ON MODAL BE-ALGEBRAS

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Abstract. In this paper, we introduce modal BE-algebra and study some structural properties of modal BE-algebra. The notions of modal upper set, modal BE-filter, BE - □-tautology filter, dual modal BE-algebra and quotient modal BE-algebra are introduced and their basic properties are investigated. We will prove that every self-distributive BE-algebra, induce a dual modal BE-algebra. Finally, we will prove that every dual modal BE-algebra is a modal BE-algebras under special conditions.

1. Introduction and Preliminaries

Modal logic is a theoretical field that is important not only in philosophy, where logic in general is commonly studied, but also in mathematics, linguistics, computer and information sciences as well. Classical modal logics have been a matter of growing interest in the last decades due to their role in the formalization of several aspects of computer science. The earliest paper on a many-valued modal logic appears to have been Segerberg (1967), which specifies some 3-valued modal logics.

Modal logics and many-valued logics were both historically introduced in order to free oneself from the rigidity of propositional logic. With many-valued logics, the logician can choose the truth values of the propositions in a set with more than two elements. With modal logics, the logician introduce a new connector whose aim is, for instance, to model the possibility. Many systems with various kind of modal operators have been constructed in order to provide effective formalisms for talking about time, space, knowledge, beliefs, actions, obligations, temporal, spatial, epistemic, dynamic, deontic, and so forth. However, modern applications often require rather complex formal models and corresponding languages that are capable of reflecting different features of the application domain [1, 9, 10].

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Furthermore, the study of $BCK/BCI$–algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic. There exist several generalization of $BCK/BCI$–algebras, such as $BCH$–algebras, $d$–algebras, $B$–algebras, $BH$–algebras, etc. Especially, the notion of $BE$–algebras was introduced by H. S. Kim and Y. H. Kim [13], in which was deeply studied by S. S. Ahn and et al. in [2, 3, 4], Walendziak in [18], A. Rezaei and et al. in [7, 8, 15, 17].

The idea of introducing modal operators in residuated lattices and other algebraic structures has been adapted by some researchers, for several purpose: Belohlavek and Vychodil [6] defined a so-called ”truth stresser” $\nu$ for a residuated lattice $(L, \cup, \cap, *, \to, 0, 1)$ as a unary operator on $L$ such that

- $\nu x \leq x$,
- $\nu 1 = 1$,
- $\nu(x \to y) \leq \nu x \to \nu y$, for all $x, y \in L$.

Ono [14] defined modal structures $(L, \cup, \cap, *, \to, \nu, 0, 1)$ in which $(L, \cup, \cap, *, \to, 0, 1)$ is a residuated lattice and $\nu$ is a unary operator on $L$ such that:

- $\nu x \leq x$,
- $\nu x \leq \nu \nu x$,
- $\nu 1 = 1$,
- $\nu(x \cap y) \leq \nu x$,
- $\nu x \ast \nu y \leq \nu(x \ast y)$, for all $x, y \in L$.

Hajek [11] used a unary operator $\triangledown$ on the $BL$–algebra $L$ to get the algebra $BL_{\triangledown}$ such that axioms of $BL_{\triangledown}$ are those of $BL$ plus:

- $\triangledown \phi \lor \neg \triangledown \phi$,
- $\triangledown(\phi \lor \psi) \implies (\triangledown \phi \lor \triangledown \psi)$,
- $\triangledown \phi \implies \phi$,
- $\triangledown \phi \implies \Delta \Delta \phi$,
- $\triangledown(\phi \lor \psi) \implies (\triangledown \phi \implies \triangledown \psi)$.

The axioms evidently resemble modal logic with $\Delta$ as necessity; but in the axiom on $\Delta(\phi \lor \psi)$, $\Delta$ be haves as possibility rather than necessity.

Magdalena and Rachunek [12] defined a unary operator $f$ on an $MV$–algebra $A$ as follows: If $A = (A, \oplus, \neg, 0)$ is an $MV$–algebra where $x \circ y = \neg(\neg x \oplus \neg y)$, then $f : A \to A$ is called a modal operator on $A$ satisfying:

- $x \leq f(x)$,
- $f(f(x)) = f(x)$,
- $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in A$.

In fact the modal operator $f$ be haves as possibility $\Diamond$ in modal logics. All above motivates us to introduce a modal operator on $BE$–algebra to get a modal $BE$–algebra as an algebraic structure.

This paper has been organized in three sections. In section 1, we give some definitions and some previous results. In section 2 we define modal $BE$–algebras and modal $BE$–filters. Finally, in section 3 we construct quotient modal $BE$–algebra via the modal normal $BE$–filter.
Definition 1.1 ([13]). An algebra \((X; *, 1)\) of type \((2, 0)\) is called a BE-algebra if following axioms hold:

(BE1) \(x * x = 1\),
(BE2) \(x * 1 = 1\),
(BE3) \(1 * x = x\),
(BE4) \(x * (y * z) = y * (x * z)\), for all \(x, y, z \in X\).

We introduce a relation \(\leq\) on \(X\) by \(x \leq y\) if and only if \(x * y = 1\).

Definition 1.2 ([13]). A BE-algebra \(X\) is said to be self distributive if

\[x * (y * z) = (x * y) * (x * z),\]

for all \(x, y, z \in X\).

Proposition 1.3 ([16]). Let \(X\) be a self distributive. If \(x \leq y\), then

(i) \(z * x \leq z * y\), and \(y * z \leq x * z\),
(ii) \(y * z \leq (z * x) * (y * x)\), for all \(x, y, z \in X\).

Definition 1.4 ([15, 18]). A BE-algebra \(X\) is said to be commutative if

\[(x * y) * y = (y * x) * x,\]

for all \(x, y \in X\).

Proposition 1.5 ([18]). If \(X\) is a commutative BE-algebra, then for all \(x, y \in X\), \(x * y = 1\) and \(y * x = 1\) imply \(x = y\).

Proposition 1.6 ([13]). Let \(X\) be a BE-algebra. Then

(i) \(x * (y * x) = 1\),
(ii) \(y * ((y * x) * x) = 1\), for all \(x, y \in X\).

Definition 1.7 ([13]). A subset \(F\) of \(X\) is called a filter of \(X\) if

(F1) \(1 \in F\),
(F2) \(x \in F\) and \(x * y \in F\) imply \(y \in F\), for all \(x, y \in X\).

Definition 1.8 ([19]). A filter \(F\) is said to be normal if it satisfies the following condition:

\[(NF)\quad x * y \in F \Rightarrow [(z * x) * (z * y) \in F \text{ and } (y * z) * (x * z) \in F],\]

for all \(x, y, z \in X\).

2. Modal BE-algebras

Definition 2.1. An algebra \((X; *, \Box, 1)\) of type \((2, 1, 0)\) is called a modal BE-algebra if it satisfies the following:

(BE) \((X; *, 1)\) is a BE-algebra,
(MBE1) \(\Box 1 = 1\),
(MBE2) \(\Box x \leq x\),
(MBE3) \(\Box x = \Box \Box x\),
(MBE4) \(\Box (x * y) = \Box x * \Box y\),
From now on, for simply in this section $X$ is a modal $BE$–algebra, unless otherwise is stated.

**Example 2.2.** (i) Let $X = \{1, a, b, c\}$. Define the operations "$*$" and "$\Box$" on $X$ as follows:

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Then, $(X; *, \Box, 1)$ is a modal $BE$–algebra.

(ii). Let $X = \mathbb{N}$ and "$*$" be the binary operation on $X$ defined by

$$x * y = \begin{cases} y, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1. \end{cases}$$

Then, $(X; *, 1)$ is a $BE$–algebra. Now, if we define the unary operation "$\Box$" such as:

$$\Box \cdots \Box x = \begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if } x = 2, \\ (x - n) + (n - 1), & \text{if } x \neq 1, 2. \end{cases}$$

Then, $(X; *, \Box, 1)$ is a modal $BE$–algebra.

**Proposition 2.3.** Let $X$ be a modal $BE$–algebra. Then

(i) if $x \leq y$, then $\Box x \leq \Box y$,

(ii) $\Box x * \Box x = 1$,

(iii) $\Box x * 1 = 1$,

(iv) $1 * \Box x = \Box x$,

(v) $\Box x * (\Box y * \Box z) = \Box y * (\Box x * \Box z)$, for all $x, y, z \in X$.

For every modal $BE$–algebra $X$, put $\Box X = \{\Box x : x \in X\}$.

If $X$ is a modal $BE$–algebra, then $\Box X = X$ does not hold, necessary. Indeed, in Example 2.2 (i) we have $\Box X = \{1, a, c\} \neq X$.

**Theorem 2.4.** Let $(X; *, 1)$ be a $BE$–algebra. Then $(\Box X; *, 1)$ is a $BE$–algebra.

**Proof.** By using Proposition 2.3, the proof is clear.

**Definition 2.5.** Let $(X; *, \Box, 1)$ be a modal $BE$–algebra and $x, y \in X$. Modal upper set of $x, y$ is denoted by $mA(x, y)$ and defined as follows:

$$mA(x, y) = \{z \in X : x * (y * \Box z) = 1\}.$$  

Obviously, it is a non empty set. Because $1 \in mA(x, y)$.

**Remark 2.6.** The upper set $A(x, y)$ does not equal to modal upper set $mA(x, y)$.

Indeed, in the Example 2.2(i), $mA(1, b) = \{1, a\} \neq \{1, a, b\} = A(1, b)$. 


Proposition 2.7. If □y = y, then A(1, y) = mA(1, y), for all x ∈ X.

Proof. Let y ∈ X. Then we have
\[
A(1, y) = \{ z ∈ X : y * z = 1 \} = \{ z ∈ X : □(y * z) = 1 \} = \{ z ∈ X : □y * □z = 1 \} = \{ z ∈ X : y * □z = 1 \} = mA(1, y).
\]

Proposition 2.8. mA(x, 1) ⊆ mA(x, y), for all x, y ∈ X.

Proof. Let z ∈ mA(x, 1). Then 1 = x * (1 * □z) = x * □z. Now, we get that
\[
x * (y * □z) = y * (x * □z) = 1.
\]
Therefore, z ∈ mA(x, y).

Theorem 2.9. Let X be a modal BE–algebra and x, y ∈ X. Then
(i) mA(□x, 1) ⊆ mA(□x, y),
(ii) if mA(□x, 1) is a filter of X and y ∈ mA(□x, 1), then
\[
mA(□x, y) ⊆ mA(□x, 1).
\]

Proof. (i). Let z ∈ mA(□x, 1). Then □x * (1 * □z) = 1, i.e. □x * □z = 1. Hence □x * (y * □z) = y * (□x * □z) = y * 1 = 1, i.e. z ∈ mA(□x, y).

(ii). Since □x * (1 * □x) = 1, we can see that □x ∈ mA(□x, y). Now, let y ∈ mA(□x, 1), then we have 1 = □x * □y = □x * (1 * □y) ∈ mA(□x, 1). Thus □y ∈ mA(□x, 1).

Let z ∈ mA(□x, y). Then by using (BE4)
\[
1 = □x * (y * □z) = y * (□x * □z).
\]

Now, by (MBE1), (MBE3) and (MBE4) we get that
\[
1 = □1 = □(y * (□x * □z)) = □y * □(□x * □z) = □y * (□x * □z) ∈ mA(□x, 1).
\]
Hence □x * □z ∈ mA(□x, 1). Thus □z ∈ mA(□x, 1) and so
\[
1 = □x * (1 * □z) = □x * (1 * □z).
\]

Therefore, z ∈ mA(□x, 1).

Proposition 2.10. Let F be a filter of X. Then □mA(x, y) ⊆ F, for all x, y ∈ F.

Proof. Let z ∈ □mA(x, y), then there exists a c ∈ mA(x, y) such that z = □c. Hence x * (y * □c) = 1 ∈ F. Thus y * □c ∈ F. Therefore, z = □c ∈ F. □
Theorem 2.11. Let $F$ be a subset of $X$ containing 1. $\Box F$ is a modal filter if and only if $x \leq y * z$ imply $z \in \Box F$, for all $x, y \in F$.

Proof. Let $\Box F$ be a modal filter and $x \leq y * z$, for all $x, y \in \Box F$. Since $x, y \in \Box F$ and $\Box F$ is a modal filter, we have $y * z \in \Box F$ and so $z \in \Box F$.

Conversely, $1 \in \Box F$, since $1 \in F$. If $x, x * y \in \Box F$, since $x * y \leq x * y$, we can see that by hypothesis $y \in \Box F$. Then there is a $z \in F$ such that $y = \Box z$. Therefore, $\Box y = \Box (\Box z) = \Box z = y \in \Box F$. □

Theorem 2.12. Let $F$ be a subset of $X$ containing 1. $\Box F$ is a modal filter of $X$ if and only if $x \in \Box F$, $y \in X \setminus \Box F$, then $x \cdot y \in X \setminus \Box F$.

Proof. Assume that $\Box F$ is a modal filter of $X$ and let $x, y \in X$ be such that $x \in \Box F$ and $y \in X \setminus \Box F$. If $x \cdot y \notin X \setminus \Box F$. Then $x \cdot y \in \Box F$, i.e. $y \in \Box F$. which is a contradiction. Hence $x \cdot y \in X \setminus \Box F$.

Conversely, $1 \in F$ by hypothesis. Let $x, x \cdot y \in \Box F$. Let $y \notin \Box F$. By assumption $x \cdot y \in X \setminus F$. This is a contradiction. Hence $y \in \Box F$. Thus there is a $z \in F$ such that $y = \Box z$. Therefore, $\Box y = \Box (\Box z) = \Box z = y \in \Box F$. □

Theorem 2.13. Let $F$ be a modal filter. Then

$$\Box F = \bigcup_{x, y \in F} \Box mA(\Box x, y).$$

Proof. Let $F$ be a modal filter of $X$ and consider $\Box z$, for $z \in F$. Since

$$\Box z * (1 \cdot \Box z) = \Box z * (1 \cdot \Box \Box z) = 1 \text{ by } (MBE3),$$

we have $\Box z \in mA(\Box z, 1)$. Now, by Proposition 2.9, we have

$$\Box z \in mA(\Box z, 1) \subseteq mA(\Box z, y).$$

Thus $\Box z = \Box \Box z \in \Box mA(\Box z, 1) \subseteq \Box mA(\Box z, y)$. Therefore,

$$\Box F \subseteq \bigcup_{y \in F} \Box mA(\Box z, y).$$

Now, by Theorem 2.10, $\Box mA(x, y) \subseteq F$, for all $x, y \in F$. Thus $\Box mA(\Box x, y) \subseteq \Box F$, for all $x, y \in F$. Therefore, $\bigcup_{x, y \in F} \Box mA(\Box x, y) \subseteq \Box F$. □

Definition 2.14. A (normal)filter $F$ of a modal $BE$–algebra $X$ is called a modal (normal)$BE$–filter if it closed under $\Box$ (i.e. if $x \in F$, then $\Box x \in F$, for all $x \in X$).

Example 2.15. In Example 2.2(i), $F_1 = \{1, a\}$ is a modal $BE$–filter of $X$ and $F_2 = \{1, b\}$ is a filter but it is not a modal $BE$–filter.

Theorem 2.16. If $\{F_i\}_{i \in I}$ is a family of modal $BE$–filters of $X$, then $\bigcap_{i \in I} F_i$ is a modal $BE$–filter of $X$, too.
Proposition 2.17. Let \( X \) be a modal \( BE \)-algebra and \( \ker(\Box) := \{ x \in X : \Box x = 1 \} \). Then

(i) \( \ker(\Box) \) is a filter of \( X \),

(ii) \( \ker(\Box) \) is closed under \( \Box \).

Proof. (i). Since \( \Box 1 = 1 \), we have \( 1 \in \ker(\Box) \). Hence \( \ker(\Box) \) is a non-empty set. Now, let \( x \star y \in \ker(\Box) \) and \( x \in \ker(\Box) \). Thus \( \Box(x \star y) = \Box(x) = 1 \). By using \( (MBE4) \) and \( (BE3) \) we have

\[
1 = \Box(x \star y) = \Box x \star \Box y = 1 \star \Box y = \Box y.
\]

Therefore, \( y \in \ker(\Box) \).

(ii). Let \( x \in \ker(\Box) \). Then \( \Box x = 1 \). Using \( (MBE1) \) and \( (MBE3) \) we have

\[
1 = \Box 1 = \Box(\Box x).
\]

Therefore, \( \Box x \in \ker(\Box) \).

Definition 2.18. The \( \ker(\Box) \) is called the \( \Box \)-tautology filter related to \( BE \)-algebra \( X \) or is called a \( BE \)-\( \Box \)-tautology filter.

Example 2.19. In Example 2.2(i), \( F_1 = \{1\} \) is a \( BE \)-\( \Box \)-tautology filter.

Proposition 2.20. Let \( [\alpha, 1] = \{ x \in X : \alpha \leq x \leq 1 \} \), where \( X \) is a commutative self distributive \( BE \)-algebra and \( \alpha \in X \). Then \( \ker(\Box_\alpha) = [\alpha, 1] \), where \( \Box_\alpha(x) = \alpha \star x \).

Proof. Let \( x \in [\alpha, 1] \). Then, \( \alpha \leq x \leq 1 \). Hence by using self distributivity and commutativity \( 1 = \alpha \star \alpha \leq \alpha \star x \leq \alpha \star 1 = 1 \) and so \( \Box_\alpha(x) = \alpha \star x = 1 \). Therefore, \( x \in \ker(\Box_\alpha) \).

Conversely, let \( x \in \ker(\Box_\alpha) \). Then \( \Box_\alpha(x) = 1 \), i.e. \( \alpha \star x = 1 \). Hence \( \alpha \leq x \) and so \( x \in [\alpha, 1] \). Therefore, \( [\alpha, 1] \) is a \( BE \)-\( \Box \)-tautology filter.

Definition 2.21. An algebra \((X; \ast, \Box, 1)\) of type \((2, 1, 0)\) is called a dual modal \( BE \)-algebra if it satisfies the following:

\[
\begin{align*}
(BE) &
\quad (X; \ast, 1) \text{ is a } BE \text{-algebra}, \\
(MBE1) &
\quad \Box 1 = 1, \\
(dMBE2) &
\quad x \leq \Box x, \\
(MBE3) &
\quad \Box x = \Box \Box x, \\
(MBE4) &
\quad \Box(x \ast y) = \Box x \ast \Box y, \text{ for all } x, y \in X.
\end{align*}
\]

Example 2.22. (i). Let \( X = \{1, a, b, c\} \). Define the operations "\( \ast \)" and "\( \Box \)" on \( X \) as follows:

\[
\begin{array}{c|ccc}
\ast & 1 & a & b \\
1 & 1 & a & b \\
a & 1 & 1 & 1 \\
b & 1 & a & 1 \\
c & 1 & b & 1 \\
\end{array}
\]

Then, \((X; \ast, \Box, 1)\) is a dual modal \( BE \)-algebra.
(ii). Let \( X = \mathbb{N} \) and " \(*" be the binary operation on \( X \) defined by
\[
x * y = \begin{cases} 
y, & \text{if } x = 1 \\
1, & \text{if } x \neq 1
\end{cases}
\]
Then, \( (X; *, 1) \) is a \( BE \)-algebra. Now, we define the unary operation " \( \square \)" on \( X \) as:
\[
(\square \cdots \square) x = \begin{cases} 
1, & \text{if } x = 1 \\
(x + n) - (n - 1), & \text{if } x \neq 1
\end{cases}
\]
Therefore, \( (X; *, \square, 1) \) is a dual modal \( BE \)-algebra.

**Proposition 2.23.** Let \( X \) be a self distributive \( BE \)-algebra. Define \( \square \alpha(x) = \alpha * x \), for all \( x \in X \). Then \( (X; *, \square \alpha, 1) \) is a dual modal \( BE \)-algebra.

**Proof.** Clearly, \( (X; *, 1) \) is a \( BE \)-algebra. For \((MBE1)\), we have \( \square \alpha(1) = \alpha * 1 = 1 \), by \((BE3)\). Since \( x * (\alpha * x) = \alpha * (x * x) = \alpha * 1 = 1 \), we have \( x \leq \alpha * x \), i.e. \( x \leq \square \alpha(x) \). Hence \((dMBE3)\) is valid. For \((MBE4)\),
\[
\square \alpha(\square \alpha(x)) = \alpha * (\alpha * x) = (\alpha * \alpha) * (\alpha * x) = 1 * (\alpha * x) = \alpha * x = \square \alpha(x).
\]
Also, \( \square \alpha(x * y) = \alpha * (x * y) = (\alpha * x) * (\alpha * y) = \square \alpha(x) * \square \alpha(y) \).

3. **Quotient Modal \( BE \)-algebra**

For a modal normal \( BE \)-filter \( F \) of \( X \) we define the binary relation \( \sim _F \) in the following way:
\[
x \sim _F y \iff x * y \in F \text{ and } y * x \in F.
\]
Clearly \( \sim _F \) is reflexive and symmetric. Now, let \( x \sim _F y \) and \( y \sim _F z \). Then \( x * y, y * x, y * z, z * y \in F \). Since \( F \) is a normal filter, \( (y * z) * (x * z) \in F \). Hence \( x * z \in F \). By a similar way, \( z * x \in F \). Consequently, \( x \sim _F z \). So, \( \sim _F \) is a transitive relation. Thus \( \sim _F \) is an equivalence relation on \( X \).

**Theorem 3.1.** \cite{19} Let \( F \) be a normal filter of a \( BE \)-algebra \( X \). Then \( \sim _F \) is a congruence relation on \( X \).

We have
\[
F_x = \{ y \in X : x \sim _F y \}
\]
Also, define the operations " \( \square \)" and " \( * \)" on congruence classes as follows:
\[
\square F_x = F_{\square x} \text{ and } F_x * F_y = F_{x * y}.
\]
We show that \( \square \) and \( * \) on congruence classes are well defined. Let \( F_x = F_y \).
Then \( x \sim _F y \) and \( y \sim _F x \), i.e. \( x * y, y * x \in F \). We get that \( \square x * \square y = \square (x * y) \in F \) and \( \square y * \square x = \square (y * x) \in F \), since \( F \) is a modal filter. Therefore, \( \square x \sim _F \square y \), i.e. \( F_{\square x} = F_{\square y} \). Equivalently, \( \square F_x = \square F_y \). Also, let \( F_x = F_y \) and \( F_u = F_v \), i.e. \( x \sim _F y, y \sim _F x \) and \( u \sim _F v, v \sim _F u \). Hence \( x * y, y * x \in F \) and \( u * v, v * u \in F \). Since \( F \) is a normal filter, \( (z * x) * (z * y), (z * y) * (z * x) \in F \).
Therefore, \( z * x \sim _F z * y \). By a similar way \( x * z \sim _F y * z \). Now, \( x * u \sim _F y * u \)
and $y * u \sim_F y * v$. Since $\sim_F$ is transitive, we have $x * u \sim_F y * v$. Therefore, $F_x * u = F_y * v$. Set $\mathcal{X}_F := \{x \in X : F_x \}$. It can be easily seen that $F_1 = F$. Since:

$$x \in F_1 \iff x \sim_F 1$$

$$\iff x * 1 = 1, 1 \in F$$

$$\iff x \in F,$$

we define a binary operation "*" on $\mathcal{X}_F$ as follows:

$$F_x * F_y = F_{x * y} \text{ and } \Box F_x = \Box F_x.$$  

We saw above, this binary operation is well-defined.

We can define an order such as "\leq" on $\mathcal{X}_F$ as follows:

$$F_x \leq F_y \iff x \leq y = 1.$$  

**Theorem 3.2.** $(\mathcal{X}_F; *, \Box, F)$ is a modal BE-algebra.

**Proof.** By Proposition 3.11 of [19], $(\mathcal{X}_F; *, F_1)$ is a BE-algebra,

$(MBE1)$ $\Box F_1 = F_1 = F = 1_X$,

$(MBE2)$ $\Box F_x = F_1 = F_x$, since $\Box x \leq x = 1,$

$(MBE3)$ $\Box F_x = \Box F_1 = \Box F_1 = F_1 = \Box F_1 = \Box F_x.$

$(MBE4)$ $\Box (F_x \Box F_y) = \Box F_{x * y} = F_{x \Box y} = F_{x \Box x * y} = \Box F_x * \Box F_y = \Box F_x * \Box F_y.$

**Theorem 3.3.** Let $F$ be a modal normal BE-filter of a commutative modal BE-algebra $X$. Then $(\mathcal{X}_F; *, \Box, F)$ is a commutative modal BE-algebra.

**Proof.** Let $F_x, F_y \in \mathcal{X}_F$. Then

$$(F_x * F_y) * F_y = (F_{x * y}) * F_y$$

$$= F_{(x * y) * y}$$

$$= F_{y * x} * F_x$$

$$= (F_y * F_x) * F_x.$$  

**Example 3.4.** Let $X = \{1, a, b, c\}$. Define the operations "*" and "\Box" on $X$ as follow:

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Then $(X; *, \Box, 1)$ is a modal BE-algebra, $F = \{1, b\}$ is a modal normal BE-filter, $F_1 = \{b, 1\} = F$, $F_a = \{a, c\}$, $F_b = \{b, c, 1\}$ and $F_c = \{a, c\}$. Hence
\((\mathbin{\ltimes}^F; \ast, \Box, F_1 = F)\) is a modal \(BE\)-algebra, where \(\mathbin{\ltimes}^F = \{\{b, 1\}, \{a, c\}, \{b, c, 1\}\}\) with the following table:

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Let \((X; \ast, \Box, 1)\) be a dual modal \(BE\)-algebra. We define

\[F_x = \{y \in X : x \sim_F y\}\]

Also, define the operations "\(\Box\)" and "\(\ast\)" on congruence classes as follows:

\[\Box F_x = F_\Box x \text{ and } F_x \ast F_y = F_{xy} \ast\]

Then by a similar way \((\mathbin{\ltimes}^F; \ast, \Box, F)\) is a dual modal \(BE\)-algebra. Because, it remains only to prove the condition \((dMBE2)\). Since \(x \ast \Box x = 1\), we get

\[F_x \leq \Box F_x = F_\Box x\]

**Theorem 3.5.** Let \(X\) be a modal \(BE\)-algebra. Then \((\mathbin{\ltimes}^F; \ast, \Box, F)\) is a dual modal \(BE\)-algebra if and only if the relation \(\leq\) has been defined as follows:

\[F_x \leq F_y \iff y \ast x = 1\]

**Proof.** Since \(X\) is a modal \(BE\)-algebra, we have \(\Box x \ast x = 1\). Thus \(F_x \leq F_\Box x = \Box F_x\). Therefore, the condition \((dMBE2)\) is valid. \(\square\)

**Theorem 3.6.** Let \((X; \ast, \Box, 1)\) be a dual modal \(BE\)-algebra. Let the relation \(\leq\) has been defined as

\[F_x \leq F_y \iff y \ast x = 1\]

Then \((\mathbin{\ltimes}^F; \ast, \Box, F)\) is a modal \(BE\)-algebra. In particular, if \((X; \ast, \Box, 1)\) is a commutative dual modal \(BE\)-algebra and the operator \(\Box\) is one-to-one, then the structure \((X; \ast, \Box, 1)\) is a modal \(BE\)-algebra.

**Proof.** Clearly \((\mathbin{\ltimes}^F; \ast, \Box, F)\) is a modal \(BE\)-algebra by Theorem 3.2. Since \(X\) is a commutative \(BE\)-algebra then every filter is a normal filter. Hence, \(F_1 = \{1\}\) is a normal filter. Now, let

\[F_x = \{y \in X : x \sim_F y \text{ and } \Box x = \Box y\}\]

Hence the equivalence class \(F_x = \{x\}\) (in particular \(F_1 = \{1\}\)), since \(\Box\) is one-to-one. Thus the natural map \(\pi : X \to \mathbin{\ltimes}^F_\Box F_1\) where \(\pi(x) = [x]_{F_1}\), is an isomorphism. Now, we can easily see that \((X; \ast, \Box, 1)\) is a modal \(BE\)-algebra. \(\square\)

4. **Conclusion and future research**

In this paper, we introduced the notion of modal \(BE\)-algebras and get some results.

In the future work, we try assemble of calculus relative to different kinds of modal algebraic structure.
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References


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