ON THE SHIFT-WINDOW PHENOMENON OF
SUPER-FUNCTIONS

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ABSTRACT. Using exploded numbers we consider the exploded Descartes-plane
$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ in which have the graphs of super-functions. Of
course certain parts of these graphs are visible in the traditional Descartes-
plane $\mathbb{R}^2$, while the other parts may be invisible. To show the invisible parts we
introduce the super shift - and screw transformations which result the shifted -
and screwed Descartes - coordinate systems, for the sake of understanding
the paper contains some examples, too.

1. Computation with exploded numbers

In [1] we introduced the exploded numbers with the operations of super-addition
and super-multiplication
\begin{equation}
\mathcal{F} \circ \mathcal{F} = x + y, \quad x, y \in \mathbb{R},
\end{equation}
and
\begin{equation}
\mathcal{F} \circ \mathcal{F} = x - y, \quad x, y \in \mathbb{R},
\end{equation}
respectively. Moreover, the rules of super-subtraction and super-division
\begin{equation}
\mathcal{F} \circ \mathcal{F} = \frac{x}{y}, \quad x, y \in \mathbb{R},
\end{equation}
and
\begin{equation}
\mathcal{F} \circ \mathcal{F} = x \cdot y, \quad x, y \in \mathbb{R}, \quad y \neq 0,
\end{equation}
were presented, too. The number $\mathcal{F}$ was called the exploded of the real number
$x \in \mathbb{R}$ and the set of exploded numbers is denoted by $\mathbb{R}_e$. Sometimes we say that
$\mathbb{R}_e$ is the exploded real axis. For any real number $x \in (-1,1)$ explosion means
\begin{equation}
\mathcal{F} = \text{area} \times x.
\end{equation}

Clearly, $\mathbb{R}$ is isomorphic with $\mathbb{R}_e$ by the transformation
\begin{align*}
x \rightarrow \mathcal{F}, \\
+ \rightarrow \circ, \\
\cdot \rightarrow \div.
\end{align*}
Moreover,
\begin{equation}
0 = \mathfrak{U}
\end{equation}
is the unit element of super-addition and the unit element of super-multiplication is \( \mathfrak{U} \).

Considering that by (1.5) for any \( x \in (-1,1) \) we have
\begin{equation}
\overline{-x} = -\mathfrak{U}
\end{equation}
for any \( x \in \mathbb{R} \) its additive inverse is denoted by \((-x)\).

For any pair \( x, y \in \mathbb{R} \) the arrangement
\begin{equation}
x < y \quad \text{if and only if} \quad \overline{z} < \overline{y}
\end{equation}
was also introduced in [1] (see [1], Definition 1.7), where \( \overline{z} \) is called the compressed of \( x \), defined by the identity
\begin{equation}
\overline{(\overline{x})} = x, \quad x \in \mathbb{R}.
\end{equation}
Clearly, for any \( x \in \mathbb{R} \) we have \( x \in R \). The inverse of explosion is called compression. Moreover,
\begin{equation}
\overline{x} = \text{th}_{\overline{x}}, \quad \text{if} \quad -\infty < x < \infty,
\end{equation}
and we also use the identity
\begin{equation}
\overline{(\overline{x})} = x, \quad x \in R.
\end{equation}
Clearly, for any \( x \in R \) and \( \bar{x} \in \mathbb{R} \)
\begin{equation}
\overline{-x} = -\mathfrak{U} \quad \text{and} \quad \overline{-\bar{x}} = -\mathfrak{U}
\end{equation}
respectively. If \( x \in \mathbb{R} \) and \( x \leq -1 \) or \( x \geq 1 \) we say that \( x \) is invisible. The visible subset of the exploded real axis \( \mathbb{R} \) is the real axis \( R \) itself. So, if \( x \in R \) then
\(-1 \leq x \leq 1\).

2. Super-functions and super-curves

The concept of super-function was introduced in [1]. (See [1], (part 4).) Considering a basic function \( f \) with its definition-domain belonging to \( R \) we said that an \( x \in \mathbb{R} \) belongs to the definition-domain of super-function spr \( f \) if \( f(\overline{x}) \) is defined. Moreover,
\begin{equation}
spr f(x) = f(\overline{x}).
\end{equation}

Remark 2.2. If \( S \subset \mathbb{R} \) and for any \( x \in S \)
\begin{equation}
y = F(x)
\end{equation}
is unambiguous, then \( F \) is a super-function.

Really, considering
\[
 f_F(x) = \overline{F(\overline{x})}, \quad x \in S \subset R
\]
where \( S \) is the set of the compressed numbers of elements belonging to \( S \) as a basic function, we have by (2.1) and (1.9) that
\[
\text{spr} f_F(x) = \overline{f_F(\overline{x})} = \overline{(F(\overline{x}))} = F(x).
\]

The graphs of super-functions are situated in the exploded Descartes-plane \( \mathbb{R}^2 \) which is a super-linear space with the operations
\[
 X \oplus \overline{Y} = (x_1 - y_1, x_2 - y_2), \quad X = (x_1, x_2), \quad Y = (y_1, y_2)
\]
and
\[ c \circ \bigcirc X = (c \circ \bigcirc x_1, c \circ \bigcirc x_2), \quad x_1, x_2, y_1, y_2 \text{ and } c \in \mathcal{H} \]
as well as
\[ X \circ \bigcirc Y = (x_1 \circ \bigcirc y_1, x_2 \circ \bigcirc y_2). \]
Moreover, \( \mathcal{R}^2 \) is a super-euclidian space with the super-inner-product
\[ X \circ \bigcirc Y = (x_1 \circ \bigcirc y_1) \circ \bigcirc (x_2 \circ \bigcirc y_2) \]
and it is super-normed with
\[ ||X||_{\mathcal{R}^2} = \text{spr} \sqrt{X \circ \bigcirc X}. \]

If \( X, Y, Z \in \mathcal{R}^2 \) and \( X \neq Z, Y \neq Z \) then we say
\[
\text{meas} \text{ spr} \angle XYZ = \text{spr} \arccos \left( \left( \frac{\langle X \circ \bigcirc Z \rangle \circ \bigcirc \langle Y \circ \bigcirc Z \rangle}{\langle X \circ \bigcirc Z \rangle \circ \bigcirc \langle X \circ \bigcirc Z \rangle_{\mathcal{R}^2} \circ \bigcirc \langle Y \circ \bigcirc Z \rangle_{\mathcal{R}^2, 75} \rangle \right) \right).
\]

The super-curve \( G \) is defined as follows:
\[
G = \{ X = (x, y) \in \mathcal{R}^2 : x = x(t), y = y(t), \quad \alpha \leq t \leq \beta, \quad (\alpha, \beta) \in \mathcal{R} \}
\]
such that \( u = x(t) \) and \( v = y(t) \) are continuously differentiable functions of the parameter \( \varphi = \frac{t}{\mathcal{L}} \) on the interval \( [\alpha, \beta] \). Moreover, we require for any \( \varphi \in [\alpha, \beta] \) that
\[(u'(\varphi))^2 + (v'(\varphi))^2 \neq 0.\]

An important special case is the super-line \( L_C(V) \) defined by the equation
\[
X = C \circ \bigcirc (t \circ \bigcirc V), \quad t \in \mathcal{R},
\]
where \( C = (c_1, c_2), V = (v_1, v_2) \in \mathcal{R}^2 \) are given such that
\[
(v_1 \circ \bigcirc v_1) \circ \bigcirc (v_2 \circ \bigcirc v_2) = \mathcal{T}.
\]
The super measure of super-angle of super-lines \( L_C(V) \) and \( L_C(U) \) with
\[
||U||_{\mathcal{R}^2} = \mathcal{T}
\]
is defined as follows:
\[
\text{meas} \text{ spr} \angle (L_C(V), L_C(U)) = \min(\text{spr} \arccos(V \circ \bigcirc U), \text{spr} \arccos(-V \circ \bigcirc U)).
\]

3. Window phenomenon and shift-window phenomenon

Considering a super-curve \( G \) (which may be a graph of a super-function) the intersection
\[
G \cap \mathcal{R}^2
\]
is called the window phenomenon of \( G \) (or a super-function of \( f \)). For example, the window phenomenon of the super-unit-circle with the centre origo \( ||x||_{\mathcal{R}^2} = \mathcal{T} \) is itself the super-unit-circle except for points \((-\mathcal{T}, 0), (0, \mathcal{T}), (\mathcal{T}, 0)\) and \((0, -\mathcal{T})\) which are invisible points in the window \( \mathcal{R}^2 \).
Considering a fixed point \((x_0, y_0) \in \overline{R^2}\) we introduce the super-shift transformation
\[
x = \xi - x_0
\]
\[
y = \eta - y_0
\]
which moves the exploded Descartes-coordinate system “\(x, y\)” into the system “\(\xi, \eta\)” having the new coordinate axes “\(\xi\)” and “\(\eta\)” with the new origo \((x_0, y_0)\). While the Descartes-plane
\[
-\Upsilon < x < \Upsilon
\]
\[
-\Upsilon < y < \Upsilon
\]
is moving on the exploded Descartes-plane \(\overline{R^2}\) into the Descartes-plane
\[
x_0 - \Upsilon < x < x_0 + \Upsilon
\]
\[
y_0 - \Upsilon < y < y_0 + \Upsilon
\]
we can see the another subset of \(\overline{R^2}\). This transformation gives a new window on the wall \(\overline{R^2}\).

We can observe this transformation on the compressed model where the compressed wall \(\overline{R^2}\) is \(R^2\) and the compressed window \(R^2\) is \(\overline{R^2}\), that is, the set of points \((x, y)\) for which
\[
-1 < x < 1
\]
\[
-1 < y < 1 \quad (x, y) \in R^2.
\]

By the compression the super-shift transformation becomes the familiar shift-transformation
\[
\xi = \xi + x_0
\]
\[
y = \eta + y_0
\]
and the new sub-window is the set of points \((\xi, \eta)\) for which
\[
x_0 - 1 < \xi < x_0 + 1
\]
\[
y_0 - 1 < \eta < y_0 + 1.
\]

**Example 3.2.** In the window we can see the visible parts of graphs of super-parabola
\[
y = x - x^2, \quad x \in \overline{R}
\]
and of super-square-root function \(\sqrt[3]{x} = \sqrt[3]{x}, x \geq 0\). Their super-curves have two common points \((0, 0)\) and \((1, 1)\) such that the latter is invisible. Using the super-shift transformation (3.1) with \((x_0, y_0) = (1, 1)\), the new equation of super-parabola is
\[
\eta = (\xi - 1) - \xi, \quad \xi \in \overline{R}
\]
while the super-square-root has the equation
\[
\eta = \sqrt[3]{\xi - 1} - \xi, \quad -\Upsilon \leq \xi \leq \Upsilon
\]
having the equations of window-curves (window phenomena)
\[
\eta = \text{areath}(\text{th } \xi)(\text{th } \xi + 2)), \quad -\infty < \xi < \text{areath}(\sqrt{2} - 1)
\]
and
\[ \eta = \text{area th}(\sqrt{\sinh x + 1} - 1), \quad -\infty < x < \infty, \]
respectively. Their intersection is visible in the new window:

**Example 3.3.** Considering the super-function

\[ (3.4) \quad y = (2 - \sqrt{x}) - \sqrt{(1 - x)} (x - \sqrt{x}), \quad x \in H \]

by (1.6), (1.9), (1.2) (1.1), (1.4), (1.10) and (1.5) we can see that for any \( x \in R \)

\[ (2 - \sqrt{x}) - \sqrt{(1 - x)} (x - \sqrt{x}) = \left( \frac{2x}{1 + x^2} \right) = \]

\[ = \frac{2 \sinh x}{1 + \sinh^2 x} = \text{th} 2x = \text{area th}(\text{th} 2x) = 2x \]

is obtained. This means that the (familiar) line, having the equation

\[ (3.5) \quad y = 2x \quad (x \in R), \]

is the window-curve of the super-function defined under (3.4). In the window
the graph of super-function appears to be a line. Applying the super-shift transforma-
tion (3.1) with \( (x_0, y_0) = (1, 1) \) we have the new equation of the original
super-function (3.4):

\[ (3.6) \quad \eta = -((\xi - \sqrt{\xi} - \xi) - \sqrt{(\eta - \sqrt{\eta} - (\xi - \sqrt{\xi} - \eta)))}, \quad \xi \in H. \]

The window phenomenon of (3.6) has the equation

\[ (3.7) \quad \eta = \frac{1}{2} \ln \frac{2(e^{4\xi} + e^{2\xi})}{3e^{4\xi} + 1}, \quad \xi \in R \]

and we can see it in the shift-window which is the following subset of \( \mathbb{H} \):

\[ (3.8) \begin{align*}
0 < x < \sqrt{n} \\
0 < y < \sqrt{n}
\end{align*} \]

Let us consider the points \( O = (0, 0), P_1 = (1, 1), P_2 = (1, 2) \) and \( \Omega = (1, 1) \) of
the graph of super-function (3.4). Points \( O, P_1, \) and \( P_2 \) are situated on the line
(3.5) but $\Omega$ is invisible. On the other hand $P_1$, $P_2$ and $\Omega$ has new coordinates in
the Descartes-system $\{\xi, \eta\}$:

$$\left(\frac{1}{2}, \frac{1}{2}, \eta \right), (1, \frac{1}{2}, \eta), (2, \frac{1}{2}, \eta) \text{ and } (0, 0)$$

respectively, so we can see them in the shift-window (3.8) on the window-phenomenon (3.7):

In Fig. 3.9 the point $O$ is invisible. However, another super-shift transformation, namely (3.1) with

$$(x_0, y_0) = \left(\frac{\eta}{2}, \frac{\eta}{2}\right),$$

results that $O, P_1, P_2$ and $\Omega$ are visible in the new window. In this case, the new equation of the super-function defined under (3.4) is

$$\eta = \left(\left(\xi - \frac{1}{2}\right) - \xi \right) \left(\eta - \frac{1}{2}\right) \left(\xi - \frac{1}{2}\right) \left(\eta - \frac{1}{2}\right), \quad \xi \in \mathbb{R}$$

(3.10)

The window phenomenon of (3.10) has the equation

$$\eta = \operatorname{arctanh} \frac{\frac{1}{2}(\operatorname{th} \xi)^2 + \frac{3}{2} \operatorname{th} \xi + \frac{3}{2}}{(\operatorname{th} \xi)^2 + \operatorname{th} \xi + \frac{3}{2}}, \quad \operatorname{arctanh}(\sqrt{3} - \frac{5}{2}) < \xi < \infty$$

(3.11)

and we have:

\[\text{Figure 3.12.}\]
The case of super-shift transformation (3.1) with \((x_0, y_0) = (0, \xi)\):

\[
x = \xi \quad \text{and} \quad y = \eta \frac{\xi}{\xi + 1}.
\]

The new equation of super-function defined under (3.4) is the following:

\[
(3.13) \quad \eta = -\left(\left((\xi - \xi - \xi) - \xi - (\xi - \xi)\right) - \xi - (\xi - \xi - (\xi - \xi))\right), \quad \xi \in \mathbb{R}
\]

The window phenomenon of (3.13) is

\[
(3.14) \quad \eta = \frac{1}{2} \ln \frac{e^{2\xi} - 1}{e^{2\xi} + 3}, \quad 0 < \xi < \infty,
\]

which shows that both \(O\) and \(\Omega\) are invisible:

![Figure 3.15.](image)

The super-shift transformation (3.1) with \((x_0, y_0) = (\xi, 0)\) is the most interesting. Now, the shift-window is the following subset of \(\mathbb{R}^2\):

\[
(3.16) \quad 0 < x < \xi, \quad -\infty < y < \infty.
\]

Similarly to the case mentioned above, \(O\) and \(\Omega\) are invisible again. The new equation of super-function defined under (3.4) is the following:

\[
(3.17) \quad \eta = \left((\xi - \xi - \xi) - \xi - (\xi - \xi)\right) - \xi - \left((\xi - \xi - \xi) - \xi - (\xi - \xi)\right), \quad \xi \in \mathbb{R}
\]

The window phenomenon of (3.17) has the equation

\[
(3.18) \quad \eta = \arctan \frac{2\xi + 1}{1 + (\xi + 1)^2}, \quad -\infty < \xi < \infty, \quad \xi \neq 0
\]

By Fig. 3.9, 3.12, 3.15 and 3.19 it seems that \(\Omega\) is the maximum point of the graph of the super-function (3.4). Really, we have

**Remark 3.20.** Considering that for any \(x \in \mathbb{R}\) the inequalities

\[
-1 \leq \frac{2x}{1 + (x^2)} \leq 1
\]
are valid such that the equalities are valid if and only if the left-hand side, $z = -1$ and, on the right-hand side, $z = 1$. Hence (1.6), (1.7) and (1.8) yield
\[(3.21) \quad -T \leq (y - (x - x)) - (1 - (x - x)) \leq T, \quad x \in \mathbb{R}
\]
with the equalities on the left-hand or right-hand sides in the cases $x = -T$ or $x = T$, respectively.

Remark 3.22. By (3.21) we can see that the super-function defined under (3.4) is bounded but not constant. So, we have that (3.4) is not a super-line though its window phenomenon is line (3.5).

Example 3.23. Let us consider the super-square determined by the super-lines
\[
\begin{align*}
L_{\omega, 1} : y &= x \\
L_{-\infty, \infty} : y &= 1 - (x) \\
L_{\infty, -\infty} : y &= x - T \\
L_{-\infty, 0} : y &= -x
\end{align*}
\]
\[(3.23)\]

Using the super-shift transformation (3.1) with $(x_0, y_0) = ((\frac{T}{2}), 0)$ we can see the symmetric form of the super-square in the shift window:
\[
(\frac{T}{2}) < x < (\frac{T}{2}), \\
-\infty < y < \infty
\]
where the new equations of super-lines are:
\[
\begin{align*}
L_{\omega, 0} : \eta &= \xi - (\frac{T}{2}) \\
L_{-\infty, \infty} : \eta &= (\frac{T}{2}) - \xi \\
L_{\infty, -\infty} : \eta &= \xi - (\frac{T}{2}) \\
L_{-\infty, 0} : \eta &= -\xi - (\frac{T}{2})
\end{align*}
\]
with the equations of window phenomena $y = \text{areath}(\theta \xi + \frac{T}{2})$, $y = \text{areath}(\frac{T}{2} - \theta \xi)$, $y = \text{areath}(\theta \xi - \frac{T}{2})$, and $y = -\text{areath}(\theta \xi + \frac{T}{2})$, respectively.
Example 3.25. That the super-lines

\[ L_{\infty, \infty} : y = x \]

\[ L_{2, 0} : y = \frac{x}{2} \]

have the common (invisible) point \( M = (1, 2) \). It is easy to see that point \( Q = (-1, 0) \) lies on \( L_{\infty, \infty} \) and \( O = (0, 0) \) lies on \( L_{2, 0} \). Using (2.4), by (1.1) - (1.4), (1.6), (1.7) and (2.1) we compute

\[ \text{meas } spr \angle QMO = \text{area th} \left( \arccos \frac{3}{\sqrt{10}} \right). \]

By the super-shift transformation (3.1) with \((x_0, y_0) = M\), we can see the intersection in the shift-window:

\[ 0 < x < \frac{y}{2}, \frac{y}{2} < y < 3. \]

Now, the new equations of super-lines are

\[ L_{\infty, \infty} : \eta = \xi \]

\[ L_{2, 0} : \eta = \frac{y}{2} - \xi \]

with the equations of shift-window phenomena \( \eta = \xi, (\xi \in R) \) and \( \eta = \text{area th}(2 \text{ th } \xi), (|\xi| < \text{area th } \frac{1}{2}) \) respectively.
Example 3.27. Let us consider super-lines

\[(3.28) \quad L_C(V) : X = C \circledast (t \circledast V)\]

and

\[(3.29) \quad L_C(U) : X = C \circledast (t \circledast U),\]

where

\[C = \left( \frac{2 + \sqrt{3}}{2} , \frac{\sqrt{3}}{2} \right), \quad V = \left( \sqrt{\frac{2 + \sqrt{3}}{4}} , \sqrt{\frac{2 - \sqrt{3}}{4}} \right)\]

and

\[U = \left( \frac{1}{\sqrt{2}} , \frac{1}{\sqrt{2}} \right)\]

Computing their super-angle by (1.1), (1.2), (1.8), (1.9), (1.12), (2.1) and (2.7)

\[
\text{meas supp}(L_C(V), L_C(U)) = \left( \frac{\pi}{6} \right)
\]

is obtained, but the intersection of super-lines cannot be seen in the window.

Now, using the super-shift transformation (3.1) with \((x_0, y_0) = C\), we can see it in the shift-window:

\[
(\frac{\sqrt{3}}{2}) < x < (2 + \frac{\sqrt{3}}{2})
\]

\[
(\frac{\sqrt{3}}{2} - 1) < y < (1 + \frac{\sqrt{3}}{2})
\]

Considering that the equations under (3.28) and (3.29) are equivalent with the equations:

\[y = \frac{\sqrt{3}}{2} \circledast (2 - \sqrt{3} \circledast (x \circledast \frac{2 + \sqrt{3}}{2})))\]

and

\[y = x \circledast 1,\]

respectively. Hence, the new equations of super-lines

\[L_C(V) : \eta = 2 - \sqrt{3} \circledast \xi, \quad \xi \in \mathbb{R}\]

and

\[L_C(U) : \eta = \xi, \quad \xi \in \mathbb{R}\]

Moreover, their shift-window phenomena have equations

\[\eta = \text{areath}(2 - \sqrt{3} \text{th} \xi), \quad \xi \in \mathbb{R}\]

and

\[\eta = \xi, \quad \xi \in \mathbb{R},\]

respectively, and have the graphs below:
4. Screw transformation

Having the exploded Descartes-coordinate system “x, y”, for any point \((x, y)\) in \(\mathbb{R}^2\) we introduce the screw coordinates \(\omega\) and \(\vartheta\) by the equation system

\[
\begin{align*}
x &= (\omega - \gamma - \text{spr cos } \varphi_0 - \theta - \gamma - \text{spr sin } \varphi_0) \\
y &= (\omega - \gamma - \text{spr sin } \varphi_0 - \theta - \gamma - \text{spr cos } \varphi_0)
\end{align*}
\]  

(4.1)

where \(\varphi_0 \in \mathbb{R}\) is a given (exploded) number. The transformation given by the equation system (4.1) is called screw transformation. The name of coordinate system “\(\omega, \vartheta\)” is the screw system.

By (1.1), (1.2), (1.6), (1.9) and (2.1) for any \(\varphi \in \mathbb{R}\) we easily prove the identity

\[
\text{spr sin } \varphi - \text{spr cos } \varphi = 1
\]

and using it, we can solve the equation system (4.1) for \(\omega\) and \(\vartheta\), so

\[
\begin{align*}
\omega &= (x - \gamma - \text{spr cos } \varphi_0 - \theta - \gamma - \text{spr sin } \varphi_0) \\
\vartheta &= (y - \gamma - \text{spr cos } \varphi_0 - \theta - \gamma - \text{spr sin } \varphi_0).
\end{align*}
\]  

(4.2)

The equation systems (4.1) and (4.2) show the mutual and unambiguous connection between coordinates \((x, y)\) and \((\omega, \vartheta)\) which characterise the same point of \(\mathbb{R}^2\). The systems “x, y” and “\(\omega, \vartheta\)” have a common origo.

Considering the points characterised by

\[
\vartheta = 0
\]  

(4.3)

the equation system (4.1) yields

\[
\begin{align*}
x &= \omega - \gamma - \text{spr cos } \varphi_0 \\
y &= \omega - \gamma - \text{spr sin } \varphi_0.
\end{align*}
\]  

(4.4)

If

\[
\varphi_0 = \frac{2\pi}{k}, \quad k = 0, \pm 1, \pm 2, \ldots
\]

then by (4.4)

\[
\vartheta = 0 \quad \text{if and only if} \quad y = 0,
\]

and the screw transformation is identical.
If

$$\varphi_0 = \pi - \frac{\pi}{n} \left( \frac{k}{n} - \frac{2\pi}{n} \right), \quad k = 0, \pm 1, \pm 2, \ldots$$

then by (4.4)

$$\vartheta = 0 \quad \text{if and only if} \quad y = 0,$$

but the point \((x, y) = (\frac{1}{\pi}, 0)\) has the new coordinates \((\omega, \vartheta) = (0, 0)\).

If

$$\varphi_0 = \left( \frac{\pi}{2} - \frac{\pi}{n} \left( \frac{k}{n} - \frac{2\pi}{n} \right) \right), \quad k = 0, \pm 1, \pm 2, \ldots$$

then

\[(4.5)\]

$$\vartheta = 0 \quad \text{if and only if} \quad x = 0,$$

and the point \((x, y) = (\frac{1}{\pi}, 0)\) has the new coordinates \((\omega, \vartheta) = (1, 0)\).

In general, except for the cases (4.5) and (4.6), the equation system (4.4) yields

\[(4.7)\]

$$\vartheta = 0 \quad \text{if and only if} \quad y = (\text{sprt} \varphi_0) - \frac{\pi}{n} x,$$

that is, the “\(\omega\)-axis” of the screwed system is a super-line in the exploded Descartes-coordinate system “\(x, y\)”.  

Similarly, considering the points characterised by

\[(4.8)\]

$$\omega = 0,$$

the equation system (4.1) yields

\[(4.9)\]

$$x = - \left( \vartheta - \frac{\pi}{n} \right) \text{spr} \sin \varphi_0$$

$$y = \vartheta \text{spr} \cos \varphi_0.$$

Hence, except for cases

$$\varphi_0 = \frac{\pi}{n} - \frac{\pi}{n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

we have that

\[(4.10)\]

$$\omega = 0 \quad \text{if and only if} \quad y = (\text{sprct} \varphi_0) - \frac{\pi}{n} x,$$

that is, the “\(\vartheta\)-axis” of the screwed system is a super-line in the exploded Descartes system “\(x, y\)”.  

Moreover, for any \(\varphi_0 \in \mathbb{R}\) we have

\[(4.11)\]

$$\text{mes} \ \text{spr} < \left( \left. \omega - \text{axis} \right|, \left. \vartheta - \text{axis} \right\rangle \right) = \left( \frac{\pi}{2} \right).$$

The window phenomena of screwed axes in the cases

$$\varphi_0 = \frac{k}{n} - \frac{\pi}{4}, \quad k = 0, \pm 1, \pm 2, \ldots$$

are lines.
In the special case \( \gamma_0 = \left( \frac{2}{3} \right) \), the window phenomena of screwed axes having the equations

\[
\text{"} \omega - \text{axis} \quad \gamma = \text{areath}(\sqrt{3}\text{th } x), \quad |x| < \text{areath} \frac{1}{\sqrt{3}}
\]

and

\[
\text{"} \vartheta - \text{axis} \quad \gamma = -\text{areath}(\frac{\text{th } x}{\sqrt{3}}), \quad x \in \mathbb{R}
\]

are situated in the following figure:

![Graph](image)

**Figure 4.12.**

5. **Criterion for super-lines**

Now turning to the lines with equations

(5.1) \[ y = -x, \quad x \in \mathbb{R} \]

and

(5.2) \[ y = 1 - x, \quad x \in \mathbb{R} \]

we can consider as the window phenomena of some kind of super-functions. Clearly, the equation

(5.3) \[ y = -x, \quad x \in \mathbb{R} \]

represents a super-line and its window phenomenon has the equation under (5.1). The case of (5.2) is more complicated. Using (1.1) - (1.4) we can write for any \( x \in \mathbb{R} \)

\[ 1 - x = (1 - \frac{1}{\text{th} 1} x) - \frac{1}{\text{th} 1} (1 - \frac{1}{\text{th} 1} x) \]

which means that the super-function represented by the equation

(5.4) \[ y = (1 - \frac{1}{\text{th} 1} x) - \frac{1}{\text{th} 1} (1 - \frac{1}{\text{th} 1} x), \quad x \in \mathbb{R}, \quad x \neq \frac{1}{\text{th} 1} (> \infty), \]

has the window phenomenon with equation (5.2). The graph of this super-function is not a super-line because starting from

(5.5) \[ f(x) = \frac{\text{th} 1 - x}{1 - (\text{th} 1)x}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{\text{th} 1} \]

by (1.2)-(1.4), (1.6), (1.10) and (2.1) we have

\[ \text{spr } f(x) = (1 - \frac{1}{\text{th} 1} x) - \frac{1}{\text{th} 1} (1 - \frac{1}{\text{th} 1} x), \quad x \neq \frac{1}{\text{th} 1}. \]
Having that (5.5) gives a hyperbola we can say that the super-function (5.4) represents a super-hyperbola.

To see this super-hyperbola in a better position, we use the super-shift transformation (3.1) with

\[(x_0, y_0) = \left(\frac{1}{\text{th} 1}, \frac{1}{\text{th} 1}\right)\]

Now the equation (instead of (5.4)) of super-hyperbola is:

\[(5.6) \quad \eta = \left(\frac{1}{\text{th} 1}\right)^2 - 1 - \frac{\xi}{\zeta}, \quad \xi \in \mathbb{H}, \quad \xi \neq 0\]

and we can see its window phenomenon

\[\eta = \text{area th} \left(\frac{1}{\text{th} \xi} \right)^2 - 1, \quad |\xi| > \text{area th} (\frac{1}{\text{th} 1})^2 - 1\]

in the shift-window:

\[
\begin{align*}
\text{area th} (\frac{1}{\text{th} 1} - 1) < x < 1 + \frac{1}{\text{th} 1} \\
\text{area th} (\frac{1}{\text{th} 1} - 1) < y < 1 + \frac{1}{\text{th} 1}.
\end{align*}
\]

\[\text{Figure 5.7.}\]

Considering the sets

\[S_{x_0} = \{(x, y) \in \mathbb{H}^2 : x = x_0, \quad y \in \mathbb{H} \text{ is arbitrary}\}\]

and

\[S_{y_0} = \{(x, y) \in \mathbb{H}^2 : x \in \mathbb{H} \text{ is arbitrary, } y = y_0\}\]

We see at a glance, that their points form super-lines. (The cases \(x_0 = 0\) and \(y_0 = 0\) are the exploded coordinate-axes.)

If \(x\) is running over \(\mathbb{H}\), and \(y = F(x)\) represents a mutual and unambiguous connection between \(x\) and \(y \in \mathbb{H}\), it is not trivial whether the set

\[(5.8) \quad S = \{(x, y) \in \mathbb{H}^2 : x \in \mathbb{H} \text{ is arbitrary and } y = F(x)\}\]

forms a super-line or not. (Exceptions are \(F(x) = x\) and \(F(x) = -x\).) Now we give a criterion in
Theorem 5.9. Let us assume that $x$ is running over $\mathbb{R}$ and $y = F(x)$ is a mutual and unambiguous connection between $x$ and $y \in \mathbb{R}$. The set (5.8) forms a super-line if and only if there exists one and only one $x_0 \in \mathbb{R}$ so that

\begin{equation}
F(x_0) = 0,
\end{equation}

moreover, there exists a $\varphi_0 \in \mathbb{R}$, $\varphi_0 \neq k - \frac{\pi}{2} \left( \frac{1}{k} \right)$, $k = 0, \pm 1, \ldots$ such that the transformations first, the super-shift transformation

\begin{equation}
x = \xi - \bigcirc \bigcirc \ x_0
\end{equation}

second, the screw transformation

\begin{equation}
\xi = (\omega \bigcirc \bigcirc \ \text{spr cos } \varphi_0 - \bigcirc \bigcirc \ (\theta \bigcirc \bigcirc \ \text{spr sin } \varphi_0)
\end{equation}

\begin{equation}
\eta = (\omega \bigcirc \bigcirc \ \text{spr sin } \varphi_0 - \bigcirc \bigcirc \ (\theta \bigcirc \bigcirc \ \text{spr cos } \varphi_0)
\end{equation}

result that the graph of super-function $F$ becomes the “$\omega$-axis” of the coordinatesystem “$\omega, \theta$” with the origo $(x_0, 0)$.

Example 5.13. Let us consider the set (5.8) with

\begin{equation}
F(x) = \xi - \bigcirc \bigcirc \ x, \quad x \in \mathbb{R}.
\end{equation}

In this case $x_0 = 0$ (see (5.10)), so the super-shift transformation is identical. (See (3.1) and (5.11).) Moreover, the screw transformation (5.12) with the necessary condition $\theta = 0$ has the form

\begin{equation}
x = \omega \bigcirc \bigcirc \ \text{spr cos } \varphi_0
\end{equation}

\begin{equation}
\xi - \bigcirc \bigcirc \ x = \omega \bigcirc \bigcirc \ \text{spr sin } \varphi_0.
\end{equation}

Hence $\varphi_0 = \text{spr arc tg } \frac{\xi}{x}$. So, the set (5.8) forms a super-line.

Example 5.14. Let us consider the set (5.8) with

\begin{equation}
F(x) = \text{spr arc tg } x, \quad x \in \mathbb{R}. \quad (\text{See } (2.1)).
\end{equation}

In this case $x_0 = 0$ (see (5.10)), so the super-shift transformation is identical. (See (3.1) and (5.11).) Moreover, the screw transformation (5.12) with the necessary condition $\theta = 0$ yields

\begin{equation}
x = \omega \bigcirc \bigcirc \ \text{spr cos } \varphi_0
\end{equation}

\begin{equation}
\text{spr arc tg } x = \omega \bigcirc \bigcirc \ \text{spr sin } \varphi_0
\end{equation}

Hence,

\begin{equation}
\text{spr arc tg } \varphi_0 = \left( \text{spr arc tg } x \right) - \bigcirc \bigcirc \ \frac{1}{x} = \frac{1}{\text{arc tg } x}
\end{equation}

depends on $x$, so $\varphi_0$ does not exists.
6. Proof of Theorem 5.9.

We need the following

**Lemma 6.1.** If \( v_1 \in \mathbb{R}^r \), \( v_1 \) and \( v_2 \) satisfy condition (2.6) then we have

\[
\text{spr} \sin(\text{spr} \arctg(v_2 - \bigcirc \cdot v_1)) = v_2
\]

and

\[
\text{spr} \cos(\text{spr} \arctg(v_2 - \bigcirc \cdot v_1)) = v_1.
\]

**Proof of the Lemma.** Using (2.1), (1.9), (1.4) and (1.11) we get

\[
\text{spr} \arctg(v_2 - \bigcirc \cdot v_1) = \arctg(v_2 - \bigcirc \cdot v_1) =
\]

\[
= \arctg \left( \frac{v_2}{v_1} \right) = \arctg \left( \frac{v_2}{v_1} : v_1 \right) =
\]

Moreover, (2.6) implies

\[
(v_1)^2 + (v_2)^2 = 1.
\]

Hence, using that \( v_1 > 0 \) (see (1.8)) we get

\[
\sin \left( \arctg \frac{v_2}{v_1} \right) = \frac{v_2}{v_1}.
\]

Moreover, (2.1), (1.11) and (1.9) yield

\[
\text{spr} \sin(\text{spr} \arctg(v_2 - \bigcirc \cdot v_1)) = \sin \left( \text{spr} \arctg(v_2 - \bigcirc \cdot v_1) \right)
\]

\[
= \sin(\arctg(v_2 : v_1)) = \left( \frac{v_2}{v_1} \right) = v_2,
\]

that is we have (6.2).

Similarly,

\[
\cos(\arctg \frac{v_2}{v_1}) = v_1,
\]

implies

\[
\text{spr} \cos(\text{spr} \arctg(v_2 - \bigcirc \cdot v_1)) = v_1
\]

that is we have (6.3).

\[\Box\]

**Proof of Theorem. Necessity.** Let us assume that set \( S \) (see (5.8)) forms a super-line which has the equation system

\[
x = c_1 - \bigcirc - (t \bigcirc \cdot v_1)
\]

\[
y = c_2 - \bigcirc - (t \bigcirc \cdot v_2)
\]

where \( C = (c_1, c_2) \) and \( V = (v_1, v_2) \) with (2.6) belong to \( \mathbb{R}^2 \). As the connection between \( x \) and \( y \) is mutual and unambiguous, \( v_1 \) and \( v_2 \) are different from 0. Moreover, we can assume that \( v_1 > 0 \). Hence, we have that

\[
y = (m - \bigcirc \cdot x) - \bigcirc \cdot b
\]

where

\[
m = v_2 - \bigcirc \cdot v_1, (m \neq 0)
\]
and 

\[ b = c_2 \bigtriangleup \bigtriangleup (m \bigtriangleup \bigtriangleup c_1), \]

that is

\[ F(x) = (m \bigtriangleup \bigtriangleup x) - \bigtriangleup b \]

is a super-linear function. Now, by (6.6) we have

\[ \varphi_0 = -b \bigtriangleup \bigtriangleup m \]

as the unique point such that

\[ F(\varphi_0) = 0, \]

so (5.10) is fulfilled. Let now

\[ \varphi_0 = \text{sp} \arctgm. \]

Clearly, \( \varphi_0 \neq \text{sp} \arctgm \), \( k = 0, \pm 1, \pm 2, \ldots \). Using transformation (5.11) with \( x_0 = \varphi_0 \), by (6.4) and (6.7) we have the equation super-line (5.8) with (6.6) in the system \( \xi, \eta \) with the origo \((\varphi_0, 0)\):

\[ \eta = m - \bigtriangleup \bigtriangleup \xi. \]

Considering transformation (5.12) with \( \varphi_0 \) under (6.8), by (6.5) and Lemma 6.1 we have

\[ \xi = (\omega \bigtriangleup \bigtriangleup v_1) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup v_2) \]

\[ \eta = (\omega \bigtriangleup \bigtriangleup v_2) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup v_1) \]

Using (6.9) the equation

\[ \omega \bigtriangleup \bigtriangleup v_1 - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup v_1) = m - \bigtriangleup ((\omega \bigtriangleup \bigtriangleup v_1) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup v_2)) \]

is the equation of super-line (5.8) in the coordinate system \( \omega, \vartheta \) with the origo \((\varphi_0, 0)\). Spermuting both sides of the equation (6.10) by \( v_1 \) and using (6.5) we can write

\[ v_2 \bigtriangleup (\omega \bigtriangleup \bigtriangleup v_2) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup v_2) = \]

\[ (v_2 - \omega \bigtriangleup \bigtriangleup v_1) - \bigtriangleup (v_2 - \vartheta \bigtriangleup \bigtriangleup v_2)). \]

Hence

\[ \vartheta \bigtriangleup \bigtriangleup (v_1 - \omega \bigtriangleup \bigtriangleup v_1) - \bigtriangleup (v_2 - \vartheta \bigtriangleup \bigtriangleup v_2) = 0 \]

and (2.6) gives that \( \vartheta = 0 \). This shows that super-line (5.8) lies on the \( \omega \)-axis\) of the coordinate-system \( \omega, \vartheta \).

**Sufficiency.** Now we assume that there exists a (unique) \( x_0 \) with (5.10) and

\[ \varphi_0 \neq \text{sp} \arctgm \] such that after the transformations (5.11) and (5.12) the points of the set \( S \) lie on the \( \omega \)-axis\) of the coordinate system \( \omega, \vartheta \) with the origo \((x_0, 0)\). These mean that if \((x, y) \in S \) having the coordinates \( \omega \) and \( \vartheta \) such that

\[ x = \omega \bigtriangleup \bigtriangleup x_0 = (\omega \bigtriangleup \bigtriangleup \text{sp} \cos \varphi_0) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup \text{sp} \sin \varphi_0) \]

\[ y = (\omega \bigtriangleup \bigtriangleup \text{sp} \sin \varphi_0) - \bigtriangleup (\vartheta \bigtriangleup \bigtriangleup \text{sp} \cos \varphi_0) \]
is valid with \( \varphi = 0 \). Hence, for any \( x \in \mathbb{R} \)
\[
x - x_0 = \omega - spr \cos \varphi_0
\]
\[
F(x) = \omega - spr \sin \varphi_0
\]

Having that \( spr \cos \varphi_0 \neq 0 \) an assuming that \( x \neq x_0 \) (that is \( \omega \neq 0 \)) by (2.1) we can write
\[
F(x) = \omega - (x - x_0) = spr \tan \varphi_0 (\neq 0).
\]

Moreover,
\[
F(x) = (spr \tan \varphi_0) - (x - x_0)
\]

which, by (5.10), remains valid for \( x = x_0 \), too. As \( F \) is a super-linear function, the points of \( S \) form a super-line. \( \square \)

References


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