Some Remarks On Traveling Wave Solutions In A Time-Delayed Population System With Stage Structure

Liang Zhang†, Pengyan Liu, Shitao Liu

Received 26 August 2016

Abstract

We show the existence of two traveling wave solutions in a time-delayed population system with stage structure by using the cross-iteration method.

1 Introduction

This work is a sequel to [1], we continue study the existence of traveling wave solutions for the two-species Lotka-Volterra competition model with age structure in the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} + \alpha_1 \int_{\mathbb{R}} G_1(y) u(t - \tau_1, x - y) dy - \eta_1 u^2 - p_1 uv, \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_2(y) v(t - \tau_2, x - y) dy - \eta_2 v^2 - p_2 uv.
\end{align*}
\]

(1)

Here \( u(t, x) \) and \( v(t, x) \) represent densities of adult members of two species \( u \) and \( v \) at time \( t \) and point \( x \), respectively. \( d_1 > 0 \) (\( d_2 > 0 \)) is the diffusion coefficient of the adult population \( u \) (\( v \)). \( \alpha_1 \) (\( \alpha_2 \)) is made up of two factors, the per capita birth rate and the survival rate of immature for the population \( u \) (\( v \)) during the immature stage. The two probability kernels \( G_1 \) and \( G_2 \) are given by

\[
G_1(y) = \frac{e^{-y^2/4d_i(\cdot)\tau_1}}{\sqrt{4\pi d_i(\cdot)\tau_1}}, \quad G_2(y) = \frac{e^{-y^2/4d_i(\cdot)\tau_2}}{\sqrt{4\pi d_i(\cdot)\tau_2}}.
\]

For more details of model (1) see [1] and the references cited therein.

Model (1) has the trivial equilibrium \( E_0 = (0, 0) \), the mono-culture equilibria \( E_u = (u^*, 0) \) and \( E_v = (0, v^*) \) with

\[
u^* = \frac{\alpha_1}{\eta_1}, \quad v^* = \frac{\alpha_2}{\eta_2}.
\]

*This work was supported by Chinese Universities Scientific Fund (Grant No.20452015086, 2014YB023) and the National Natural Science Foundation of China(Grant No. 11601405).
†Institute of Applied Mathematics, College of Science, Northwest A&F University, Yangling, Shaanxi 712100, PR China.
Table 1: Summary of local stability of system (1)

<table>
<thead>
<tr>
<th>Steady state</th>
<th>Criteria for existence</th>
<th>Criteria for asymptotic stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>always exists</td>
<td>unstable</td>
</tr>
<tr>
<td>$E_u$</td>
<td>always exists</td>
<td>$\alpha_1 p_2 &gt; \alpha_2 \eta_1$</td>
</tr>
<tr>
<td>$E_v$</td>
<td>always exists</td>
<td>$\alpha_2 p_1 &gt; \alpha_1 \eta_2$</td>
</tr>
<tr>
<td>$E_+$</td>
<td>$\alpha_2 p_1 &lt; \alpha_1 \eta_2 \alpha_2 p_2 &lt; \alpha_2 \eta_1$ or $\alpha_2 p_1 &gt; \alpha_1 \eta_2 \alpha_2 p_2 &gt; \alpha_2 \eta_1$</td>
<td>$\alpha_1 p_2 &lt; \alpha_2 \eta_1$ and $\alpha_2 p_1 &lt; \alpha_1 \eta_2$</td>
</tr>
</tbody>
</table>

and the coexistence equilibrium $E_+ = (e_1, e_2)$ with

$$
e_1 = \frac{\alpha_2 p_1 - \alpha_1 \eta_2}{p_1 p_2 - \eta_2 \eta_2}, \quad e_2 = \frac{\alpha_3 p_2 - \alpha_2 \eta_1}{p_1 p_2 - \eta_1 \eta_2}. $$

$E_+$ exists if and only if $\alpha_2 p_1 < \alpha_1 \eta_2$ and $\alpha_2 p_2 < \alpha_2 \eta_1$ or $\alpha_2 p_1 > \alpha_1 \eta_2$ and $\alpha_2 p_2 > \alpha_2 \eta_1$. We showed that if $\alpha_1 p_2 < \alpha_2 \eta_1$ and $\alpha_2 p_1 < \alpha_1 \eta_2$, then the unique coexistence $E_+$ is globally asymptotically stable. We summarized the stability of the equilibria in Table 1.1. A traveling wave solution of (1) connecting $E_0$ to $E_+$ takes the form of $u(t, x) = \phi(x + ct)$, $v(x, t) = \psi(x + ct)$, where $(\phi, \psi) \in C^2(\mathbb{R}, \mathbb{R}^2)$ with $\phi(\xi)$ and $\psi(\xi)$ satisfying

$$d_1 \phi''(\xi) - c \phi'(\xi) + \alpha_1 \int_{\mathbb{R}} G_1(y) \phi(\xi - y - ct)dy - \eta_1 \phi^2(\xi) - p_1 \phi(\xi)\psi(\xi) = 0, \quad (2)$$

$$d_2 \psi''(\xi) - c \psi'(\xi) + \alpha_2 \int_{\mathbb{R}} G_2(y) \psi(\xi - y - ct)dy - \eta_2 \psi^2(\xi) - p_2 \phi(\xi)\psi(\xi) = 0, \quad (3)$$

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = E_0 \text{ and } \lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = E_+. $$

We substitute $\phi(\xi) = e^{\lambda \xi}$ and $\psi(\xi) = e^{\lambda \xi}$ into the linearization equation of (2)–(3) to obtain the characteristic equations as follows

$$\Delta_i(\lambda, \xi) := d_i \lambda^2 - c\lambda + \alpha_i e^{-\lambda \tau_i} \int_{\mathbb{R}} G_i(y) e^{-\lambda y}dy, \quad i = 1, 2.$$

Then it is easy to verify the following properties:

i. $\Delta_i(0, c) = \alpha_i \int_{\mathbb{R}} G_i(y) e^{-\lambda y}dy > 0$;

ii. $\lim_{\lambda \to \infty} \Delta_i(\lambda, c) = \infty$ for all $c \geq 0$;

iii. $\frac{\partial^2 \Delta_i(\lambda, c)}{\partial \lambda^2} = 2d_i > 0$ and

$$\frac{\partial \Delta_i(\lambda, c)}{\partial c} = -\lambda - \lambda \alpha_i \tau_i \int_{\mathbb{R}} G_i(y) e^{-\lambda y}dy < 0$$

for all $\lambda > 0$;
iv. \( \lim_{t \to -\infty} \Delta_i(\lambda, c) = -\infty \) for all \( \lambda > 0 \) and \( \Delta_i(\lambda, 0) > 0 \).

By the properties of \( \Delta_i(\lambda, c) \) we know that there exist \( c_i^* > 0, i = 1, 2 \) such that the following statements are valid.

i. If \( c \geq c_i^* \), then there exist four positive numbers \( \Lambda_{i1}, \Lambda_{i2}, i = 1, 2 \) (which are independent on \( c \)) with \( \Lambda_{i1} \leq \Lambda_{i2} \) such that \( \Delta_i(\Lambda_{i1}, c) = \Delta_i(\Lambda_{i2}, c) = 0 \).

ii. If \( c < c_i^* \), then \( \Delta_i(\lambda, c) > 0 \) for all \( \lambda > 0 \).

iii. If \( c = c_i^* \), then \( \Lambda_{i1} = \Lambda_{i2} \); and if \( c > c_i^* \), then \( \Lambda_{i1} < \Lambda_{i2} \), \( \Delta_i(\lambda, c) < 0 \) for all \( \lambda \in (\Lambda_{i1}, \Lambda_{i2}), \Delta_i(\lambda, c) > 0 \) for all \( \lambda \in [0, \infty) \setminus [\Lambda_{i1}, \Lambda_{i2}] \).

Define

\[
 c^* = \max\{c_1^*, c_2^*\}.
\]

By Liang and Zhao [2], \( c^* \) may be viewed as the spreading speeds of species \( u \) if \( i = 1 \) and species \( v \) if \( i = 2 \) in the absence of its rival. The existence of one traveling wave solution has been studied in [1]. In this remark, we show the existence of two traveling wave solutions by employing the cross-iteration method, which has been successfully used in many literatures, see e.g., [1, 3, 4, 5] and the references cited therein.

Our main theorem is now in the following:

**THEOREM 1.** Suppose that \( \alpha_1 p_2 < \alpha_2 \eta_1 \), and \( \alpha_2 p_1 < \alpha_1 \eta_2 \) in (1). Then for \( c > c^* \), there exist two traveling wave solution \( (u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)) \) with

\[
 \lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, 0), \quad \lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (e_1, e_2),
\]

and

\[
 \lim_{\xi \to -\infty} \phi(\xi) e^{\Lambda_{11}(\xi)} = \lim_{\xi \to -\infty} \psi(\xi) e^{\Lambda_{21}(\xi)} = 1,
\]

where \( \Lambda_{11}(\xi) \) and \( \Lambda_{21}(\xi) \) are small eigenvalues of \( \Delta_1(\lambda, c) \) and \( \Delta_2(\lambda, c) \), respectively.

## 2 Proofs

In [1], in order to prove the existence of traveling wave solutions for model (1), we constructed two pairs of functions \( (\tilde{\phi}(\xi), \tilde{\psi}(\xi)) \) and \( (\bar{\phi}(\xi), \bar{\psi}(\xi)) \) as follows:

\[
 \tilde{\phi}(\xi) = \begin{cases} 
 e^{\Lambda_{11}\xi} & \text{for } \xi \leq \xi_3, \\
 e_1 + \epsilon_3 e^{-\lambda \xi} & \text{for } \xi \geq \xi_3,
\end{cases}
\]

\[
 \tilde{\psi}(\xi) = \begin{cases} 
 e^{\Lambda_{21}\xi} & \text{for } \xi \leq \xi_4, \\
 e_2 + \epsilon_4 e^{-\lambda \xi} & \text{for } \xi \geq \xi_4,
\end{cases}
\]

\[
 \bar{\phi}(\xi) = \begin{cases} 
 e^{\Lambda_{11}\xi} - q_1 e^{\eta \Lambda_{11}\xi} & \text{for } \xi \leq \xi_1, \\
 e_1 - \epsilon_1 e^{-\lambda \xi} & \text{for } \xi \geq \xi_1
\end{cases},
\]

\[
 \bar{\psi}(\xi) = \begin{cases} 
 e^{\Lambda_{21}\xi} - q_2 e^{\eta \Lambda_{21}\xi} & \text{for } \xi \leq \xi_2, \\
 e_2 - \epsilon_2 e^{-\lambda \xi} & \text{for } \xi \geq \xi_2,
\end{cases}
\]

where each \( q_i > 1 \) is sufficiently large and \( \lambda > 0 \) is sufficiently small.

We use the usual Banach space \( \mathcal{B} := C(\mathbb{R}, \mathbb{R}^2) \) of bounded continuous functions endowed with the maximum norm \( \| (\phi, \psi) \| = \sup_{\xi \in \mathbb{R}} (|\phi(\xi)| + |\psi(\xi)|) \). For any \( c > c^* \), let

\[
 \mathcal{C}_c = \{ (\phi, \psi) : (\phi, \psi) \in \mathcal{B}, \bar{\phi}(\xi) \leq \phi(\xi) \leq \tilde{\phi}(\xi), \bar{\psi}(\xi) \leq \psi(\xi) \leq \tilde{\psi}(\xi) \}.
\]
Clearly, $\mathcal{S}_c$ is a bounded nonempty closed convex subset of $\mathcal{B}$.

Define the operator $F = (F_1, F_2) : \mathcal{S}_c \to \mathcal{B}$ by

$$F_1(\phi, \psi)(\xi) := \alpha_1 \int_{\mathbb{R}} G_1(y)\phi(\xi - y - ct_1)dy - \eta_1\phi^2(\xi) - p_1\phi(\xi)\psi(\xi) + \beta_1\phi(\xi),$$

$$F_2(\phi, \psi)(\xi) := \alpha_2 \int_{\mathbb{R}} G_2(y)\psi(\xi - y - ct_1)dy - \eta_2\psi^2(\xi) - p_2\phi(\xi)\psi(\xi) + \beta_2\psi(\xi),$$

where each $\beta_i$ is a large positive number. Then system (2)–(3) now can be rewritten as

$$\begin{cases}
  d_1\phi''(\xi) - c\phi'(\xi) - \beta_1\phi(\xi) + F_1(\phi, \psi)(\xi) = 0, \\
  d_2\psi''(\xi) - c\psi'(\xi) - \beta_2\psi(\xi) + F_2(\phi, \psi)(\xi) = 0.
\end{cases}$$

(4)

Let

$$\lambda_{11} = \frac{c - \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \quad \lambda_{12} = \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1},$$

$$\lambda_{21} = \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, \quad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}.$$

Clearly,

$$\lambda_{11} < 0 < \lambda_{12}, \quad \lambda_{21} < 0 < \lambda_{22},$$

$$d_1\lambda_{1j}^2 - c\lambda_{1j} - \beta_1 = 0 \quad \text{and} \quad d_2\lambda_{2j}^2 - c\lambda_{2j} - \beta_2 = 0 \quad \text{for} \ j = 1, 2.$$

Define the operator $Q = (Q_1, Q_2) : \mathcal{S}_c \to \mathcal{B}$ by

$$Q_1(\phi, \psi)(\xi) = \frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left( \int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)}F_1(\phi, \psi)(s)ds + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi-s)}F_1(\phi, \psi)(s)ds \right)$$

$$+ \int_{-\infty}^{\xi} e^{\lambda_{12}(\xi-s)}F_1(\phi, \psi)(s)ds$$

(5)

$$Q_2(\phi, \psi)(\xi) = \frac{1}{d_2(\lambda_{22} - \lambda_{21})} \left( \int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)}F_2(\phi, \psi)(s)ds + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi-s)}F_2(\phi, \psi)(s)ds \right)$$

$$+ \int_{-\infty}^{\xi} e^{\lambda_{22}(\xi-s)}F_2(\phi, \psi)(s)ds$$

(6)

It is easily verified that the operator $Q$ is well defined for $(\phi, \psi) \in \mathcal{S}_c$ and

$$\begin{cases}
  d_1Q_1(\phi, \psi)''(\xi) - cQ_1(\phi, \psi)'(\xi) - \beta_1Q_1(\phi, \psi)(\xi) + F_1(\phi, \psi)(\xi) = 0, \\
  d_2Q_2(\phi, \psi)''(\xi) - cQ_2(\phi, \psi)'(\xi) - \beta_2Q_2(\phi, \psi)(\xi) + F_2(\phi, \psi)(\xi) = 0.
\end{cases}$$

Thus the fixed of $Q$ is the solution of (4), which is the travelling solution of (1).

We showed that $(\bar{\phi}(z), \bar{\psi}(z))$ is an upper solution and $(\underline{\phi}(z), \underline{\psi}(z))$ is a lower solution of the operator $Q$ defined by (5) and (6) in the sense that

$$Q_1(\bar{\phi}, \bar{\psi})(\xi) \leq \bar{\phi}(\xi), \quad Q_2(\bar{\phi}, \bar{\psi})(\xi) \leq \bar{\psi}(\xi),$$

(7)
We also showed that for any \((\phi, \psi) \in \mathcal{R}\), \(Q_1(\phi, \tilde{\psi})(\xi)\) is nondecreasing in \(\phi\) and nonincreasing in \(\psi\), and \(Q_2(\phi, \psi)\) is nondecreasing in \(\psi\) and nonincreasing in \(\phi\). Define
\[
(\phi, \psi)(\xi) = (\phi_0, \psi_0)(\xi), \quad (\phi, \tilde{\psi})(\xi) = (\tilde{\phi}_0, \tilde{\psi}_0)(\xi),
\]
\[
\tilde{\phi}_i(\xi) = Q_1[\tilde{\phi}_i, \tilde{\psi}_0](\xi), \quad \phi_1(\xi) = Q_1[\phi_0, \psi_1]\)
\]
\[
\psi_1(\xi) = Q_2[\phi_0, \psi_1](\xi), \quad \tilde{\psi}_1(\xi) = Q_2[\tilde{\phi}_0, \tilde{\psi}_0](\xi).
\]
By (5)–(8), it follows that
\[
(\phi, \psi_0)(\xi) \leq (\phi_0, \psi_0)(\xi) \leq (\phi_1, \psi_1)(\xi) \leq (\phi_0, \tilde{\psi}_0)(\xi).
\]
For general cases we define
\[
\begin{aligned}
\phi_{k+1}(\xi) &= Q_1[\phi_k, \psi_0](\xi), \quad \phi_{k+1}(\xi) = Q_1[\phi_k, \tilde{\psi}_0](\xi), \\
\psi_{k+1}(\xi) &= Q_2[\phi_k, \psi_0](\xi), \quad \tilde{\psi}_{k+1}(\xi) = Q_2[\phi_k, \tilde{\psi}_0](\xi),
\end{aligned}
\]
for \(k = 0, 1, 2, \ldots\). The inductive method show that
\[
(\phi_k, \psi_k)(\xi) \leq (\phi_{k+1}, \psi_{k+1})(\xi) \leq (\phi_{k+1}, \tilde{\psi}_{k+1})(\xi) \leq (\phi_k, \tilde{\psi}_k)(\xi),
\]
for \(k = 0, 1, 2, \ldots \) and \(\xi \in \mathbb{R}\).

One can easily check that \((\phi_k(\xi), \psi_k(\xi), \tilde{\phi}_k(\xi))\) and \((\tilde{\psi}_k(\xi))\) are equicontinuous for \(k = 0, 1, 2, \ldots\) and \(\xi \in \mathbb{R}\). Furthermore, for \(\xi \in \mathbb{R}\), the monotonicity of function sequences \(\{\phi_k(\xi)\}_{k=0}^\infty\), \(\{\psi_k(\xi)\}_{k=0}^\infty\), \(\{\tilde{\phi}_k(\xi)\}_{k=0}^\infty\), and \(\{\tilde{\psi}_k(\xi)\}_{k=0}^\infty\) implies that there exist two pairs of continuous functions \((\phi_*, \psi_*)(\xi)\) and \((\phi^*, \psi^*)(\xi)\) such that
\[
\lim_{k \to \infty} \phi_k(\xi) = \phi_*(\xi), \quad \lim_{k \to \infty} \psi_k(\xi) = \psi_*(\xi),
\]
\[
\lim_{k \to \infty} \tilde{\phi}_k(\xi) = \phi^*(\xi), \quad \lim_{k \to \infty} \tilde{\psi}_k(\xi) = \psi^*(\xi),
\]
convergence uniformly for all \(\xi \in \mathbb{R}\) with respect to the super norm. In fact, for given \(\varepsilon > 0\), by the construction of \((\phi(\xi), \tilde{\psi}(\xi))\) and \((\phi(\xi), \psi(\xi))\) there exists \(M(\varepsilon) > 0\) such that
\[
\sup_{|\xi| > M(\varepsilon)} |\phi(\xi) - \phi_*(\xi) + \tilde{\psi}(\xi) - \psi_*(\xi)| < \varepsilon.
\]
Since \(\tilde{\phi}(\xi), \tilde{\psi}(\xi), \phi(\xi), \) and \(\phi_*(\xi)\) are equicontinuous, there exists \(N(\varepsilon) > 0\) such that for any \(m, n > N(\varepsilon)\),
\[
\max_{|\xi| \leq T(\varepsilon)} \left\{|\phi_m(\xi) - \phi_n(\xi)| + |\phi_n(\xi) - \phi_*(\xi)| + |\tilde{\psi}_m(\xi) - \tilde{\psi}_n(\xi)| + |\tilde{\psi}_m(\xi) - \tilde{\psi}_n(\xi)|\right\} < \varepsilon.
\]
Hence,
\[
\sup_{\xi \in \mathbb{R}} \left\{|\phi_m(\xi) - \phi_n(\xi)| + |\phi_m(\xi) - \phi_*(\xi)| + |\tilde{\psi}_m(\xi) - \tilde{\psi}_n(\xi)|
\right\} < \varepsilon.
\]
Some Remarks on Traveling Wave Solutions

\[ + \left| \psi_m(\xi) - \psi_n(\xi) \right| < \varepsilon \]

It follows from the dominated convergence theorem and (9) that

\[ \begin{cases} 
\phi_\ast(\xi) = Q_1[\phi_\ast, \psi_\ast](\xi), & \phi^\ast(\xi) = Q_1[\phi^\ast, \psi_\ast](\xi), \\
\psi_\ast(\xi) = Q_2[\phi_\ast, \psi_\ast](\xi), & \psi^\ast(\xi) = Q_2[\phi_\ast, \psi^\ast](\xi) 
\end{cases} \]

for all \( \xi \in \mathbb{R} \). By (10) we obtain that

\[ (\phi, \psi)(\xi) \leq (\phi_\ast, \psi_\ast)(\xi) \leq (\phi^\ast, \psi^\ast)(\xi) \leq (\bar{\phi}, \bar{\psi})(\xi). \]

The operator \( Q \) defined by (5) and (6), and (11) show that the wave system (2)–(3) has two traveling wave solutions \((\phi_\ast, \psi_\ast)(\xi)\) and \((\phi^\ast, \psi^\ast)(\xi)\) between the super solution \((\bar{\phi}, \bar{\psi})(\xi)\) and the lower solution \((\phi, \psi)(\xi)\). Moreover, by the definitions of \((\bar{\phi}, \bar{\psi})(\xi)\) and \((\bar{\phi}, \bar{\psi})(\xi)\) one can see that the traveling waves have the following decay rate

\[ \lim_{z \to -\infty} \phi_\ast(z)R_1(\xi) = \lim_{z \to -\infty} \psi_\ast(z)R_2(\xi) = \lim_{z \to -\infty} \phi^\ast(z)R_1(\xi) = \lim_{z \to -\infty} \psi^\ast(z)R_2(\xi) = 1 \]

with \( R_1(\xi) = e^{-\Lambda_{11}(c)\xi} \) and \( R_2(\xi) = e^{-\Lambda_{21}(c)\xi} \). The proof for Theorem 1 is complete.

Acknowledgements. This work was supported by Chinese Universities Scientific Fund (Grant No.2452015086, 2014YB023) and the National Natural Science Foundation of China(Grant No. 11601405). The authors are grateful to the editors and the anonymous referee for their helpful comments which lead to an improvement of our original manuscript.

References


