: $x=|u^2-2v^2|, y=2uv, z=u^2+v^2$,$\gcd(u,v)=1$, where $u$ is an odd integer. Substitute these into the first part of (5)

$$\left(u^2-2v^2\right)^2-4Du^2v^2=1.$$

It can be written in the form

$$[u^2-(2+2D)\,v^2]^2-\left(8D+4D^2\right)v^4=1. \tag{8}$$

The diophantine equation $x^2-Dy^4=1$ has at most two solutions (Mordell [11]), $8D+4D^2=(2D+2)^2-4$ is not perfect square. Thus (8) holds for at most two pairs $(u,v)$. If $p=2$ than the equation system (5) has at most two solutions.

**Theorem 2.** *Let $D$ and $p$ be a fixed positive integer and a prime, respectively, such that neither $D$ nor $D+p$ is perfect square. Further let $a,b,c,d$ be non-zero integers satisfying the equations $a^2-Db^2=1$ and $c^2-(D+p)\,d^2=1$. Then the sequences $M\,(2a,-1,0,b)$ and $N\,(2c,-1,0,d)$, apart from the zero initial terms have at most three common terms if $p>2$.*

**Proof.** If the sequences $M$ and $N$ have some common terms then the equation system

$$x^2-Dy^2=1,$$
$$z^2-(D+p)\,y^2=1$$

has at most three solutions. It follows from the first Proof. It is enough to prove that this equation system have at most three solutions if $y\neq 0$. It follows from the result of M. A. Bennett which was published in [1]. Our simultaneous Pell' s equation system has at most three solutions in positive integers $(x,y,z)$. If $p>2$ then the sequences $M(2a,-1,0,b)$ and $N(2c,-1,0,d)$ apart from the zero initial terms have at most three common terms.

**Remark:** If we use the result of Mordell [10] then it can be proved that the number of common terms at most four.

If $p>2$ then the primitive solutions of (6)

$$x=|pm^2-n^2|, y=2mn, z=pm^2+n^2 \tag{9}$$

or

$$x=\left|\frac{pu^2-v^2}{2}\right|, y=uv, z=\frac{pu^2+v^2}{2} \tag{10}$$

where $m$ and $n$ are different and $\gcd(m,n) = 1$, $\gcd(u,v) = 1$. Substitute these into the first equation of (5) and we get from (9)

$$\left(pm^2 - n^2\right)^2 - 4Dm^2n^2 = 1,$$

whereas from (10)

$$\left(\frac{pu^2 - v^2}{2}\right)^2 - Du^2v^2 = 1.$$

These can be formed as

$$\left(n^2 - (p+2D)m^2\right)^2 - (4D^2 + 4pD)m^4 = 1, \tag{11}$$

$$\left(\frac{v^2 - (p+2D)u^2}{2}\right)^2 - (D^2 + pD)u^4 = 1. \tag{12}$$

It can be shown that neither $4D^2 + 4pD$ nor $D^2 + pD$ are perfect squares. Equations (11) and (12) have at most 2 solutions. So the equation system of (5) has at most 4 integer solutions.

**Theorem 3.** *Let $L$ be a fixed positive integer such that neither $L$ nor $L+8$ is perfect square and $8 \mid L$. Further let $r, s, k, t$ be non-zero integers satisfying the equations*

$$r^2 - Ls^2 = 1$$

*and*

$$k^2 - (L+16)\,t^2 = 1.$$

*Then the sequences $H = H\,(2r, -1, 0, s)$ and $K = K\,(2k, -1, 0, t)$ apart from the zero initial terms, have at most 2 common terms.*

**Proof.** The proof is based on the proof of the Theorem 1.

**Remarks**

1. Let $D$ be a positive integer which is not a perfect square. Pell's equation

$$x^2 - Dy^2 = 1$$

has infinitely many integer solutions pairs of $(x, y)$. It can be seen that there are infinitely many $a, b, c, d$ or $r, s, k, t$ integers for which our conditions hold.
2. If $L = 8$, then J. Binz's theorem follows from the Theorem 3. In this case we can determine the common terms of the sequences $G(6, -1, 0, 6)$ and $H(10, -1, 0, 10)$.
3. In particular, it would be interesting to prove a similar result for any sequence of $G(A, B, G_0, G_1)$ and $H(C, D, H_0, H_1)$ for which there are finitely many common terms. But the upper bound of the common terms would be too large.

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