PLANAR BEURLING TRANSFORM
AND GRUNSKY INEQUALITIES

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Abstract. In recent work with Baranov, it was explained how to view the classical Grunsky inequalities in terms of an operator identity, involving a transferred Beurling operator induced by the conformal mapping. The main property used is the fact that the Beurling operator is unitary on $L^2(\mathbb{C})$. As the Beurling operator is also bounded on $L^p(\mathbb{C})$ for $1 < p < +\infty$ (with so far unknown norm), an analogous operator identity was found which produces a generalization of the Grunsky inequalities to the $L^p$ setting. Here, we consider weighted Hilbert spaces $L^2_\theta(\mathbb{C})$ with weight $|z|^{2\theta}$, for $0 \leq \theta \leq 1$, and find that the Beurling operator perturbed by adding a Cauchy-type operator acts unitarily on $L^2_\theta(\mathbb{C})$. After transferring to the unit disk $D$ with the conformal mapping, we find a generalization of the Grunsky inequalities in the setting of the space $L^2_\theta(D)$; this generalization seems to be essentially known, but the formulation is new. As a special case, the generalization of the Grunsky inequalities contains the Prawitz theorem used in a recent paper with Shimorin. We also mention an application to quasiconformal maps.

1. Introduction

Beurling and Fourier transforms. In this note, we shall study a perturbation of the Beurling transform in the complex plane $\mathbb{C}$. The Fourier transform of an appropriately area-integrable function $f$ is

$$\hat{f}([\xi]) = \int_{\mathbb{C}} e^{-2\pi i \text{Re}[\xi] z} f(z) \, dA(z), \quad \xi \in \mathbb{C},$$

while the Beurling transform is the singular integral operator

$$\mathcal{B}_C[f](z) = \text{pv} \int_{\mathbb{C}} \frac{f(w)}{(w - z)^2} \, dA(w), \quad z \in \mathbb{C};$$

here “pv” stands for “principal value”, and

$$dA(z) = \frac{dxdy}{\pi}, \quad z = x + iy,$$

2000 Mathematics Subject Classification: Primary 30C55, 30C60; Secondary 42B10, 42B20, 46E22.

Key words: Beurling transform, Grunsky inequalities.

Research supported by the Göran Gustafsson Foundation.
is normalized area measure. The two transforms are connected via
\[ \mathfrak{B}_C[f](\xi) = -\overline{\xi} f(\xi), \quad \xi \in \mathbb{C}. \]
By the Plancherel identity, \( \mathfrak{B} \) is a unitary transformation on \( L^2(\mathbb{C}) \), which is supplied with the standard norm
\[ \|f\|_{L^2(\mathbb{C})}^2 = \int_{\mathbb{C}} |f(z)|^2 \, dA(z). \]
It is clear from this and the above relationship that \( \mathfrak{B}_C \) is unitary on \( L^2(\mathbb{C}) \) as well. We recall that an operator \( T \) acting on a complex Hilbert space \( \mathcal{H} \) is unitary if \( T^*T = TT^* = \text{id} \), where \( T^* \) is the adjoint and “id” is the identity operator. Expressed differently, that \( T \) is unitary means that \( T \) is a surjective isometry.

The Cauchy transform. The Cauchy transform \( \mathfrak{C}_C \) is the integral transform
\[ \mathfrak{C}_C[f](z) = \int_{\mathbb{C}} \frac{f(w)}{w - z} \, dA(w), \]
defined for appropriately integrable functions. It is related to Beurling transform \( \mathfrak{B}_C \) via
\[ \mathfrak{B}_C[f](z) = \partial_z \mathfrak{C}_C[f](z), \]
where both sides are understood in the sense of distribution theory. Here, we use the notation
\[ \partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial}_z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \]

The perturbed Beurling transform. For real \( \theta \), let \( L^2_\theta(\mathbb{C}) \) denote the Hilbert space of square integrable functions on \( \mathbb{C} \) with norm
\[ \|f\|_{L^2_\theta(\mathbb{C})}^2 = \int_{\mathbb{D}} |f(z)|^2 |z|^{2\theta} \, dA(z) < +\infty. \]
Moreover, let \( \mathfrak{X}_C \) denote the operator
\[ \mathfrak{X}_C[h](z) = \frac{1}{z} \mathfrak{C}_C[h](z), \]
for suitably integrable functions \( h \). It turns out that it is enough to require that \( h \in L^2_\theta(\mathbb{C}) \) for some positive \( \theta \) for \( \mathfrak{X}_C[h] \) to be well-defined. We also need the operator \( \mathfrak{X}'_C \), as defined by
\[ \mathfrak{X}'_C[h](z) = \mathfrak{C}_C \left[ \frac{h}{z} \right](z). \]
We introduce, for \( 0 \leq \theta \leq 1 \), the perturbed Beurling transform
\[ \mathfrak{B}^\theta_C = \mathfrak{B}_C + \theta \mathfrak{X}_C, \]
while for \( -1 \leq \theta \leq 0 \), we instead write
\[ \mathfrak{B}^\theta_C = \mathfrak{B}_C + \theta \mathfrak{X}'_C. \]

Theorem 1.1. For \( -1 \leq \theta \leq 1 \), the operator \( \mathfrak{B}^\theta_C \) acts unitarily on \( L^2_\theta(\mathbb{C}) \).
The proof of this theorem is supplied in the next section.

Acknowledgement. The author thanks Lennart Carleson for suggesting the application to quasiconformal maps.

2. The perturbed Beurling transform

For \( N = 1, 2, 3, \ldots \), let \( \mathcal{A}_N \) denote the \( N \)-th roots of unity, that is, the collection of all \( \alpha \in \mathbb{C} \) with \( \alpha^N = 1 \). For \( n = 1, \ldots, N \), we consider the closed subspace \( L^2_{n,N}(\mathbb{C}) \) of \( L^2(\mathbb{C}) \) consisting of functions \( f \) having the invariance property

\[
(2.1) \quad f(\alpha z) = \alpha^n f(z), \quad z \in \mathbb{C}, \quad \alpha \in \mathcal{A}_N.
\]

It is easy to see that \( f \in L^2_{n,N}(\mathbb{C}) \) if and only if \( f \in L^2(\mathbb{C}) \) is of the form

\[
(2.2) \quad f(z) = z^n g(z^N), \quad z \in \mathbb{C},
\]

where \( g \) some other complex-valued function.

We shall now study the Beurling transform on the subspaces \( L^2_{n,N}(\mathbb{C}) \).

The Beurling transform and root-of-unity invariance. Fix an \( N = 1, 2, 3, \ldots \) and an \( n = 1, \ldots, N \). We suppose \( f \in L^2_{n,N}(\mathbb{C}) \). Then, by the change of variables formula,

\[
\mathcal{B}_C[f](z) = \text{pv} \int_{\mathbb{C}} \frac{f(w)}{(w - z)^2} dA(w) = \text{pv} \int_{\mathbb{C}} \frac{\alpha^n}{(\alpha w - z)^2} f(w) dA(w)
\]

\[
= \alpha^{-2} \mathcal{B}_C[f](\bar{\alpha}z), \quad z \in \mathbb{C},
\]

for \( \alpha \in \mathcal{A}_N \). Taking the average over \( \mathcal{A}_N \), we get the identity

\[
\mathcal{B}_C[f](z) = \frac{1}{N} \text{pv} \int_{\mathbb{C}} \sum_{\alpha \in \mathcal{A}_N} \frac{\alpha^n}{(\alpha w - z)^2} f(w) dA(w), \quad z \in \mathbb{C}.
\]

A symmetric sum. Next, we study the sum

\[
F(z) = \frac{1}{N} \sum_{\alpha \in \mathcal{A}_N} \frac{\alpha^n}{1 - \alpha z}.
\]

This sum has the symmetry property

\[
F(\beta z) = \bar{\beta}^n F(z), \quad \beta \in \mathcal{A}_N,
\]

which means that \( F \) has the form

\[
F(z) = z^{N-n} G(z^N).
\]

The function \( G \) then has a simple pole at 1, and is analytic everywhere else in the complex plane. Moreover, \( F \) vanishes at infinity, so \( G \) vanishes there, too. This leaves us but one possibility, that \( G \) has the form

\[
G(z) = \frac{C}{1 - z},
\]
where $C$ is a constant. It is easily established that $C = 1$. It follows that
\begin{equation}
\label{F}
F(z) = \frac{1}{N} \sum_{\alpha \in \mathcal{S}_N} \frac{\alpha^n}{1 - \alpha z} = \frac{z^{N-n}}{1 - z^N}, \quad z \in \mathbb{C}.
\end{equation}
As a consequence, we get that
\begin{align*}
H(z) &= F(z) + zF'(z) = [zF(z)]' = \frac{1}{N} \sum_{\alpha \in \mathcal{S}_N} \frac{\alpha^n}{(1 - \alpha z)^2} \\
&= z^{N-n} \left\{ \frac{N}{(1 - z^N)^2} - \frac{n - 1}{1 - z^N} \right\},
\end{align*}
where the left hand side identity is used to define the function $H(z)$. This allows us to compute the sum we need:
\begin{equation*}
\frac{1}{N} \sum_{\alpha \in \mathcal{S}_N} \frac{\alpha^n}{(\alpha w - z)^2} = \frac{1}{z^2} H\left(\frac{w}{z}\right) = z^{n-2} w^{N-n} \left\{ \frac{N z^N}{(z^N - w^N)^2} - \frac{n - 1}{z^N - w^N} \right\}.
\end{equation*}
For $f \in L^2_{\mathcal{S}_N}(\mathbb{C})$, we thus get the representation
\begin{equation*}
\mathcal{B}_C[f](z) = z^{n-2} \mathrm{pv} \int_{\mathbb{C}} \left\{ \frac{N z^N}{(z^N - w^N)^2} - \frac{n - 1}{z^N - w^N} \right\} w^{N-n} f(w) \, dA(w), \quad z \in \mathbb{C}.
\end{equation*}
Let $f$ and $g$ be connected via \eqref{Cauchy}, and implement this relationship into the above formula:
\begin{equation}
\mathcal{B}_C[f](z) = z^{n-2} \mathrm{pv} \int_{\mathbb{C}} \left\{ \frac{N z^N}{(z^N - w^N)^2} - \frac{n - 1}{z^N - w^N} \right\} w^{N-n} g(w^N) \, dA(w), \quad z \in \mathbb{C}.
\end{equation}
A similar expression may be found for the Cauchy transform as well:
\begin{equation}
\mathcal{C}_C[f](z) = z^{n-N-1} \int_{\mathbb{C}} \frac{w^N}{w^N - z^N} g(w^N) \, dA(w), \quad z \in \mathbb{C}.
\end{equation}
It is easy to check that with
\begin{equation*}
h(z) = \frac{z^N}{|z|^{2-2/N}},
\end{equation*}
where $g$ is connected to $f$ via \eqref{Cauchy}, we have
\begin{equation*}
\mathcal{B}_C[f](z) = z^{N+n-2} \mathcal{B}_C^{(n-1)/N}[h](z^N), \quad z \in \mathbb{C}.
\end{equation*}
The fact that $\mathcal{B}_C$ is an isometry becomes the norm identity
\begin{equation}
\int_{\mathbb{C}} |h(z)|^2 |z|^{2\theta} \, dA(z) = \int_{\mathbb{C}} |\mathcal{B}_C^\theta[h](z)|^2 |z|^{2\theta} \, dA(z),
\end{equation}
where we suppose that $\theta = (n - 1)/N$. However, fractions of this type are dense in the interval $[0, 1]$, so that \eqref{norm} extends to all $\theta$ with $0 \leq \theta \leq 1$. In other words, for $0 \leq \theta \leq 1$, the operator $\mathcal{B}_C^\theta$ is unitary on the space $L^2_\theta(\mathbb{C})$, which was defined earlier. But then, considering that
\begin{equation*}
\mathcal{B}_C^\theta = \mathcal{M}_z \mathcal{B}_C^{\theta+1} \mathcal{M}_z^{-1}, \quad -1 \leq \theta \leq 0,
\end{equation*}
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which follows immediately from the fact that
\[
\frac{1}{(w-z)^2} + \frac{\theta}{w(w-z)} = \frac{z}{w} \left\{ \frac{1}{(w-z)^2} + \frac{\theta+1}{z(w-z)} \right\},
\]
we conclude that \( \mathcal{B}_\theta \) is unitary on \( L^2_\theta (\mathbb{C}) \) for \(-1 \leq \theta \leq 0 \) as well.

This completes the proof of Theorem 1.1.

**Remark 2.1.** It is known [8] that \( \mathcal{B}_\theta \) is a bounded operator on \( L^2_\theta (\mathbb{C}) \) for \(-1 < \theta < 1 \) (but not for \( \theta = \pm 1 \)). This means that for \(-1 < \theta < 1 \), both terms in (1.1) are bounded operators on \( L^\theta (\mathbb{C}) \). We suspect that the second term in (1.1), the operator \( \mathcal{T}_\theta \), is compact on \( L^2_\theta (\mathbb{C}) \) with small spectrum for \( 0 < \theta < 1 \). The analogous statement for \( \mathcal{T}_\theta ' \) is essentially equivalent.

**Extension to real \( \theta \).** We first note that \( M_z \), multiplication by the independent variable, is an isometric isomorphism \( L^2_\theta +1 (\mathbb{C}) \to L^2_\theta (\mathbb{C}) \) for all real \( \theta \). Therefore, for integers \( k \) and \( 0 \leq \theta \leq 1 \), the operator \( \mathcal{B}_\theta^+ + k \mathcal{M}_z \mathcal{B}_\theta \mathcal{M}_z^k \)
is unitary on \( L^2_{\theta+k} (\mathbb{C}) \). It supplies an extension of \( \mathcal{B}_\theta \) to all real \( \theta \) which coincides with the previously defined notion for \(-1 \leq \theta \leq 1 \).

3. Applications of Beurling transforms to conformal mapping

Grunsky identity and inequalities. It was shown in [1] that if \( \varphi : D \to \Omega \) is a conformal mapping where \( \Omega = \varphi (D) \subset \mathbb{C} \), then
\[
\mathcal{B}_\varphi f(z) = \text{pv} \int_D \frac{\varphi'(z)\varphi'(w)}{(\varphi(w)-\varphi(z))^2} f(w) \, dA(w), \quad z \in D,
\]
is a contraction on \( L^2(D) \); as a matter of fact, this follows from the fact that \( \mathcal{B}_\varphi \) is unitary on \( L^2(\mathbb{C}) \). Moreover, it was shown that if \( e \) denotes the function \( e(z) = z \), so that
\[
\mathcal{B}_e f(z) = \text{pv} \int_D \frac{1}{(w-z)^2} f(w) \, dA(w), \quad z \in D,
\]
we have the Grunsky identity
\[
(3.1) \quad \mathcal{B}_\varphi - \mathcal{B}_e = \mathcal{P} \mathcal{B}_\varphi = \mathcal{B}_\varphi \mathcal{P} = \mathcal{P} \mathcal{B}_\varphi \mathcal{P},
\]
where \( \mathcal{P} \) and \( \mathcal{P} ' \) are the associated Bergman projections
\[
\mathcal{P} [f](z) = \int_D \frac{f(w)}{(1-z\bar{w})^2} \, dA(w), \quad z \in D,
\]
and
\[
\mathcal{P} ' [f](z) = \int_D \frac{f(w)}{(1-\bar{z}w)^2} \, dA(w), \quad z \in D.
\]
As \( \mathcal{P} \) and \( \mathcal{P} ' \) are contractions on \( L^2(D) \), we find that
\[
(3.2) \quad \| (\mathcal{B}_\varphi - \mathcal{B}_e) [f] \|_{L^2(D)} \leq \| f \|_{L^2(D)}, \quad f \in L^2(D).
\]
In [1], it is explained how (3.2) expresses the Grunsky inequalities in a compact manner.

We shall now try to carry out the same considerations in the weighted situation.

**Transfer to the unit disk.** We need to introduce some general notation. Let \( M_F \) denote the operator of multiplication by the function \( F \). We also need the Hilbert space \( L^2_\theta(X) \) with the norm

\[
\|h\|_{L^2_\theta(X)}^2 = \int_X |h(z)|^2 |z|^{2\theta} \, dA(z),
\]

where \( X \) is some Borel measurable subset of \( \mathbb{C} \) with positive area. In the sequel, we restrict \( \theta \) to the interval \( 0 \leq \theta \leq 1 \). Fix a simply connected domain \( \Omega \) in \( \mathbb{C} \), which contains the origin and is not the whole plane, and let \( \varphi : \mathbb{D} \to \Omega \) denote the conformal mapping with \( \varphi(0) = 0 \) and \( \varphi'(0) > 0 \). Let \( f \in L^2(\Omega) \), and extend it to the whole complex plane so that it vanishes on \( \mathbb{C} \setminus \Omega \). Let \( \mathcal{B}_\Omega[f] \) denote the restriction to \( \Omega \) of \( \mathcal{B}_\mathbb{C}[f] \), and do likewise to define the operators \( \mathcal{C}_\Omega, \mathcal{T}_\Omega, \mathcal{F}_\Omega, \mathcal{B}_\mathbb{C}^\theta \), as well as \( \mathcal{B}_\Omega^\theta \). We introduce transferred operators on spaces over the unit disk in the following fashion. First, we suppose \( f \in L^2_\theta(\Omega) \). Then the associated function

\[
g(z) = \varphi'(z) \left[ \frac{\varphi(z)}{z} \right]^\theta f \circ \varphi(z), \quad z \in \mathbb{D},
\]

belongs to \( L^2_\theta(\mathbb{D}) \), with equality of norms:

\[
\|g\|_{L^2_\theta(\mathbb{D})} = \|f\|_{L^2_\theta(\Omega)}.
\]

The transferred Cauchy transform is defined as follows:

\[
(3.3) \quad \mathcal{C}_\varphi^\theta[g](z) = \left[ \frac{\varphi(z)}{z} \right]^\theta \mathcal{C}_\Omega[f] \circ \varphi(z) = \int_{\mathbb{D}} \left[ \frac{w \varphi(z)}{z \varphi(w)} \right]^\theta \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} g(w) \, dA(w).
\]

The transferred perturbed Beurling transform is defined analogously:

\[
(3.4) \quad \mathcal{B}_\varphi^\theta[g](z) = \varphi'(z) \left[ \frac{\varphi(z)}{z} \right]^\theta \mathcal{B}_\Omega^\theta[f] \circ \varphi(z)
\]

\[
= \varphi'(z) \left[ \frac{\varphi(z)}{z} \right]^\theta \left\{ \mathcal{B}_\Omega[f] \circ \varphi(z) + \frac{\theta}{\varphi(z)} \mathcal{C}_\Omega[f] \circ \varphi(z) \right\}
\]

\[
= \mathcal{B}_\varphi^{\theta,0}[g](z) + \theta \frac{\varphi'(z)}{\varphi(z)} \mathcal{C}_\varphi^\theta[g](z),
\]

where

\[
\mathcal{B}_\varphi^{\theta,0}[g](z) = \text{p.v.} \int_{\mathbb{D}} \left[ \frac{w \varphi(z)}{z \varphi(w)} \right]^\theta \frac{\varphi'(z)\varphi'(w)}{(\varphi(w) - \varphi(z))^2} g(w) \, dA(w).
\]

It is clear that \( \mathcal{B}_\varphi^\theta \) is a norm contraction on \( L^2_\theta(\mathbb{D}) \). Let \( \mathfrak{B}_\theta \) be the integral operator

\[
\mathfrak{B}_\theta[f](z) = \int_{\mathbb{D}} \left[ \frac{1}{(1-z\bar{w})^2} + \frac{\theta}{1-z\bar{w}} \right] f(w) \, |w|^{2\theta} \, dA(w);
\]
it is the orthogonal projection to the subspace of analytic functions in $L^2_\theta(D)$. As both $\mathcal{B}_\varphi^\theta$ and $\mathcal{P}_\varphi$ are contractions on $L^2_\theta(D)$, so is their product $\mathcal{P}_\varphi \mathcal{B}_\varphi^\theta$. It remains to represent the operator $\mathcal{P}_\varphi \mathcal{B}_\varphi^\theta$ in a reasonable fashion. The main observation is that

$$
\left[ \frac{w \varphi'(z)}{z \varphi'(w)} \right]^\theta \frac{\varphi'(z) \varphi'(w)}{(\varphi(w) - \varphi(z))^2} = \frac{1}{(w - z)^2} - \theta \left[ \frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z} \right] \frac{1}{w - z} + O(1)
$$

near the diagonal $z = w$, so that

$$(3.5) \quad \frac{w \varphi(z)}{z \varphi(w)} = \frac{\varphi'(z) \varphi'(w)}{(\varphi(w) - \varphi(z))^2} + \theta \frac{\varphi'(z)}{\varphi(z)} \left[ \frac{w \varphi(z)}{z \varphi(w)} \right] \frac{\varphi'(w)}{\varphi(w) - \varphi(z)}$$

again near the diagonal. We observe that in view of (3.5), we get the Grunsky-type identity

$$(3.6) \quad \mathcal{P}_\varphi \mathcal{B}_\varphi^\theta = \mathcal{B}_\varphi^\theta - \mathcal{B}_D + \mathcal{P}_\varphi \mathcal{B}_D + \theta \mathcal{P}_\varphi \mathcal{I}_D - \theta \mathcal{I}_D.$$

To make the involved operators $\mathcal{P}_\varphi \mathcal{B}_D$ and $\mathcal{P}_\varphi \mathcal{I}_D$ appearing in the right hand side of (3.6) more concrete, it is helpful to know that for $\lambda \in D$,

$$\mathcal{P}_\varphi [f_\lambda](z) = \overline{\lambda} |\lambda|^{2\theta} \int_0^1 \left[ \frac{1}{(1 - t\lambda z)^2} + \frac{\theta}{1 - t\lambda z} \right] t^{\theta} \, dt, \quad f_\lambda(z) = \frac{1}{\lambda - z},$$

while

$$\mathcal{P}_\varphi [g_\lambda](z) = -\theta \overline{\lambda} |\lambda|^{2\theta - 2} \int_0^1 \left[ \frac{1}{(1 - t\lambda z)^2} + \frac{\theta}{1 - t\lambda z} \right] t^{\theta} \, dt, \quad g_\lambda(z) = \frac{1}{(\lambda - z)^2}.$$ 

In view of these relations, we quickly verify that

$$\mathcal{P}_\varphi \mathcal{B}_D + \theta \mathcal{P}_\varphi \mathcal{I}_D = 0.$$

The Grunsky-type identity (3.6) thus simplifies a bit:

$$(3.7) \quad \mathcal{P}_\varphi \mathcal{B}_\varphi^\theta = \mathcal{B}_\varphi^\theta - \mathcal{B}_D - \theta \mathcal{I}_D = \mathcal{B}_\varphi^\theta - \mathcal{B}_D.$$

The corresponding Grunsky-type inequality reads

$$(3.8) \quad \| (\mathcal{B}_\varphi^\theta - \mathcal{B}_D^\theta) [f] \|_{L^2_\theta(D)} \leq \| f \|_{L^2_\theta(D)}, \quad f \in L^2_\theta(D).$$

To get a concrete example of how the Grunsky-type inequality works, we pick

$$f_\lambda(z) = |z|^{-2\theta} \left( \frac{1}{(1 - z\lambda)^2} - \frac{\theta}{1 - z\lambda} \right), \quad z \in D,$$

and compute

$$\left( \mathcal{B}_\varphi^\theta - \mathcal{B}_D^\theta \right) [f](z) = \left[ \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \right]^\theta \frac{\varphi'(z) \varphi'(\lambda)}{(\varphi(\lambda) - \varphi(z))^2} - \frac{1}{(\lambda - z)^2}$$

$$+ \theta \frac{\varphi'(z)}{\varphi(z)} \left[ \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \right]^\theta \frac{\varphi'(\lambda)}{\varphi(\lambda) - \varphi(z)} - \frac{\theta}{z(\lambda - z)}.$$
We see that (3.8) in this case assumes the form \((0 \leq \theta \leq 1)\)

\[
\int_{D} \left| \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \right|^\theta \frac{\varphi'(z) \varphi'(\lambda)}{\varphi(\lambda) - \varphi(z)}^2 - \frac{1}{(\lambda - z)^2} + \theta \frac{\varphi'(z)}{\varphi(z)} \left( \frac{\lambda \varphi(z)}{z \varphi(\lambda)} \right)^\theta \frac{\varphi'(\lambda)}{\varphi(\lambda) - \varphi(z)} - \frac{\theta}{z(\lambda - z)} \right| |z|^{2\theta} dA(z)
\]

(3.9)

\[
\leq \int_{D} |f(\lambda)(z)|^2 |z|^{2\theta} dA(z) = \int_{D} \left| \frac{1}{(1 - |z|^2)^2} - \frac{\theta}{1 - \lambda^2} \right|^2 |z|^{-2\theta} dA(z)
\]

\[
= \frac{1}{(1 - |\lambda|^2)^2} - \frac{\theta}{1 - |\lambda|^2}.
\]

The special case \(\lambda = 0\) gives us the inequality of Prawitz (see [6] and [7]; we assume \(\varphi'(0) = 1\)):

\[
\int_{D} \left| \varphi'(z) \left( \frac{\varphi(z)}{z} \right)^{\theta - 2} - 1 \right|^2 |z|^{2\theta} dA(z) \leq \frac{1}{1 - \theta}.
\]

**A dual version.** We carry out the corresponding calculations on the basis of the fact that \(B_{\varphi}^\theta\) is unitary on \(L^2_{-\theta}(\mathbb{C})\) for \(0 \leq \theta \leq 1\). In analogy with the above treatment, we connect two functions \(f, g\) via

\[
g(z) = \varphi'(z) \left( \frac{\varphi(z)}{z} \right)^{-\theta} f \circ \varphi(z), \quad z \in D.
\]

Then \(f \in L^2_{-\theta}(\Omega)\) if and only if \(g \in L^2_{-\theta}(D)\), with equality of norms:

\[
\|g\|_{L^2_{-\theta}(D)} = \|f\|_{L^2_{-\theta}(\Omega)}.
\]

The corresponding transferred Beurling transform assumes the form

\[
\mathcal{B}_{\varphi}^{-\theta}[g](z) = \varphi'(z) \left( \frac{\varphi(z)}{z} \right)^{-\theta} \mathcal{B}_{\Omega}^{-\theta}[f] \circ \varphi(z)
\]

\[
= \varphi'(z) \left( \frac{\varphi(z)}{z} \right)^{-\theta} \left\{ \mathcal{B}_{\Omega}[f] \circ \varphi(z) - \theta \mathcal{C}_{\Omega} \left[ \frac{f}{z} \right] \circ \varphi(z) \right\}
\]

\[
= \mathcal{B}_{\varphi}^{-\theta,0}[g](z) - \theta \varphi'(z) \mathcal{C}_{\varphi}^{-\theta} \left[ \frac{g}{\varphi} \right](z),
\]

where \(\mathcal{B}_{\varphi}^{-\theta,0}\) and \(\mathcal{C}_{\varphi}^{-\theta}\) are as before (just plug in \(-\theta\) in place of \(\theta\) in the corresponding formulas). It is clear that \(\mathcal{B}_{\varphi}^{-\theta}\) is a contraction on \(L^2_{-\theta}(D)\).

To cut a long story short, the Grunsky-type identity analogous to (3.7) reads

\[
(3.11) \quad \mathcal{P}_{-\theta} \mathcal{B}_{\varphi}^{-\theta} = \mathcal{B}_{\varphi}^{-\theta} - \mathcal{B}_D^{-\theta}.
\]

Let \(\mathcal{P}_{-\theta}^*\) be the operator

\[
\mathcal{P}_{-\theta}^*[g](z) = |z|^{-2\theta} \int_{D} \left( \frac{1}{(1 - wz)^2} - \frac{\theta}{1 - wz} \right) g(w) dA(w);
\]
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it is a contraction on $L^2_\theta(D)$, which can be written
\[ \mathcal{P}^*_{-\theta} = M|z|^{-2\theta} \mathcal{P}_{-\theta} M|z|^{2\theta}, \]
where $\mathcal{P}_{-\theta}$ denotes the orthogonal projection onto the antiholomorphic functions in $L^2_{-\theta}(D)$. By forming adjoints, we find that (3.11) states that
\[ (3.12) \quad \mathcal{B}^\theta \mathcal{P}^*_{-\theta} = \mathcal{B}^\theta - \mathcal{B}^\theta_D. \]

We now combine (3.7) with (3.12), and arrive at the following.

**Theorem 3.1.** $(0 \leq \theta \leq 1)$ We have the Grunsky identity
\[ (3.13) \quad \mathcal{B}^\theta \mathcal{P}^*_{-\theta} = \mathcal{B}^\theta - \mathcal{B}^\theta_D. \]
Moreover, we also have the Grunsky-type inequality
\[ \| (\mathcal{B}^\theta - \mathcal{B}^\theta_D) [f] \|_{L^2_{\theta}(D)} \leq \| f \|_{L^2_{\theta}(D)}, \quad f \in L^2_{\theta}(D), \]
with equality if and only if $\varphi$ is a full mapping and $f(z)$ is of the form $|z|^{-2\theta}$ times an antianalytic function.

**Remark 3.2.** (a) It follows that (3.9) is an equality for full mappings.
(b) The above Grunsky-type inequality probably follows from the estimate mentioned by de Branges [2] as his point of departure for obtaining the more general results that led to the solution of the Bieberbach conjecture.
(c) It is possible to consider weighted $L^p$ spaces of the type $L^p_\theta(C)$, and obtain norm estimates of perturbed Beurling transforms on such spaces from well-known estimates of the Beurling operator on $L^p(C)$. This then leads to appropriate Grunsky-type identities and inequalities in the weighted $L^p$ setting.

### 4. Applications to quasiconformal maps

**Quasiconformal maps.** Here, we suppose that $\varphi : D \to \Omega$ is quasiconformal, which means that it is a homeomorphism which is one-to-one and onto, with
\[ (4.1) \quad \partial_z \varphi(z) = \mu(z) \partial_z \varphi(z), \quad z \in D, \]
where $\mu$ is an Borel measurable function on $D$ with
\[ \| \mu \|_{L^\infty(C)} = \text{ess sup} \{ |\mu(z)| : z \in D \} < 1. \]
As before, $\Omega$ is a simply connected domain in $C$ other than $C$ itself, which contains the origin. We assume that $\varphi(0) = 0$ and that $\mu$ vanishes on a (small) neighborhood of the origin. The function $\varphi$ is then analytic near the origin. In the sequel, we shall think of the Beltrami coefficient $\mu$ as fixed. We plan to derive some information regarding the mapping $\varphi$.

**The mapping $\phi = \phi_\mu$.** We extend $\mu$ to all of $C$ by declaring it to be
\[ \mu(z) = \bar{\mu} \left( \frac{1}{z} \right), \quad z \in D, \]
where
\[ D_e = \{ z \in \mathbb{C} : 1 < |z| < +\infty \} \]
is the (punctured) exterior disk, and by declaring it to vanish on the unit circle \( T \). Clearly, the extended \( \mu \) has compact support.

The material mentioned here is largely a condensed version of Section 1.7 of [3]; we refer to that book for details. Let \( F = F_\mu : \mathbb{C} \to \mathbb{C} \) solve the equation
\[(\text{id} + \mathfrak{B}_C \mathfrak{M}_\mu)[F] = \mathfrak{B}_C[\mu];\]
A solution \( F \) exists and is unique, and it belongs to \( L^p(\mathbb{C}) \) for \( p \) in some open interval containing the point 2. We define
\[ \Phi(z) = z + \bar{\mathfrak{C}}_C[F](z) - \bar{\mathfrak{C}}_C[F](0), \]
and obtain a quasiconformal map \( \Phi = \Phi_\mu : \mathbb{C} \to \mathbb{C} \) which solves the Beltrami equation
\[ \bar{\partial}_z \Phi(z) = \mu(z) \partial_z \Phi(z), \quad z \in \mathbb{C}. \]
Here, \( \bar{\mathfrak{C}}_C \) is the conjugate Cauchy transform
\[ \bar{\mathfrak{C}}_C[f](z) = \int_C \frac{f(w)}{\bar{w} - \bar{z}} \, dA(w), \quad z \in \mathbb{C}. \]
A calculation shows that the related mapping
\[ \Psi(z) = \frac{1}{\Phi(z)}, \quad z \in \mathbb{C} \setminus \{0\}, \]
solves the same Beltrami equation
\[ \bar{\partial}_z \Psi(z) = \mu(z) \partial_z \Psi(z), \quad z \in \mathbb{C}. \]
As \( \Psi \)—like \( \Phi \)—fixes the points 0 and \( \infty \), it follows that
\[ \Psi(z) = \lambda \Phi(z), \quad z \in \mathbb{C}, \]
for some complex parameter \( \lambda \). Since we must have
\[ \frac{\Phi(z)}{\Psi(z)} = |\Phi(z)|^2 = \frac{1}{\lambda}, \quad z \in T, \]
it follows that \( 0 < \lambda < +\infty \). As a consequence, we have that
\[ \phi(z) = \phi_\mu(z) = \sqrt{\lambda} \Phi(z), \quad z \in D, \]
maps \( D \) onto itself, and preserves the origin. Moreover, \( \phi \) solves the same Beltrami equation (4.1) as does \( \varphi \).

**The induced transform.** The parameter \( \theta \) is assumed to be confined to the interval \( 0 \leq \theta \leq 1 \). It is easy to see that it is possible to define a single-valued logarithm
\[ \log \frac{\varphi(z)}{z}, \quad z \in D. \]
Planar Beurling transform and Grunsky inequalities

One just checks that the associated differential is exact. This allows us to define real (and complex) powers of the function \( \varphi(z)/z \). Next, we suppose \( f \in L^2_\theta(\Omega) \), and associate to it the function \( g \):

\[
g(z) = (1 - |\mu(z)|^2)^{1/2} \partial_\varphi \varphi(z) \left[ \frac{\varphi(z)}{z} \right] \theta f \circ \varphi(z), \quad z \in \mathbb{D}.
\]

It is a consequence of the change-of-variables formula

\[
\int_\Omega |F(z)|^2 \, dA(z) = \int_\mathbb{D} |F \circ \varphi(z)|^2 \left( 1 - |\mu(z)|^2 \right) |\partial_\varphi \varphi(z)|^2 \, dA(z)
\]

that

\[
\|g\|_{L^2_\theta(\mathbb{D})} = \|f\|_{L^2_\theta(\Omega)}.
\]

We define the transferred Beurling transform to be

\[
\mathcal{B}^{\theta,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2} \partial_\varphi \varphi(z) \left[ \frac{\varphi(z)}{z} \right] \theta \mathcal{B}^{\theta}_\Omega[f \circ \varphi(z), \quad z \in \mathbb{D},
\]

so that \( \mathcal{B}^{\theta,\mu}_\varphi \) acts contractively on \( L^2_\theta(\mathbb{D}) \). In case \( \theta = 0 \), the formula simplifies pleasantly:

\[
\mathcal{B}^{0,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2} \partial_\varphi \varphi(z) \left[ \frac{\varphi(z)}{z} \right] \theta \int_\mathbb{D} \left( \frac{1 - |\mu(w)|^2}{\varphi(w) - \varphi(z)} \right) g(w) \, dA(w), \quad z \in \mathbb{D}.
\]

The differentiation is in the sense of distribution theory.

The Grunsky-type identity and inequality. Since \( \varphi \) and \( \phi \) have the same Beltrami coefficient \( \mu \), there is a conformal mapping \( \psi : \mathbb{D} \to \Omega \) fixing the origin such that \( \varphi = \psi \circ \phi \). Next, we connect \( h \) and \( f \) via

\[
h(z) = \overline{\psi}'(z) \left[ \frac{\psi'(z)}{z} \right] \theta f \circ \psi(z), \quad z \in \mathbb{D},
\]

so that

\[
\|h\|_{L^2_\theta(\mathbb{D})} = \|f\|_{L^2_\theta(\Omega)} = \|g\|_{L^2_\theta(\mathbb{D})}
\]

and

\[
g(z) = (1 - |\mu(z)|^2)^{1/2} \partial_z \phi(z) \left[ \frac{\phi(z)}{z} \right] \theta h \circ \phi(z), \quad z \in \mathbb{D},
\]

while

\[
\mathcal{B}^{\theta,\mu}_\varphi[g](z) = (1 - |\mu(z)|^2)^{1/2} \partial_z \phi(z) \left[ \frac{\phi(z)}{z} \right] \theta \mathcal{B}^{\theta}_\psi[h \circ \phi(z), \quad z \in \mathbb{D}.
\]

To simplify the notation, let \( \mathcal{U}^{\theta,\mu}_\psi \) denote the unitary transformation on \( L^2_\theta(\mathbb{D}) \) given by

\[
\mathcal{U}^{\theta,\mu}_\psi[g](z) = (1 - |\mu(z)|^2)^{1/2} \partial_z \phi(z) \left[ \frac{\phi(z)}{z} \right] \theta g \circ \phi(z), \quad z \in \mathbb{D}.
\]
so that $\mathcal{B}_{\psi}^{\theta,\mu} = \mathcal{U}^{\theta,\mu} \mathcal{B}_{\phi}^{\theta}$. Next, let the orthogonal projection $\mathcal{P}_{\theta,\mu}$ on $L_2^2(D)$ be defined by

$$
\mathcal{P}_{\theta,\mu} = \mathcal{U}^{\theta,\mu} \mathcal{P}_\theta (\mathcal{U}^{\theta,\mu})^{-1}.
$$

It now follows from the results of the previous section that

$$(4.3) \quad \mathcal{P}_{\theta,\mu} \mathcal{B}_{\phi}^{\theta,\mu} = \mathcal{B}_{\phi}^{\theta,\mu} - \mathcal{B}_{\phi}^{\theta,\phi},$$

and since the left hand side is a contraction, we conclude that

$$(4.4) \quad \| (\mathcal{B}_{\psi}^{\theta,\mu} - \mathcal{B}_{\phi}^{\theta,\mu})[g] \|_{L_2^2(D)} \leq \| g \|_{L_2^2(D)}, \quad g \in L_2^2(D).$$

References


Received 12 March 2007