ON CONCIRCULAR AND TORSE-FORMING VECTOR FIELDS ON COMPACT MANIFOLDS

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Abstract. In this paper we modify the theorem by E. Hopf and found results and conditions, on which concircular, convergent and torse-forming vector fields exist on (pseudo-) Riemannian spaces. These results are applied for conformal, geodesic and holomorphically projective mappings of special compact spaces without boundary.

1. Introduction

Concircular and torse-forming vector fields on Riemannian manifolds and manifolds with affine connection were studied by K. Yano [21, 22]. We studied the theorem by E. Hopf (see [23], p. 26) about the existence of solutions of differential equations in partial derivations and we found some interesting result. If we modify this theorem we can prove that on a compact Riemannian manifold $V^n$ with an indefinite metric there are no global torse-forming vector fields and we can also determine other examples for which these fields do not exist.

Immediately we can find these results in the theory of conformal, geodesic, holomorphically projective and almost geodesic mappings and transformations. For example, conformal transformations of Einstein spaces [2, 16, 12], geodesic, holomorphically projective mappings and transformations of semisymmetric, Ricci-semisymmetric and other spaces are connected with the existence of concircular vector fields on them, see [11, 12, 13, 17]. We generalize and intensify the results which were achieved in [18, 19, 20].

Examples of the existence of global concircular, convergent and torse-forming vector fields are introduced in [14, 15].

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First we modify the Theorem by E. Hopf (see [23], p. 26).

**Theorem 1.** Let $M_n$ be a compact $n$-dimensional manifold and $\varphi \in C^2$ be a scalar function on $M_n$. Let at every point $P_0 \in M_n$ there exists a coordinate neighbourhood $U(x_1, x_2, \ldots, x_n) \subset M_n$, on which there exist continuous functions $A^{ij}(x)$ and $B^i(x)$ at a point $P(x) \in U$ such that

\[
A^\alpha\beta(x) \frac{\partial^2 \varphi(x)}{\partial x^\alpha \partial x^\beta} + B^\alpha(x) \frac{\partial \varphi(x)}{\partial x^\alpha} \geq 0 \ (\leq 0),
\]

holds all over $U$, and $A^{\alpha\beta}(x)z_\alpha z_\beta$ is a positive definite quadratic form on $U$.

Then $\varphi \equiv \text{const}$ on $M_n$.

**Remarks.**
- Signs “$\leq$” or “$\geq$” in the inequalities (1) for every neighbourhood are identical. Here and further we consider that the studied spaces are connected and boundless.
- Evidently we can replace partial derivatives by covariant derivatives on spaces with affine connection and on Riemannian spaces [3, 23].
- In Theorem 1 it is not demanded that functions $A^{ij}(x)$ and $B^i(x)$ are defined on every coordinate neighbourhood by geometric objects globally on $M_n$ as it is demanded in the theorem by S. Bochner and K. Yano [23], p. 26, and its applications [19].

**Proof.** On a compact manifold $M_n$ there is possible to select a set of coordinate neighbourhoods $U$ so that their union covers $M_n$ and on each of them there hold equations (1); due to the determinacy we give the sign “$\geq$”.

Because a function $\varphi$ is continuous and a manifold $M_n$ is compact, $\varphi$ reaches its maximum at a point $P_0 \in U_0$ where $U_0$ is one of the neighbourhoods. Hence $\varphi(P) \leq \varphi(P_0)$ for all $P \in U_0$. On $U_0$ there holds every conditions of the theorem by E. Hopf (see [23], p. 26) and according to it holds $\varphi(P) = \varphi(P_0)$ for all $P \in U_0$.

Further we take a coordinate neighbourhood $U_1$ which covers $U_0$. Obviously $\varphi(P) \leq \varphi(P_0)$ for all $P \in U_1$, however $\varphi(P) = \varphi(P_0)$ for all $P \in U_1$. Similarly we can choose all selected neighbourhoods $U$. Because the number of these neighbourhoods is finite and $M_n$ is connected, we verify that $\varphi(P) = \varphi(P_0)$ for all $P \in M_n$.\qed

This theorem is possible to prove as well as in case if functions $A^{ij}$ and $B^i$ depend also on $\varphi(x)$. We have the following theorem.

**Theorem 2.** Let $M_n$ be a compact $n$-dimensional manifold and $\varphi \in C^2$ be a scalar function on $M_n$. Let at every point $P_0 \in M_n$ there exist a coordinate neighbourhood $U(x_1, x_2, \ldots, x_n) \subset M_n$, on which there exist continuous functions $A^{ij}(x, \varphi(x))$ and $B^i(x, \varphi(x))$ at a point $P(x) \in U$ such that

\[
A^{\alpha\beta}(x, \varphi(x)) \frac{\partial^2 \varphi(x)}{\partial x^\alpha \partial x^\beta} + B^\alpha(x, \varphi(x)) \frac{\partial \varphi(x)}{\partial x^\alpha} \geq 0 \ (\leq 0),
\]
3.

Concircular and torse-forming vector fields

A vector field $\Phi$ defined on a space $A_n$ with affine connection is called \textit{torse-forming} if it holds:

\begin{equation}
\nabla_X \Phi = \nu X + \mu(X) \Phi
\end{equation}

where $\nu$ is a function, $\mu(X)$ is a linear form, $X$ is an arbitrary vector field, and $\nabla_X$ is a covariant derivation on $A_n$ respectively to a vector $X$.

In local transcriptions we can write

\begin{equation}
\varphi^h_{,i} = \nu \delta^h_i + \mu_i \varphi^h
\end{equation}

where $\varphi^h$ and $\mu_i$ are components of $\Phi$ and $\mu(X)$, respectively, “,” means a covariant derivation on $A_n$, and $\delta^h_i$ is the Kronecker symbol.

A torse-forming field we call \textit{concircular} if $\mu(x) = 0$ and $\nu$ is an arbitrary function. If $\mu(X) = 0$ and $\nu = \text{const}$ this field is \textit{convergent}.

If $\mu(X)$ is a gradient-like (i.e. it holds $\mu(X) = \nabla_X \hat{\mu}$, where $\hat{\mu}$ is a function) then a vector field $\Theta = \exp(-\hat{\mu})\Phi$ satisfying equations $\nabla_X \Theta = \rho X$ where $\rho = \nu \exp(-\hat{\mu})$ is concircular too. Vice versa, if $\Theta$ is concircular then every vector field which is collinear with $\Theta$ is torse-forming too.

On a Riemannian space $V_n$ with a metric tensor $g(X,Y)$ we consider a linear form $\varphi(X) = g(X,\Phi)$. Locally this form $\varphi(X)$ is always gradient.

A torse-forming (including concircular and convergent) vector field will be called \textit{gradient} if a linear form $\varphi(X)$ is gradient, i.e. that on $V_n$ there exists a scalar function $\varphi$ for which $\varphi(X) = \nabla_X \varphi$. A form $\mu(X)$ corresponding to gradient torse-forming fields $\Phi$ is collinear to a form $\varphi(X)$. Subsequently, we can write equations (3) for these vector fields as follows:

\begin{equation}
\nabla_X \nabla_Y \varphi = \nu g(X,Y) + \tau \nabla_X \varphi \nabla_Y \varphi,
\end{equation}
where \( \nu, \tau \) are functions.

For concircular and torse-forming fields it is naturally required that

\[
\Phi, \varphi \in C^1; \quad \varphi^* \in C^2; \quad A_n \in C^0; \quad V_n \in C^1.
\]

See a local expression

\[
\Phi^h(x) \in C^1; \quad \varphi(x) \in C^1; \quad \varphi^*(x) \in C^2; \quad \Gamma^h_{ij}(x) \in C^0; \quad g_{ij}(x) \in C^1,
\]

where \( \Gamma^h_{ij} \) are components of affine connection \( \nabla \) on \( A_n \), \( g_{ij} \) are components of a metric tensor on Riemannian space \( V_n \), \( C^r \) is the class of the continuity.

Naturally, \( \nu \in C^0 \) and \( \tau \in C^0 \).

4. Concircular and torse-forming vector fields globally on compact manifolds

Below, we study some sufficient conditions on which torse-forming fields on compact spaces do not exist.

It holds

**Theorem 3.** On compact pseudo-Riemannian spaces \( V_n \in C^1 \) there exist only vanishing torse-forming fields.

**Proof.** Let on \( V_n \in C^1 \) there exist the mentioned vector fields, i.e. there exists a scalar function \( \varphi \in C^2 \) satisfying (5). We assume that \( \varphi (\neq \text{const}) \), \( \nu \) and \( \tau \) are continuous functions. We prove that this solution does not exist.

Remark that the pseudo-Riemannian space \( V_n \) has an indefinite metric \( g \).

Let \( P_0 \) be some point on \( V_n \). There exist such a coordinate neighbourhood \( U^*(x) \) as

\[
g_{ij}(P_0) = \text{diag}(1, -1, e_3, \ldots, e_n), \quad (e_i = \pm 1).
\]

We get

\[
A(x) \overset{\text{def}}{=} \frac{-1}{g_{11}(x)} (n g_{22}(x) + g_{33}(x) + \cdots + g_{nn}(x)).
\]

Evidently, \( A(P_0) \geq 2 \). Therefore there exists a domain \( U \subset U^* \) which includes a point \( P_0 \) for which \( A(x) > 0 \). Hence

\[
A_i^j(x) \overset{\text{def}}{=} \text{diag}(A(x), n, 1, \ldots, 1)
\]

determine on a domain \( U \) the positive form \( A^\alpha_\beta(x) z_\alpha z_\beta \) as well as on this domain there holds

\[
A^\alpha_\beta(x) g_{\alpha\beta}(x) = 0.
\]

The local expression of (5) has the following form

(6)

\[
\varphi,_{ij} - \tau \varphi,_{i} \varphi,_{j} = \nu g_{ij}.
\]

Contracting the last formula with \( A^i_j \), we get

(7)

\[
A^\alpha_\beta \varphi,_{\alpha\beta} - B^\alpha \varphi,_{\alpha} = 0,
\]

where \( B^\alpha \equiv \tau A^i_j \varphi,_{\beta}. \)
Evidently, according to Theorem 2 it holds $\varphi \equiv \text{const}$ on $V_n$.

**Theorem 4.** On compact Riemannian spaces $V_n \in C^1$ there do not exist torse-forming fields satisfying condition $\nu \geq 0$ (or $\nu \leq 0$) on whole $V_n$.

**Proof.** According to Theorem 3 it remains to study only the case if on $V_n$ there is a positive definite metric.

We consider equations (5) respective to a function $\varphi$. On coordinate neighbourhoods $U$ these equations have the form (6) and after the contraction (6) with $A_{ij} \equiv g_{ij}$ (where $\|g_{ij}\| = \|g_{ij}\|^{-1}$) we get

$$A^\alpha_{\beta \alpha} \varphi_{, \beta} - B^\alpha \varphi_{, \alpha} = n \nu,$$

where $B^i \equiv \tau A_{i\beta} \varphi_{, \beta}$. The right side of (8) is either every non-negative or non-positive. Considering a positive definite metric the conditions of Theorem 2 hold. Hence $\varphi \equiv \text{const}$.

Evidently, the conditions for concircular fields are contained in Theorems 3 and 4. The next follows from Theorem 4:

**Lemma 1.** On compact Riemannian spaces $V_n \in C^1$ there do not exist gradient convergent vector fields.

In fact, conditions of Theorem 4 in this case are $\nu = \text{const}$ then they satisfy conditions of the Theorem 4.

**Lemma 2.** On compact Riemannian spaces $V_n \in C^1$ there do not exist covariantly non-constant convergent vector fields.

**Proof.** On $V_n$ there is a convergent vector field $\Phi$ satisfying $\nabla_X \Phi = \mu X$, $\mu$ is constant. We consider a function $\lambda \equiv g(\Phi, \Phi)$. It is easy to see, that $\nabla_X \nabla_Y \lambda = 2\mu g(X, Y)$. According to Lemma 1 it follows $\lambda \equiv \text{const}$. Hence $\mu = 0$. Then it follows $\nabla_X \Phi = 0$.

In [17] there are studied torse-forming vector fields $\Phi$ satisfying:

$$g(\Phi, \Phi) = e; \quad \nabla_X \Phi = \nu (X - e g(X, \Phi) \Phi),$$

where $e = \pm 1$.

These conditions hold on normalization non-vanishing non-isotropic torse-forming vector fields.

Therefore we will call a vector field $\Phi$ satisfying (9) a normalized torse-forming field. Naturally, the Theorems 1 and 3 hold as well as for these fields. It is shown that the Theorems 1 and 3 generalize the results introduced in [18].
5. Applications of achieved results

The above introduced results, we use for conformal, geodesic and holomorphically-projective mappings of compact manifolds without boundary.

(Pseudo-) Riemannian manifold \( V_n \) admits conformal mappings onto (pseudo-) Riemannian manifold \( V_n' \) if their metrics are connected by the following condition 
\[ \bar{g} = \varphi \cdot g \]
where \( \varphi \) is a function. If \( \varphi \equiv \text{const} \), then the mappings are homothetic. A conformal mapping \( V_n \to V_n' \) is called conformal transformation on \( V_n \).

H. W. Brinkman, see [16], proved that if an Einstein space \( V_n \) admits a conformal mapping onto Einstein space \( \bar{V}_n \), then a gradient (like a vector field) \( \nabla \varphi \) is concircular.

The next Theorem follows from Theorem 3.

**Theorem 5.** If a compact pseudo-Riemannian Einstein space \( V_n \) \( (n \geq 3) \) admits a conformal mapping onto Einstein space \( \bar{V}_n \), then this mapping is homothetic.

**Theorem 6.** If a compact pseudo-Riemannian Einstein space \( V_n \) \( (n \geq 3) \) admits a conformal transformation, then this transformation is homothetic.

**Remark.** Theorems 5 and 6 fail for classical Riemannian metrics (even if we replace light-line by usual completeness) – Möbius transformations of the standard round sphere and the stereographic map of the punctured sphere to the Euclidean space are conformal nonhomothetic mappings. One can construct other examples on warped Riemannian manifolds, see [6].

If semisymmetric, Ricci-semisymmetric, Kählerian, Ricci-flat, and flat (pseudo-) Riemannian spaces admit non-trivial geodesic mappings, then on these spaces there exists nonvanishing convergent vector field, see [9, 10, 8, 7, 11].

Analogically, if semisymmetric, Ricci-semisymmetric, and flat Kählerian spaces admit non-trivial holomorphically-projective mappings, then also on these spaces there exists nonvanishing convergent vector field, see [1, 10, 12].

According to Lemma 2, above mentioned compact special spaces do not admit global geodesic and holomorphically-projective mappings.

Finally, we remark that I. Hinterleitner and V. Kiosak introduced \( \varphi \)-Ric vector fields by the following equation \( \varphi_{i,j} = \mu R_{i,j} \), where \( \mu \) is a constant, \( R_{i,j} \) is the Ricci tensor, see [4]. One can use Theorem 1 for a gradient vector field \( \varphi_i \).

**References**


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