AN EXTENSION THEOREM
FOR SUPERTEMPERATURES

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Abstract. We present an analogue for supertemperatures of a well-known extension theorem on superharmonic functions.

1. Introduction

We call solutions of the heat equation temperatures, and the corresponding supersolutions supertemperatures. See [4] and [5] for details. The purpose of this paper is to present an analogue for supertemperatures of the following superharmonic function extension theorem.

Let $K$ be a compact subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ is connected. If $u$ is superharmonic on some open superset of $K$, then there exists a superharmonic function $\bar{u}$ on $\mathbb{R}^n$ such that $\bar{u} = u$ on a neighbourhood of $K$.

This result can be found in [1], p. 192.

For the case of supertemperatures on open subsets of $\mathbb{R}^{n+1}$, the condition that the complement of $K$ be connected is still necessary, but is no longer sufficient, as the following example shows.

We need some notation. If $p = (x, t)$ and $p_0 = (x_0, t_0)$ are two points in $\mathbb{R}^n \times \mathbb{R}$, we put

$$W(p_0, p) = (4\pi(t_0 - t))^{-\frac{n}{2}} \exp\left(-\frac{\|x_0 - x\|^2}{4(t_0 - t)}\right)$$

if $t_0 > t$, and $W(p_0, p) = 0$ if $t_0 \leq t$.

Example. Let

$$K = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|^2 + t^2 = 1, t \leq 1/2\}$$

be the part of boundary of the unit ball (centred at the origin) where $t \leq 1/2$. Put

$$u(p) = -W(p, 0) \quad \text{for all} \quad p \in \mathbb{R}^{n+1}.$$ 

Then $u$ is a temperature on $\mathbb{R}^{n+1} \setminus \{0\}$, which is an open superset of $K$. Suppose that there is a supertemperature $\bar{u}$ on $\mathbb{R}^{n+1}$ such that $\bar{u} = u$ on an open superset $D$.
of $K$. Then the function $v = \bar{u} - u$ is a supertemperature on $\mathbb{R}^{n+1}$, and is identically zero on $D$. Consider $v$ on the set

$$E = \{(x,t) \in \mathbb{R}^{n+1} : \|x\|^2 + t^2 < 1, t < 1/2\}.$$ 

Since $v \equiv 0$ on $K$, the boundary minimum principle shows that $v \geq 0$ on $E$. Since $D$ is an open superset of $K$, we can find a point $p_0 = (x_0, t_0) \in E$ such that $v(p_0) = 0$ and $t_0 > 0$. Now the strong minimum principle implies that $v \equiv 0$ on $E_0 = \{(x,t) \in E : t < t_0\}$, an open set containing the origin. So $\bar{u} = u$ on $E_0$. But $\bar{u}$ is bounded below on $E_0$, whereas $u$ is unbounded below, so we have a contradiction.

Before describing our theorem, we collect together the various pieces of notation needed for the remainder of this note. See [4] and [5] for details of these concepts.

The heat ball $\Omega(p_0; c)$ is defined for $c > 0$ by

$$\Omega(p_0; c) = \{p \in \mathbb{R}^{n+1} : W(p_0, p) > (4\pi c)^{-\frac{n}{2}}\}.$$

We shall write $\tau(c)$ for $(4\pi c)^{-\frac{n}{2}}$. We shall use the characteristic surface mean values of supertemperatures. For each $x \in \mathbb{R}^n$ and $t > 0$, we put

$$Q(x,t) = \|x\|^2(4\|x\|^2 t^2 + (\|x\|^2 - 2nt)^2)^{-1/2}.$$

Then the mean value is defined by

$$\mathcal{M}(u; x_0, t_0; c) = \tau(c) \int_{\partial \Omega(x_0, t_0; c)} Q(x_0 - x, t_0 - t) u(x, t) \, d\sigma$$

for any function $u$ such that the integral exists. Here $\sigma$ denotes surface area measure.

If $E$ is an open set in $\mathbb{R}^{n+1}$ and $p_0 \in E$, we denote by $\Lambda(p_0, E)$ (respectively $\Lambda^*(p_0, E))$ the set of all points $p \in E \setminus \{p_0\}$ that can be joined to $p_0$ by a polygonal line in $E$ along which the temporal variable $t$ is strictly increasing (respectively decreasing) as the line is described from $p$ to $p_0$. In particular, if $B = B(p_0, r)$ is an open ball with centre $p_0 = (x_0, t_0)$ and radius $r > 0$, then $\Lambda(p_0, B)$ is the open half-ball

$$\{(x,t) : \|x - x_0\|^2 + (t - t_0)^2 < r^2, t < t_0\}.$$ 

Furthermore, $\Lambda^*(p_0, \mathbb{R}^{n+1}) = \mathbb{R}^n \times [t_0, \infty[.$

If $q \in \partial E$, and there is an open ball $B = B(q, \epsilon)$ such that $\Lambda(q, B) \subseteq E$, we call $q$ an abnormal boundary point of $E$, and write $q \in \text{ab}(\partial E)$. If $\epsilon$ can be chosen so that $\Lambda(q, B) = B \cap E$, then we call $q$ an abnormal boundary point of the first kind, and write $q \in \text{ab}_1(\partial E)$. Otherwise, we call $q$ an abnormal boundary point of the second kind, and write $q \in \text{ab}_2(\partial E)$. We also put $\text{n}(\partial E) = (\partial E) \setminus \text{ab}(\partial E)$, and call its elements normal boundary points of $E$. The set $\text{ess}(\partial E)$, defined by $\text{ess}(\partial E) = \text{n}(\partial E) \cup \text{ab}_2(\partial E)$, is called the essential boundary of $E$, and is the part of the boundary that is relevant when using the minimum principle, or when considering the Dirichlet problem.

The definition of $\Lambda(p_0, E)$ can be extended in an obvious way to the case where $p_0 \in \text{ab}(\partial E)$. The definition of $\Lambda^*(p_0, E)$ can be extended in a similar way.

If $E$ is a bounded open set, and $f$ is a continuous real-valued function on $\text{ess}(\partial E)$, then there is a unique temperature on $E$ that is associated to $f$ by the PWB method.
It is denoted by $H^E_f$, and is called the Dirichlet solution for $f$ on $E$. We use the concept of Dirichlet solution in [5] because we need it to be aligned with the strongest form of the boundary minimum principle, also given in [5].

2. The theorem

So a stronger condition than the connectedness of $\mathbb{R}^{n+1}\setminus K$ is required in the present case. To motivate our condition, we first re-write the condition of connectedness of $\mathbb{R}^n\setminus K$ for the superharmonic case. Given $x_0$ in an open set $D$, let $\Gamma(x_0, D)$ denote the component of $D$ that contains $x_0$. Then obviously $K \subseteq \mathbb{R}^n = \Gamma(x_0, \mathbb{R}^n)$, and $\mathbb{R}^n\setminus K$ is connected if and only if there is a point $x_0 \in \mathbb{R}^n\setminus K$ such that $\Gamma(x_0, \mathbb{R}^n\setminus K) = \Gamma(x_0, \mathbb{R}^n)\setminus K$.

Replacing $\Gamma$ by $\Lambda^*$ (introduced above), we get the required condition.

**Definition.** Let $K$ be a compact subset of $\mathbb{R}^{n+1}$. If there is a point $p_0$ in $\mathbb{R}^{n+1}\setminus K$ such that $K \subseteq \Lambda^*(p_0, \mathbb{R}^{n+1})$ and $\Lambda^*(p_0, \mathbb{R}^{n+1}\setminus K) = \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus K$, then we say that $\mathbb{R}^{n+1}\setminus K$ is monotonically connected to $p_0$.

In general, if $p \in \Lambda^*(p_0, \mathbb{R}^{n+1}\setminus K)$, then $p \in \mathbb{R}^{n+1}\setminus K$ and can be joined to $p_0$ by a polygonal path in $\mathbb{R}^{n+1}\setminus K$ along which the temporal variable is strictly decreasing. So $p \in \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus K$, and we have the inclusion

$$\Lambda^*(p_0, \mathbb{R}^{n+1}\setminus K) \subseteq \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus K.$$ 

Equality may fail to hold. If $K$ is as in the above Example, and $p_0$ is any point such that $K \subseteq \Lambda^*(p_0, \mathbb{R}^{n+1})$, then

$$\Lambda^*(p_0, \mathbb{R}^{n+1}\setminus K) = \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus \bar{E} \subseteq \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus K.$$

Hence $\mathbb{R}^{n+1}\setminus K$ is not monotonically connected to any point $p_0$.

**Theorem.** Let $K$ be a compact subset of an open set $E$.

(a) If $\mathbb{R}^{n+1}\setminus K$ is monotonically connected to some point, then for each supertemperature $u$ on $E$ there is a lower bounded supertemperature $\bar{u}$ on $\mathbb{R}^{n+1}$ such that $\bar{u} = u$ on a neighbourhood $U$ of $K$. Furthermore, $\bar{u}$ can be chosen to be the potential of a measure supported in $\bar{U}$, plus a constant.

(b) If $\mathbb{R}^{n+1}\setminus K$ is not monotonically connected to any point, then there exists a temperature $u$ on $E$ for which there is no supertemperature $\bar{u}$ on $\mathbb{R}^{n+1}$ that coincides with $u$ on a neighborhood of $K$.

**Proof.** We begin with (b). Suppose that $\mathbb{R}^{n+1}\setminus K$ is not monotonically connected to any point. Choose a point $p_0$ such that $K \subseteq \Lambda^*(p_0, \mathbb{R}^{n+1})$. There is some point $p_1 \in \Lambda^*(p_0, \mathbb{R}^{n+1})\setminus K$ that does not belong to $\Lambda^*(p_0, \mathbb{R}^{n+1}\setminus K)$, and so the same is true of every point in the set $S = \Lambda(p_1, \mathbb{R}^{n+1}\setminus K)$. Choose a point $p^* \in S$, and put $u = -W(\cdot, p^*)$ on $\mathbb{R}^{n+1}$. Then, in particular, $u$ is a temperature on the open superset $\mathbb{R}^{n+1}\setminus \{p^*\}$ of $K$. Suppose that there is a supertemperature $\bar{u}$ on $\mathbb{R}^{n+1}$ such that $\bar{u} = u$ on an open superset $D$ of $K$. Note that, by [5] Lemma 1, $\text{ess}(\partial S) \subseteq \text{ess}(\partial(\mathbb{R}^{n+1}\setminus K)) \subseteq \partial(\mathbb{R}^{n+1}\setminus K) = \partial K \subseteq D$. 


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The function \( v = \bar{u} - u \) is a supertemperature on \( \mathbb{R}^{n+1} \) and identically zero on \( D \).
Since \( \text{ess}(\partial S) \subseteq D \), it follows from the minimum principle that \( v \geq 0 \) on \( S \).
Since \( D \) is an open superset of \( K \), for each point \( p \in S \) there is a point \( p' \in A^*(p, S) \cap D \).
Since \( v(p') = 0 \), the strong minimum principle shows that \( v(p) = 0 \) also.
So \( \bar{u} = u \) on \( S \), which is impossible because \( u \) is unbounded below on any neighbourhood of \( p^* \), and the supertemperature \( \bar{u} \) is locally bounded below on \( \mathbb{R}^{n+1} \).
So such a function \( \bar{u} \) cannot exist if \( \mathbb{R}^{n+1} \setminus K \) is not monotonically connected to any point.

The proof of part (a) of the Theorem requires several lemmas. The first of these requires the concept of a block set.

### 3. Block sets

**Definition.** An open set \( B \) in \( \mathbb{R}^{n+1} \) will be called a block set if it can be written as a union
\[
B = \bigcup_{i=1}^{m} R_i
\]
of finitely many open rectangles. (By a rectangle we mean an \((n+1)\)-dimensional interval.)

Note that, if \( B \) is a block set and \( R \) is a rectangle, then \( B \setminus \bar{R} \) is also a block set.
To see this, first choose an open rectangle \( X \) which contains \( B \cup \bar{R} \). Then \( X \setminus \bar{R} \) is a block set, because
\[
X = \prod_{i=1}^{n+1} [x_i, y_i], \quad \bar{R} = \prod_{i=1}^{n+1} [a_i, b_i], \quad x_i < a_i < b_i < y_i
\]
implies that (with a slight abuse of notation)
\[
X \setminus \bar{R} = \bigcup_{k=1}^{n+1} \left( \left( \prod_{i \neq k} [x_i, y_i] \times ]a_k, b_k[ \right) \bigcup \left( \prod_{i \neq k} [x_i, y_i] \times ]b_k, y_k[ \right) \right).
\]
Now \( B \setminus \bar{R} = B \cap (X \setminus \bar{R}) \) is an intersection of two block sets, which is itself a block set; because if
\[
B = \bigcup_{i=1}^{m} R_i \quad \text{and} \quad X \setminus \bar{R} = \bigcup_{j=1}^{q} S_j,
\]
then
\[
B \setminus \bar{R} = \left( \bigcup_{i=1}^{m} R_i \right) \cap \left( \bigcup_{j=1}^{q} S_j \right) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{q} (R_i \cap S_j),
\]
and \( R_i \cap S_j \) is a rectangle (or empty) for every \( i \) and \( j \).

It follows that, if \( B \) and \( C \) are both block sets, then \( B \setminus \bar{C} \) is also a block set.

In the proof of the superharmonic case given in [1], the relative complement \( E \setminus K \) of a compact set \( K \) in an open set \( E \), is approximated from within by Dirichlet regular sets. This technique is not available in the present case, and instead we approximate \( K \) from without by the closures of block sets. We need to be able to
do this in such a way that, if \( \mathbb{R}^{n+1} \setminus K \) is monotonically connected to a point \( p_0 \), then the approximating block sets are too. This is the purpose of our first lemma.

**Lemma 1.** Let \( E \) be an open set in \( \mathbb{R}^{n+1} \), and let \( K \) be a compact subset of \( E \). Then there is a block set \( B \) such that \( K \subseteq B \subseteq E \). Furthermore, if \( \mathbb{R}^{n+1} \setminus K \) is monotonically connected to some point \( p_0 \in \mathbb{R}^{n+1} \setminus K \), then \( B \) can be chosen so that \( \mathbb{R}^{n+1} \setminus B \) is also monotonically connected to \( p_0 \).

**Proof.** Since \( K \) is a compact subset of the open set \( E \), we can cover it with finitely many open rectangles whose closures lie in \( E \). The union \( B \) of these rectangles is a block set such that \( K \subseteq B \) and \( B \subseteq E \).

If \( \mathbb{R}^{n+1} \setminus K \) is monotonically connected to \( p_0 \), then the above choice of \( B \) may not suffice to make \( \mathbb{R}^{n+1} \setminus B \) monotonically connected to \( p_0 \). Suppose that there are points \( p_\alpha \) in \( \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus B) \) that do not belong to \( \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus K) \). Since \( \mathbb{R}^{n+1} \setminus K \) is monotonically connected to \( p_0 \), we have \( p_\alpha \in \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus K) \), so that there is a polygonal path from \( p_\alpha \) to \( p_0 \) in \( \mathbb{R}^{n+1} \setminus K \) along which time is strictly decreasing. But \( p_\alpha \) does not belong to \( \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus B) \), so any such path must meet \( B \). Let \( \Gamma(p_\alpha, p_0) \) denote the family of all such paths from \( p_\alpha \) to \( p_0 \). Then every \( \gamma \in \Gamma(p_\alpha, p_0) \) meets \( B \), and there exists

\[
t_{\alpha, \gamma} = \max\{t : (x, t) \in \gamma \cap \bar{B}\}.
\]

Put

\[
t_\alpha = \inf\{t_{\alpha, \gamma} : \gamma \in \Gamma(p_\alpha, p_0)\}.
\]

Because \( B \) is a block set, the infimum is attained. Choose a path \( \delta \in \Gamma(p_\alpha, p_0) \) such that \( t_{\alpha, \delta} = t_\alpha \) and the point \( q_\alpha = (y_\alpha, t_\alpha) \in \delta \cap \bar{B} \) is in the relative interior of \( (\mathbb{R}^n \times \{t_\alpha\}) \cap \partial B \). Then \( \Lambda^\ast(q_\alpha, \mathbb{R}^{n+1} \setminus B) \) is defined and contains \( p_\alpha \). Put

\[
I(q_\alpha) = \Lambda^\ast(q_\alpha, \mathbb{R}^{n+1} \setminus B) \setminus \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus B),
\]

which is nonempty because it contains \( p_\alpha \).

Take another point \( q_\beta \), chosen in the same way relative to another point \( p_\beta \) in \( \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus B) \) that does not belong to \( \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus B) \). If \( q_\alpha \) and \( q_\beta \) belong to the same component of \( (\mathbb{R}^n \times \{t\}) \cap \partial B \) for some \( t \), then \( I(q_\alpha) = I(q_\beta) \). Since \( B \) is a block set, there are only finitely many different values of \( t \) for which \( \mathbb{R}^n \setminus \{t\} \) contains some \( q_\alpha \), and each \( (\mathbb{R}^n \times \{t\}) \cap \partial B \) has only finitely many components. So there are only finitely many distinct sets \( I(q_\alpha) \). We choose a unique point \( q_k \) to represent each distinct set \( I(q_k) \), and thus obtain a finite set \( \{q_1, \ldots, q_m\} \) such that

\[
(\Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus \bar{B})) \setminus \Lambda^\ast(p_0, \mathbb{R}^{n+1} \setminus \bar{B}) \subseteq \bigcup_{k=1}^m \Lambda^\ast(q_k, \mathbb{R}^{n+1} \setminus \bar{B}).
\]

Since \( q_k \in \mathbb{R}^{n+1} \setminus K \) for \( k \in \{1, \ldots, m\} \), and \( \mathbb{R}^{n+1} \setminus K \) is monotonically connected to \( p_0 \), we can choose a polygonal path \( \gamma_k \) that connects \( q_k \) to \( p_0 \) along which the temporal variable is strictly decreasing. Since \( \bigcup_{k=1}^m \gamma_k \) is compact, we can cover it with finitely many open rectangles whose closures do not intersect \( K \). Let \( U \) denote
the union of the closures of these rectangles. Now $B \setminus U$ is a block set containing $K$, and $\mathbb{R}^{n+1} \setminus (B \setminus U)$ is monotonically connected to $p_0$. □

4. Preliminary extension lemmas

The remaining lemmas are all relatively minor extension results. The first is the direct analogue of a result in [1], p. 66.

**Lemma 2.** Let $v$ be a supertemperature on an open set $E$, and let $h$ be a supertemperature on an open subset $D$ of $E$. If

$$(1) \liminf_{p \to q, p \in D} h(p) \geq v(q) \quad \text{for all } q \in E \cap \partial D,$$

and $w$ is defined on $E$ by

$$w(p) = \begin{cases} (h \wedge v)(p) & \text{if } p \in D, \\ v(p) & \text{if } p \in E \setminus D, \end{cases}$$

then $w$ is a supertemperature on $E$.

**Proof.** It is clear that $w$ is a supertemperature on $E \setminus \partial D$, that $w(p) > -\infty$ for all $p \in E$, and that $w < +\infty$ on a dense subset of $E$. Condition (1) ensures that, for each point $q \in E \cap \partial D$,

$$\liminf_{p \to q} w(p) = \min \left\{ \liminf_{p \to q} h(p), \liminf_{p \to q} v(p) \right\} \geq v(q) = w(q),$$

so that $w$ is lower semicontinuous on $E$. It remains to check that the supertemperature mean value inequality is satisfied at points of $E \cap \partial D$. If $q \in E \cap \partial D$ and $\Omega(q; c) \subseteq E$, then

$$w(q) = v(q) \geq \mathcal{M}(v; q, c) \geq \mathcal{M}(w; q, c).$$

Hence $w$ is a supertemperature on $E$, by [4] Theorem 15. □

In practice, condition (1) is rarely satisfied when $q \in \text{ab}(\partial D)$, and this limits the usefulness of Lemma 2. We need a substitute result for the case where $D = E \setminus \bar{T}$ with $T$ a block set such that $\bar{T} \subseteq E$. In this case, the set of all horizontal edges of $T$ contains $E \cap \text{ab}_2(\partial D)$, and is a closed polar set, in view of [5], p. 280.

**Lemma 3.** Let $E$ be an open set, let $T$ be a block set such that $\bar{T} \subseteq E$, and let $D = E \setminus \bar{T}$. Let $v$ be a supertemperature on $E$, and let $h$ be a supertemperature on $D$. If

$$(2) \liminf_{p \to q, p \in D} h(p) \geq v(q) \quad \text{for all } q \in E \cap \text{int}(\partial D),$$

$$(3) \liminf_{p \to q, p \in D} h(p) > -\infty \quad \text{for all } q \in E \cap \text{ab}(\partial D),$$

and

$$(4) \liminf_{p \to q, p \in D} h(p) \leq v(q) \quad \text{for all } q \in E \cap \text{ab}_1(\partial D),$$

then $w$ is a supertemperature on $E$. □
then the function \( w \), defined on \( E \backslash ab_2(\partial D) \) by

\[
w(q) = \begin{cases} 
(h \wedge v)(q) & \text{if } q \in D, \\
\liminf_{p \to q, p \in D} h(p) & \text{if } q \in E \cap ab_1(\partial D), \\
v(q) & \text{if } q \in E \backslash (D \cup ab(\partial D)),
\end{cases}
\]

has a unique extension to a supertemperature on \( E \).

**Proof.** Let \( Z \) denote the closed set of all horizontal edges of \( T \). Then \( E \cap ab_2(\partial D) \subseteq Z \). Clearly \( w \) is a supertemperature on \( E \backslash \partial D \), and \( w > -\infty \) on \( E \backslash ab_2(\partial D) \), which contains \( E \backslash Z \). Furthermore, because \( T \) is a block set, \( E \cap ab(\partial D) \) is contained in the union of a finite set of hyperplanes of the form \( \mathbb{R}^n \times \{ t \} \), and so \( w < +\infty \) on a dense subset of \( E \).

Next we check the lower semicontinuity. If \( q \in E \cap n(\partial D) \), then

\[
\liminf_{p \to q} w(p) = \min \left\{ \liminf_{p \to q, p \in D} h(p), \liminf_{p \to q} v(p) \right\} \geq v(q) = w(q),
\]

in view of (2). If \( q \in E \cap ab_1(\partial D) \), then condition (4) and [5] Lemma 12 imply that

\[
\liminf_{p \to q, p \in D} h(p) \leq \liminf_{p \to q} v(p),
\]

so that

\[
\liminf_{p \to q} w(p) = \min \left\{ \liminf_{p \to q, p \in D} h(p), \liminf_{p \to q} v(p) \right\} = \liminf_{p \to q, p \in D} h(p) = w(q).
\]

Hence \( w \) is lower semicontinuous on \( E \backslash ab_2(\partial D) \), and in particular on \( E \backslash Z \).

We now check that the supertemperature mean value inequality is satisfied at every point of \( E \cap (\partial D \backslash ab_2(\partial D)) \). Because \( T \) is a block set, \( E \cap ab(\partial D) \) is contained in the union of a finite collection of hyperplanes of the form \( \mathbb{R}^n \times \{ t \} \). Therefore, if \( q \in E \cap n(\partial D) \) we have \( \Omega(q; c) \subseteq E \backslash ab(\partial D) \) for all sufficiently small values of \( c \). For those values,

\[
w(q) = v(q) \geq \mathcal{M}(v; q, c) \geq \mathcal{M}(w; q, c).
\]

On the other hand, if \( q \in E \cap ab_1(\partial D) \), then condition (3) implies that \( w \) is bounded below on some open rectangle \( R \) such that \( q \in ab(\partial R) \). Therefore we can use condition (4), Fatou’s Lemma, and the lower semicontinuity of \( h \wedge v \), to obtain

\[
w(q) = \liminf_{p \to q, p \in D} h(p) \geq \liminf_{p \to q, p \in D} (h \wedge v)(p) \geq \liminf_{p \to q, p \in D} \mathcal{M}(h \wedge v; p, c)
\]

\[
\geq \mathcal{M}(h \wedge v; q, c) = \mathcal{M}(w; q, c)
\]

for all sufficiently small values of \( c \). It follows from [4], Theorem 15, that \( w \) is a supertemperature on \( E \backslash Z \).

Since \( Z \) is a closed polar subset of \( E \), we have only to show that \( w \) is locally bounded below on \( E \) and apply [5], Theorem 29, to complete the proof. Clearly \( w \) is bounded below on compact subsets of \( E \backslash \partial D \). Condition (3) (along with the lower finiteness of \( v \)) implies that \( w \) is bounded below on some neighbourhood of
any \( q \in E \cap \text{ab}(\partial D) \), and condition (2) has a similar implication for \( q \in E \cap \text{n}(\partial D) \). So \( w \) is locally bounded below on \( E \), and the result follows. \( \square \)

In the proof of our theorem, we first extend the given supertemperature to a set of the form
\[
\Omega^*(p_0; c) = \{ p \in \mathbb{R}^{n+1} : W(p, p_0) > \tau(c) \},
\]
which is the reflection of \( \Omega(p_0; c) \) in the hyperplane \( \mathbb{R}^n \times \{t_0\} \), if \( p_0 = (x_0, t_0) \). The following lemma then gives an extension to the whole of \( \mathbb{R}^{n+1} \).

**Lemma 4.** Let \( u \) be a supertemperature on \( \Omega^* = \Omega^*(p^*; c^*) \), and let \( S \) be an open set such that \( \bar{S} \subseteq \Omega^* \). Then there is a supertemperature \( \bar{u} \) on \( \mathbb{R}^{n+1} \), such that \( \bar{u} = u \) on \( S \) and \( \bar{u} \) is lower bounded on \( \mathbb{R}^{n+1} \).

**Proof.** Let \( p^* = (x^*, t^*) \), and choose \( t_1 > t^* \) such that \( \bar{S} \subseteq \mathbb{R}^n \times [t_1, \infty[ \). Choose \( \gamma < c^* \) such that \( \bar{S} \subseteq \Omega^*(p^*; \gamma) \), and put \( \Omega^1_\gamma = \Omega^*(p^*; \gamma) \cap (\mathbb{R}^n \times [t_1, \infty[) \). Then \( \Omega^1_\gamma \) is a compact subset of \( \Omega^* \), so that we can find \( k \in \mathbb{R} \) such that \( u > k \) on \( \Omega^1_\gamma \). Let \( R^S_{\bar{u} - k} \) be the reduction of \( u - k \) relative to \( S \) in \( \Omega^1_\gamma \) (see [2] for details about reductions), and put

\[
u_1 = R^S_{\bar{u} - k} + k \quad \text{on} \quad \Omega^1_\gamma.
\]

Then \( u_1 \) is a supertemperature on \( \Omega^1_\gamma \), \( u_1 \) is a temperature on \( \Omega^1_\gamma \backslash \bar{S} \), \( k \leq u_1 \leq u \) on \( \Omega^1_\gamma \), and \( u_1 = u \) on \( S \).

Choose \( \alpha \) and \( \beta \) such that \( 0 < \alpha < \beta < \gamma \) and \( \bar{S} \subseteq \Omega^*(p^*; \alpha) \). Put \( \Omega^*(\alpha) = \Omega^*(p^*; \alpha) \), and \( \Omega^1_\alpha(\alpha) = \Omega^*(\alpha) \cap (\mathbb{R}^n \times [t_1, \infty[) \); similarly for \( \beta \). Since \( u_1 \) is continuous on \( \Omega^1_\gamma \backslash \bar{S} \), it has a maximum value \( M(\alpha) \geq k \) on \( \partial \Omega^*(\alpha) \cap (\mathbb{R}^n \times [t_1, \infty[) \). Define \( u_2 \) on \( \mathbb{R}^{n+1} \) by putting

\[
u_2(p) = \frac{M(\alpha) - k}{\tau(\alpha) - \tau(\beta)}(W(p, p^*) - \tau(\beta)) + k.
\]

Then \( u_2 \) is a supertemperature, \( u_2 = M(\alpha) \) on \( \partial \Omega^*(\alpha) \backslash \{p^*\} \), and \( u_2 = k \) on \( \partial \Omega^*(\beta) \backslash \{p^*\} \). Now define \( u_3 \) on \( \mathbb{R}^n \times [t_1, \infty[ \) by

\[
u_3 = \begin{cases} u_1 & \text{on} \quad \Omega^*(\alpha) \cap (\mathbb{R}^n \times [t_1, \infty[), \\ u_1 \wedge u_2 & \text{on} \quad \Omega^1_\beta(\alpha), \\ u_2 & \text{on} \quad (\mathbb{R}^n \times [t_1, \infty[) \backslash \Omega^1_\beta. \end{cases}
\]

We apply Lemma 2 with \( E = \Omega^1_\beta, v = u_1, D = \Omega^1_\beta \backslash \Omega^1_\alpha, \) and \( h = u_2 \), noting that for all \( q \in E \cap \partial D = \bar{\Omega}^1_\beta(\beta) \cap \partial \Omega^1_\alpha(\alpha) \) we have

\[
\lim \inf_{p \to q, p \in D} h(p) \geq u_2(q) = M(\alpha) \geq u_1(q) = v(q).
\]

Thus \( u_3 \) is a supertemperature on \( \Omega^1_\beta(\beta) \).

A second application of Lemma 2, this time with \( E = (\mathbb{R}^n \times [t_1, \infty[) \backslash \Omega^1_\alpha(\alpha) \), \( v = u_2, D = \Omega^1_\beta \backslash \Omega^1_\alpha, \) and \( h = u_1 \), so that for all \( q \in E \cap \partial D = (\mathbb{R}^n \times [t_1, \infty[) \cap \partial \Omega^1_\beta(\beta) \) we have

\[
\lim \inf_{p \to q, p \in D} h(p) \geq u_1(q) \geq k = u_2(q) = v(q).
\]
the whole of \((R^n \times [t_1, \infty[) \setminus \Omega^*_1(\alpha)\), and therefore on the whole of \((R^n \times [t_1, \infty[)\).

Since \(u_1 \geq k\) on \(\Omega^*_1(\gamma)\), and

\[
u_2 \geq \frac{M(\alpha) - k}{\tau(\alpha) - \tau(\beta)} (-\tau(\beta)) + k = \frac{-\tau(\beta)M(\alpha) + \tau(\alpha)k}{\tau(\alpha) - \tau(\beta)}
\]
on \(R^{n+1}\), \(u_3\) is lower bounded. Putting

\[
\bar{u} = \begin{cases} u_3 & \text{on } R^n \times [t_1, \infty[, \\ \inf u_3 & \text{on } R^n \times [\infty, t_1], \end{cases}
\]

we obtain a lower bounded supertemperature \(\bar{u}\) on \(R^{n+1}\) such that \(\bar{u} = u_3 = u_1 = u\) on \(S\).

5. Proof of part (a) of the theorem

Let \(K\) be a compact subset of an open set \(E\). We must prove the following statement:

If \(R^{n+1} \setminus K\) is monotonically connected to some point \(p_0\), then for each supertemperature \(u\) on \(E\) there is a lower bounded supertemperature \(\bar{u}\) on \(R^{n+1}\) such that \(\bar{u} = u\) on a neighbourhood \(U\) of \(K\). Furthermore, \(\bar{u}\) can be chosen to be the potential of a measure supported in \(U\), plus a constant.

Proof. We may suppose that \(E\) is bounded, and that \(u > 0\) on \(E\).

By Lemma 1, we can find an open (block) set \(S\) such that \(K \subseteq S\), \(\bar{S} \subseteq E\), and \(R^{n+1} \setminus \bar{S}\) is monotonically connected to \(p_0\). Let \(v = R^S_0\), the reduction of \(u\) relative to \(S\) in \(E\). Then \(v\) is a supertemperature on \(E\), \(v\) is a temperature on \(E\setminus\bar{S}\), \(0 \leq v \leq u\) on \(E\), and \(v = u\) on \(S\). Using Lemma 1 again, we can find a block set \(T\) such that \(\bar{S} \subseteq T\), \(T \subseteq E\), and \(R^{n+1}\setminus\bar{T}\) is monotonically connected to \(p_0\). Choose \(p^* \in R^{n+1}\) and \(c^* > 0\) such that \(E \cup \{p_0\} \subseteq \Omega(p^*; c^*)\), and put \(\Omega^* = \Omega(p^*; c^*)\), \(A = \Omega^* \setminus \bar{T}\). We shall extend \(u\) to a supertemperature on \(\Omega^*\), then use Lemma 4 to further extend \(u\) to \(R^{n+1}\).

Put \(g_1 = v\) on \(\partial T\), \(g_1 = 0\) on \(\partial \Omega^*\), \(g_2 = 0\) on \(\partial T\), and \(g_2 = 1\) on \(\partial \Omega^*\). Define

\[
h_k = H^A_{g_1} - kH^A_{g_2} \quad \text{for all } k \in N.
\]

Note that \(v\) is continuous on \(\partial T\), because \(v\) is a temperature on \(E\setminus\bar{S}\). For each point \((x, t) \in A\) such that \(t < \min\{s : (y, s) \in \bar{T}\}\), we have \(H^A_{g_2}(x, t) = 1\) because \(g_2 = 1\) on \(\partial \Omega^*\). In particular, \(H^A_{g_2}(p_0) = 1\). Since \(R^{n+1}\setminus\bar{T}\) is monotonically connected to \(p_0\), for all \(p \in \Lambda(p_0, R^{n+1})\setminus\bar{T}\) we have \(p_0 \in \Lambda(p, R^{n+1})\setminus\bar{T}\), and therefore \(p_0 \in \Lambda(p, A)\) if \(p \in A\). Therefore, by the strong minimum principle, \(H^A_{g_2} > 0\) on \(A\), so that \(\{h_k\}\) decreases to \(-\infty\) on \(A\) as \(k \to \infty\).

Our method of extending \(u\) to \(\Omega^*\) requires that \(h_j \leq v\) on \(E\setminus\bar{T}\) for some \(j\). Because \(\{h_k\}\) decreases to \(-\infty\) on \(A\), we can find \(j\) such that \(h_j \leq 0\) on \(\partial E\).
Therefore, for all $q \in \partial E$ we have
\[
\liminf_{p \to q, \ p \in E} v(p) \geq h_j(q) = \lim_{p \to q} h_j(p).
\]
Consider the points of $\partial T$ as boundary points in the Dirichlet problem on $A$. Because $T$ is a block set, all points of $\partial T \cap n(\partial A)$ are regular, by the parabolic tusk test in [3]. All points of $\partial T \cap ab_1(\partial A)$ can be ignored, because they are irrelevant to both the Dirichlet problem on $A$ and the use of the minimum principle on $A$. Again because $T$ is a block set, all points of $\partial T \cap ab_2(\partial A)$ are contained in the union of finitely many sets of the form $\{(x_1, \ldots, x_n, t) : t = a, x_j = b \text{ for some } j\}$, each of which is polar by [5], p. 280. So $\partial T \cap ab_2(\partial A)$ is also polar. It follows that
\[
\lim_{p \to q, \ p \in A} h_j(p) = v(q) = \lim_{p \to q, \ p \in \bar{T}} v(p),
\]
for all $q \in \partial T \cap \text{ess}(\partial A) \setminus Z$ for some polar set $Z$. Furthermore, because $g_1 \leq \max_{\partial T} v$ and $g_2 \geq 0$ on $\partial A$, we have $h_j \leq \max_{\partial T} v$ on $A$, so that $v - h_j$ is lower bounded on $E \setminus \bar{T}$. Applying the minimum principle in [5], p. 284, to $v - h_j$ on $E \setminus \bar{T}$, we obtain $v \geq h_j$.

We now put $D = E \setminus \bar{T}$ and apply Lemma 3 with $h = h_j$, noting that
\[
\lim_{p \to q, \ p \in D} h_j(p) = v(q) \quad \text{for all} \quad q \in E \cap n(\partial D),
\]
because $E \cap n(\partial D) = \partial T \cap n(\partial A)$ and all such points are regular;
\[
\liminf_{p \to q, \ p \in D} h_j(p) > -\infty \quad \text{for all} \quad q \in E \cap \text{ab}(\partial D)
\]
because $h_j \geq -\gamma$ on $A$; and
\[
\liminf_{p \to q, \ p \in D} h_j(p) \leq v(q) \quad \text{for all} \quad q \in E \cap \text{ab}_1(\partial D)
\]
because $h_j \leq v$ on $D$, $v$ is continuous on $\partial T$, and $\text{ab}_1(\partial D) \cap E = \text{ab}_1(\partial D) \cap \partial T$. Thus we see that the function $w$, defined by
\[
w = \begin{cases} h_j = h_j \wedge v & \text{on } D = E \setminus \bar{T} \\ v & \text{on } T, \end{cases}
\]
can be extended to a supertemperature $\bar{w}$ on $E$. Since $h_j$ is a temperature on $A$, the function $\bar{w}$ can be extended by $h_j$ to a supertemperature on $\Omega^*$.

Next, by Lemma 4, there is a lower bounded supertemperature $u_0$ on $\mathbb{R}^{n+1}$ such that $u_0 = w = v = u$ on the neighbourhood $S$ of $K$. Now let $U$ be any open set such that $K \subseteq U \subseteq S$. To show that $u_0$ can be taken to be the potential of a measure supported in $\bar{U}$, plus a constant, we first put $m = \inf u_0$ and $u_1 = R_{u_0-m}$, the reduction of $u_0 - m$ relative to $U$ in $\mathbb{R}^{n+1}$. Since $U$ is open, $u_1$ is a nonnegative supertemperature on $\mathbb{R}^{n+1}$, and $u_1 = u_0 - m$ on $U$. In fact, because $\bar{U}$ is compact, $u_1$ is a potential by [2], p. 319, (m). Furthermore, $u_1$ is a temperature on $\mathbb{R}^{n+1} \setminus \bar{U}$, and so its Riesz measure is supported in $\bar{U}$, by [5] Theorem 20. The function $\bar{u} = u_1 + m$ has the required form.
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References


Received 8 December 2006