ON THE PARALLEL DISPLACEMENT AND PARALLEL VECTOR FIELDS IN FINSLER GEOMETRY

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This paper is dedicated to Professor Masao Hashiguchi on the occasion of his 80-th birthday

Abstract. In Finsler geometry, the notion of parallel for Finsler tensor fields is defined, already. In this paper, however, for vector fields on a base manifold, the author studies the notion of parallel. In the last section, the obtained results are shown as compare corresponding properties to them of Riemannian geometry.

Introduction

The author gave the definition (1.1) of the parallel displacement along a curve in [7] and explained the geometrical meaning by $HTM$ (the horizontal subbundle of $TTM$) in it. Further, the author proved that the inverse vector field was parallel along the inverse curve under some conditions, in addition, that the inner product of two parallel vector fields was preserved along a path. Next, the notion of the autoparallel curves was stated and, last, the author formulated the notion of the geodesics by using the Cartan Finsler connection and the notions of the path and autoparallel curves.

In this paper, the author studies the notion of parallel displacement for vector fields on a base manifold. The notion of parallel for Finsler tensor fields is already defined([3],[5],[6]) as same as done the notion of parallel displacement along curves([2],[1]). However, the study in detail of parallel vector field on the base manifold are not done, the author thinks so. As the first step of the study in detail, the author investigates the geometrical meaning of parallel in $HTM$ (Proposition 3.1) and from the obtained results the author have the conditions for Finsler connections (Theorem 3.1). Lastly, in Riemannian geometry, parallel

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vector fields have the special properties, for examples, “be parallel along any curve” and “its norm is constant on the base manifold”. So the author checks the above two properties for the ”parallel vector field” in Finsler geometry and investigates being parallel along curves under our definition (1.1) of the parallel displacement in particular.

The terminology and notations are referred to the books [4] and [5]. The author is given very useful suggestions by Prof. T. Aikou and Prof. M. Hashiguchi frequently, and greatly appreciates their kindness.

1. THE DEFINITION OF THE PARALLEL DISPLACEMENT ALONG A CURVE

Firstly, we put the terminology and notations used in this paper. Let \( M \) be an \( n \)-dimensional differentiable manifold and \( TM \) its tangent bundle, and let \((N^i_j, F^i_{jr}, C^i_{jr})\) be a Finsler connection( or the coefficients of a Finsler connection).

Further let \( F(x, y) \) be a Finsler fundamental function and \( g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \) the Finsler metric.

Now, for a vector field along a curve \( c \), we put the definition of the parallel displacement along \( c \). It is as follows

**Definition 1.1.** For a curve \( c = (c^i(t)) \) \( (a \leq t \leq b) \) and a vector field \( v = (v^i(t)) \) along \( c \), if the equations

\[
\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c}) \dot{c}^r = 0 \quad (\dot{c}^r = \frac{dc^r}{dt})
\]

are satisfied, then \( v \) is said to be parallel along \( c \), and we call the map \( v(a) \rightarrow v(b) \) the parallel displacement along \( c \).

Let set the state as the curve \( c = (c^i(t)) \) \( (a \leq t \leq b) \) passes through two points \( p = c(a), q = c(b) \) on \( M \), and we assume that the vector field \( v = (v^i(t)) \) is parallel along \( c \) and \( A = (A^i) = v(a), B = (B^i) = v(b) \). Then we have another curve \( c^{-1} \) and vector field \( v^{-1} \) as follows

\[
(1.2) \quad c^{-1}(\tau) = (c^{-1i}(\tau)), \quad \text{where} \quad c^{-1i}(\tau) = c^i(-\tau + a + b),
\]

\[
(1.3) \quad v^{-1}(\tau) = (v^{-1i}(\tau)), \quad \text{where} \quad v^{-1i}(\tau) = v^i(-\tau + a + b)
\]

and \( t = -\tau + a + b, \quad a \leq \tau \leq b \). Then \( c^{-1}(a) = c(b) = q, \quad c^{-1}(b) = c(a) = p \) and \( v^{-1}(a) = v(b) = B, \quad v^{-1}(b) = v(a) = A \).

We have the following theorem, for the inverse curve \( c^{-1} \) and vector field \( v^{-1} \),

**Theorem 1.1.** For any differentiable curve \( c(t) \). Let \( v(t) \) be parallel vector field along the curve \( c(t) \). Then the vector field \( v^{-1} \) is parallel along the curve \( c^{-1} \) if and only if

\[
(1.4) \quad F^i_{0j}(x, y) + F^i_{0j}(x, -y) = 0
\]

is satisfied, where \((N^i_j, F^i_{jr}, C^i_{jr})\) is a Finsler connection satisfying \( F^i_{jr}(x, y) = F^i_{rj}(x, y) \).
Now, let $HTM$ be the subbundle of the bundle $TTM$. $HTM$ is the collection of horizontal vectors at every point $(x, y)$ on the tangent bundle $TM$, namely
\begin{equation}
HTM = \bigcup_{(x, y) \in TM} \{ z^i \delta_i \in T(x, y)TM \mid z^i \in \mathbb{R} \},
\end{equation}
where $\delta_i = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r}$ is the horizontal basis of $T(x, y)TM$. We put a local coordinate system $(x^i, y^i, z^i)$ of $HTM$. This system have the coordinate transformation $(x_i, y_i, z_i) \rightarrow (\bar{x}^a, \bar{y}^a, \bar{z}^a)$ attended with the coordinate transformation $(x_i) \rightarrow (\bar{x}^a)$ of the base manifold $M$, where
\begin{equation}
\left\{ \begin{array}{l}
\bar{x}^a = \bar{x}^a(x) \\
\bar{y}^a = y^j \frac{\partial \bar{x}^a}{\partial x^j} \\
\bar{z}^a = z^j \frac{\partial \bar{x}^a}{\partial x^j}.
\end{array} \right.
\end{equation}

Then we can take the derivative operator with respect to $x^i$
\begin{equation}
\delta \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r} - F^r_i \frac{\partial}{\partial z^r},
\end{equation}
where $F^i_j = z^r F^i_{jr}(x, y)$.

Next, let $c = (c^i(t))$ be a curve on $M$ and $v = (v^i(t))$ a vector field along $c$. We take the lift $\tilde{c} = (c^i, \dot{c}^i, v^i)$ to $HTM$ and the tangent vector $\frac{d\tilde{c}}{dt}$ of $\tilde{c}$ is written in
\begin{equation}
\frac{d\tilde{c}}{dt} = \frac{dc^i}{dt} \frac{\partial}{\partial x^i} + \frac{d\dot{c}^i}{dt} \frac{\partial}{\partial y^i} + \frac{dv^i}{dt} \frac{\partial}{\partial z^i}.
\end{equation}
\begin{equation}
\dot{c}^i \frac{\delta}{\delta x^i} + \left( \frac{d\dot{c}^i}{dt} + N^r_i(c, \dot{c}) \dot{c}^r \right) \frac{\partial}{\partial y^i} + \left( \frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c}) \dot{c}^r \right) \frac{\partial}{\partial z^i}.
\end{equation}

Therefore Definition 1.1 means that the lift $\tilde{c}$ is a horizontal curve in $HTM$. So we have

**Theorem 1.2.** If a vector field $v = (v^i(t))$ along the curve $c = (c^i(t))$ is parallel along $c$, then the lift $\tilde{c} = (c, \dot{c}, v)$ to $HTM$ is a horizontal curve in $HTM$. The inverse property is also true.

2. Paths and Autoparallel curves

First, we treat paths on $M$ ([4]). It is the curve $c = (c^i(t))$ satisfying
\begin{equation}
\frac{d\dot{c}^i}{dt} + N^r_i(c, \dot{c}) \dot{c}^r = 0.
\end{equation}
In other words, the canonical lift $(c, \dot{c})$ to $TM$ is a horizontal curve in $TM$.

Then we have the following theorem.
Theorem 2.1. The inverse curve \( c^{-1} \) of any path \( c \) is also a path if and only if \( N_j^i(x, -y) = -N_j^i(x, y) \) satisfies.

Next, we consider a Finsler space \( (M, F(x, y)) \) with a Finsler connection \( (N_j^i, F_i^j, C_i^j) \). We assume the curve \( c = (c(t)) \) is a path, namely, satisfies (2.1) and two vector fields \( v = (v^i(t)), \ u = (u^i(t)) \) are parallel along \( c \). Then the "inner product" \( g_{ij}(c, \dot{c}) v^i u^j \) satisfies

\[
\frac{d}{dt} (g_{ij}(c, \dot{c}) v^i u^j) - \partial_{x^i} c^r v^i u^j = \frac{\partial g_{ij}}{\partial y^r} c^r v^i u^j + g_{ij} \frac{dv^i}{dt} u^j + g_{ij} v^i \frac{du^j}{dt}
\]

(2.2)

Hence if \( g_{ij} v^i u^j \) is constant on \( c \).

Inversely, we assume that the inner product \( g_{ij}(c, \dot{c}) v^i u^j \) is preserved on the curve \( c = (c(t)) \). Then we have the following calculations

\[
\frac{d}{dt} (g_{ij}(c, \dot{c}) v^i u^j) = \frac{\partial g_{ij}}{\partial y^r} v^i u^j \left( \frac{d\dot{c}^r}{dt} + N_k^r(c, \dot{c}) \dot{c}^k \right) = 0.
\]

(2.3)

The vector fields \( v, u \) are arbitrarily. So we have

\[
\frac{\partial g_{ij}}{\partial y^r} \left( \frac{d\dot{c}^r}{dt} + N_k^r(c, \dot{c}) \dot{c}^k \right) = 0.
\]

(2.4)

Obviously, on the curve \( c \) satisfying (2.4), the inner product of the parallel vector fields \( v \) and \( u \) are preserved. So we have

Theorem 2.2. Let \( (M, F(x, y)) \) be a Finsler space with a Finsler connection \( (N_j^i, F_i^j, C_i^j) \) satisfying h-metrical \( g_{ij} v^i u^j = 0 \) and \( v = (v^i(t)), \ u = (u^i(t)) \) vector fields parallel along the curve \( c = (c(t)) \). Then the inner product \( g_{ij}(c, \dot{c}) v^i u^j \) is preserved on \( c \) if and only if the equation (2.4) is satisfied.

Furthermore if \( N_j^i(x, y) + N_j^i(x, -y) = 0, T_{jk}^i(x, y) = 0 \) and \( F_i^j(x, y) + F_i^j(x, -y) = 0 \) are satisfied, then according to Theorem 1.1 we have

Theorem 2.3. For a Finsler space \( (M, F(x, y)) \) with a Finsler connection \( (N_j^i, F_i^j, C_i^j) \) satisfying h-metrical \( g_{ij} v^i u^j = 0 \), if the curve \( c = (c(t)) \) is a path and vector fields \( v = (v^i(t)), \ u = (u^i(t)) \) are parallel along \( c \), then the inner products \( g_{ij}(c, \dot{c}) v^i u^j \) and \( g_{ij}(c^{-1}, \dot{c}^{-1}) v^{-1i} u^{-1j} \) are preserved on \( c \) and \( c^{-1} \), respectively.

Last, let’s define the autoparallel curve. It is as follows

Definition 2.1. If the canonical lift \( (c, \dot{c}, \ddot{c}) \) to \( HTM \) is horizontal of \( HTM \), then we call \( c \) an autoparallel curve.

By the above definition, autoparallel curves \( c \) satisfy

\[
\frac{d\dot{c}^i}{dt} + F_i^j(c, \dot{c})\ddot{c}^j\dot{c}^r = 0.
\]

(2.5)
In addition we consider the geodesics from the viewpoint of the parallel displacement. It is well known that for the Cartan connection \((c)_{N_i^j}, (c)_{F_i^j}, (c)_{C_i^j})\) the equations of the geodesic \(c(t)\) are written

\[
\frac{d^2 c_i}{dt^2} + N_i^j(c, \dot{c}) \frac{dc^j}{dt} = 0 \quad \text{or} \quad \frac{d^2 c_i}{dt^2} + F_i^j(c, \dot{c}) \frac{dc^r}{dt} \frac{dc^j}{dt} = 0.
\]

This means that the geodesic is path and autoparallel curve, from \(N_i^j = F_i^j_{0j}\).

### 3. Parallel vector fields

In Riemannian geometry, when the equations

\[
\frac{\partial v^i(x)}{\partial x^j} + \Gamma_{rj}^i(x) v^r(x) = 0
\]

are satisfied, we call \(v(x)\) the parallel vector field, where \(\Gamma_{jk}^i(x)\) are the coefficients of the Riemannian connection. We want the notion of parallel vector fields in Finsler geometry.

According to Section 1 and 2, we treat the lift \(\tilde{v} = (x, v, v)\) to \(HTM\) and calculate the differential with respect to \(x^i\) as follows

\[
\frac{\partial \tilde{v}}{\partial x^i} = \delta^r_i \frac{\partial}{\partial x^r} + \frac{\partial v^r}{\partial x^i} \frac{\partial}{\partial y^r} + \frac{\partial v^r}{\partial x^i} \frac{\partial}{\partial z^r}
\]

\[
= \left( \frac{\delta}{\delta x^i} + N_i^r \frac{\partial}{\partial y^r} + F_i^r \frac{\partial}{\partial z^r} \right) + \frac{\partial v^r}{\partial x^i} \frac{\partial}{\partial y^r} + \frac{\partial v^r}{\partial x^i} \frac{\partial}{\partial z^r}
\]

\[
= \left( \frac{\delta}{\delta x^i} + \left( \frac{\partial v^r}{\partial x^i} + N_i^r (x, v) \right) \frac{\partial}{\partial y^r} + \left( \frac{\partial v^r}{\partial x^i} + F_i^r (x, v) \right) \frac{\partial}{\partial z^r} \right) \frac{\partial}{\partial z^r}.
\]

So we consider the tangential case of \(\frac{\partial \tilde{v}}{\partial x^i}\), namely, we assume that \(v(x)\) satisfies

\[
\frac{\partial v^r}{\partial x^i} + N_i^r (x, v) = 0
\]

\[
\frac{\partial v^r}{\partial x^i} + F_i^r (x, v) v^j = 0.
\]

From (3.2) if the vector field \(v(x)\) on \(M\) satisfies (3.3) and (3.4), then the lift \(\tilde{v}\) on \(HTM\) is tangential to the \(n\)-dimensional subspace spanned by \(\frac{\delta}{\delta x^i}\). So we have
Proposition 3.1. Let $M$ be an $n$-dimensional differential manifold and $(N^i_j, F^i_{jr}, C^i_{jr})$ a Finsler connection. For a vector field $v(x)$ on $M$, the lift $\tilde{v} = (x, v, v)$ on $HTM$ is tangential to the $n$-dimensional subspace \[
abla \left\{ \sum_i \lambda_i \frac{\delta}{\delta x^i} \lambda_i / R, \ i = 1, 2, \cdots, n \right\} \] at every point $(x, v, v)$ if and only if $v$ satisfies (3.3) and (3.4).

Next we study the conditions in order for the vector field satisfying (3.3) and (3.4) to exist in locally at every point $(x, y) \in TM$.

First of all, we must have the necessary condition as follows

\begin{equation}
D^i_j(x, v) = 0
\end{equation}

because of\[\frac{\partial v^r}{\partial x^t} = -N^r_i(x, v) = -F^r_{ji}(x, v)v^j\] and \[D^i_j(x, v) = F^i_{0j}(x, v) - N^i_j(x, v).\]

Next, by the integrability conditions \[\frac{\partial^2 v^i}{\partial x^j \partial x^k} = \frac{\partial^2 v^i}{\partial x^k \partial x^j}\] of (3.3), the equation

\begin{equation}
R^i_{jk}(x, v) = 0
\end{equation}

are satisfied because that

\begin{equation}
\frac{\partial^2 v^i}{\partial x^j \partial x^k} = -\frac{\partial N^i_j}{\partial x^k} - \frac{\partial N^i_k}{\partial y^r} \frac{\partial v^r}{\partial x^k} = -\frac{\partial N^i_j}{\partial x^k} + \frac{\partial N^i_k}{\partial y^r} N^r_k
\end{equation}

and

\begin{equation}
\frac{\partial^2 v^i}{\partial x^j \partial x^k} = -\delta N^i_k \delta x^j
\end{equation}

so we have

\begin{equation}
R^i_{jk}(x, v) = \delta N^i_k \delta x^j - \delta N^i_k \delta x^j = 0.
\end{equation}

Furthermore by the integrability conditions \[\frac{\partial^2 v^i}{\partial x^j \partial x^k} = \frac{\partial^2 v^i}{\partial x^k \partial x^j}\] of (3.4), the equation

\begin{equation}
R^i_{ijk}(x, v) = 0
\end{equation}
are satisfied because that
\[
\frac{\partial^2 v^i}{\partial x^j \partial x^k} = - \frac{\partial (F^r_{ji} v^r)}{\partial x^k} = - \frac{\partial F^i_{rj} v^r}{\partial x^k} - F^i_{rj} \frac{\partial v^r}{\partial x^k}
\]
\[
= - \left( \frac{\partial F^i_{rj}}{\partial x^k} + \frac{\partial F^i_{rj}}{\partial y^m} \frac{\partial v^r}{\partial x^k} \right) v^r + F^i_{rj} F^r_{mk} v^m
\]
\[
= - \left( \frac{\partial F^i_{rj}}{\partial x^k} - \frac{\partial F^i_{rj}}{\partial y^m} N^m_k \right) v^r + F^i_{rj} F^r_{mk} v^m
\]
\[
= \frac{\delta F^i_{rj}}{\delta x^k} v^r + F^i_{rj} F^r_{mk} v^m = - \left( \frac{\delta F^i_{rj}}{\delta x^k} - F^i_{mk} F^r_{rk} \right) v^r
\]
(3.11)

and
\[
\frac{\partial^2 v^i}{\partial x^k \partial x^j} = - \left( \frac{\delta F^i_{rj}}{\delta x^k} - F^i_{mk} F^r_{rk} \right) v^r,
\]
(3.12)

so we have
\[
\left( \frac{\delta F^i_{rj}}{\delta x^k} - \frac{\delta F^i_{rj}}{\delta x^j} + F^i_{mk} F^r_{jm} - F^i_{mj} F^r_{mk} \right) v^r = 0.
\]
(3.13)

From (3.6), (3.13) and \( R^i_{rjk} = \frac{\delta F^i_{rj}}{\delta x^k} - \frac{\delta F^i_{rj}}{\delta x^j} + F^i_{mk} F^r_{jm} - F^i_{mj} F^r_{mk} + C^i_{jm} R^m_{jk} \), we have (3.10).

At last we have the conditions that there exist a vector field \( v(x) \) with initial value \( v(x_0) = y_0 \) satisfying (3.3) and (3.4). Of course, the solutions of the differential equations (3.3) or (3.4) are locally. So we have

**Theorem 3.1.** For any point \( x_0 \in M \) and any direction \( y_0 \) at \( x_0 \), there exists an neighborhood \( U \) of \( x_0 \) and on \( U \) the vector field \( v(x) \) with the initial value \( v(x_0) = y_0 \) satisfies (3.3) and (3.4) if and only if the Finsler connection \( (N^i_j(x, y), F^i_{jr}(x, y), C^i_{jr}(x, y)) \) satisfies
\[
D^i_j(x, y) = 0, \quad \text{(for the deflection tensor)}
\]
\[
R^i_{rjk}(x, y) = 0, \quad \text{(for the } R^1 \text{ torsion tensor)}
\]
\[
R^2_{ijk}(x, y) = 0 \quad \text{(for the } R^2 \text{ curvature tensor)}.
\]

Now, in Riemannian geometry, if a vector field \( v(x) \) on \( M \) satisfies (3.1), namely, parallel vector field, then \( v(x) \) have the following properties

1. \( v \) is parallel along any curve \( c \),
2. the norm \( ||v|| \) is constant on \( M \).

So, for the vector field \( v(x) \) stated in Theorem 3.1, we investigate the above two properties.
First of all, we consider the curve \( c(t) = (c^i(t)) \) as the solution of the following differential equation

\[
\frac{dc^i}{dt} = v^i(c(t)).
\]

Then the restriction \( v|_c = v(c(t)) \) satisfies \( \frac{dv^i|_c}{dt} = \frac{\partial v^i}{\partial x^r} \frac{dc^r}{dt} = -F^i_{jr}(c, v|_c)v^j|_c \dot{c}^r \) from (3.4). Further, from \( \dot{c} = v|_c \), we have

\[
\frac{dv^i|_c}{dt} + v^j|_c F^i_{jr}(c, \dot{c}) \dot{c}^r = 0.
\]

Thus, according to Definition 1.1, we may call \( v \) parallel along the curve \( c \).

Next the solution \( c(t) \) of (3.17) satisfies \( \frac{d\dot{c}^i}{dt} = \frac{\partial v^i}{\partial x^r} \frac{d\dot{c}^r}{dt} = -N^i_{jr}(c, v|_c) \dot{c}^r \) from (3.3). Thus we have

\[
\frac{d\dot{c}^i}{dt} + N^i_{jr}(c, \dot{c}) \dot{c}^r = 0.
\]

By the definition (2.1), the flow \( c(x) \) is a path.

Therefore let \((M, F)\) be a Finsler space with a Finsler connection satisfying (3.14), (3.15), (3.16) and \( g_{ijk}(x, y) = 0 \). According to Theorem 2.2, the inner product \( \|v\|^2 = g_{ij}(c, \dot{c}) v^i v^j \) is constant on the path \( c \) that is the solution of (3.17). This means the norm \( \|v\| \) is constant on \( c \). Thus we have

**Theorem 3.2.** Let \((M, F)\) be a Finsler space with a Finsler connection satisfying (3.14), (3.15), (3.16) and \( g_{ijk}(x, y) = 0 \). Then, at every point \( x(\in M) \), there are a neighborhood \( U \) of \( x \), a local vector field \( v \) and a certain path on \( U \), and the following properties are satisfied on \( U \)

1. \( v \) is parallel along \( c \),
2. the norm \( \|v\| \) is constant on \( c \).

**Remark 3.1.** We notice the interesting similarity when we compare the notions of geodesics and parallel vector fields with the ones of Riemannian geometry, see Table 1.

For the metrical properties (inner product, geodesic, norm), we need not only the horizontal property of \( HTM \), but also the one of \( TM \). We may define the notion of parallel vector fields on \( M \) in Finsler geometry. It is Proposition 3.1, namely, if the lift \( \tilde{v} = (x, v, v) \) to \( HTM \) is tangential to the \( n \)-dimensional subspace spanned by \( \frac{\delta}{\delta x} \), then we call \( v \) the parallel vector field on \( M \).
<table>
<thead>
<tr>
<th>Riemannian space $(M, g)$</th>
<th>Finsler space $(M, F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I) $v$ is parallel along $c$</td>
<td>$\frac{dv^r}{dt} + v^j \Gamma^r_{ji}(c) \dot{c}^i = 0$</td>
</tr>
<tr>
<td>$(c, v)$ is horizontal of $TM$</td>
<td>$(c, \dot{c}, v)$ is horizontal of $HTM$</td>
</tr>
<tr>
<td>(II) the inner product</td>
<td></td>
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<tr>
<td>it is constant on $c$</td>
<td>$(c, \dot{c})$ is horizontal of $TM$ if $c$ is a path, it is constant on $c$</td>
</tr>
<tr>
<td>(III) $c$ is a geodesic</td>
<td></td>
</tr>
<tr>
<td>$(c, \dot{c})$ is horizontal of $TM$</td>
<td>$(c, \dot{c})$ is horizontal of $TM$</td>
</tr>
<tr>
<td>$c$ is an autoparallel curve</td>
<td>$c$ is a path</td>
</tr>
<tr>
<td></td>
<td>$(c, \dot{c}, \ddot{c})$ is horizontal of $HTM$</td>
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<td>(IV) $v$ is a parallel vector field</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial v^r(x)}{\partial x^j} + \Gamma^r_{ij}(x) v^j(x) = 0$</td>
<td>$\frac{\partial v^r(x, v)}{\partial x^j} + F^r_{ji}(x, v) v^j = 0$</td>
</tr>
<tr>
<td>$(x, v)$ is horizontal of $TM$</td>
<td>$(x, v, v)$ is horizontal of $HTM$</td>
</tr>
<tr>
<td>$v$ is parallel along any $c$</td>
<td>$v$ is parallel along the curve $c$ that is the solution of $(3.17)$</td>
</tr>
<tr>
<td>the norm $|v|$ is constant on $M$</td>
<td>$\frac{\partial v^r(x, v)}{\partial x^j} + N^r_{ij}(x, v) = 0$</td>
</tr>
<tr>
<td></td>
<td>if $(x, v)$ is horizontal of $TM$, the norm $|v|$ is constant on $c$</td>
</tr>
</tbody>
</table>

**Table 1**

**References**


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