CHARACTERIZATION OF REARRANGEMENT INVARIANT SPACES WITH FIXED POINTS FOR THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

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Abstract. We characterize the rearrangement invariant spaces for which there exists a non-constant fixed point, for the Hardy–Littlewood maximal operator (the case for the spaces $L^p(\mathbb{R}^n)$ was first considered in [7]). The main result that we prove is that the space $L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is minimal among those having this property.

1. Introduction

The centered Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on the Lebesgue space $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| \, dy,$$

where $|B_r|$ denotes the measure of the Euclidean ball $B_r$ centered at the origin of $\mathbb{R}^n$.

In this paper we study the existence of non-constant fixed points of the maximal operator $\mathcal{M}$ (i.e., $\mathcal{M} f = f$) in the framework of the rearrangement invariant (r.i.) functions spaces (see Section 2 below). We will use some of the estimates proved in [7], where the case $L^p(\mathbb{R}^n)$ was studied, and show that they can be sharpened to obtain all the rearrangement invariant norms with this property (in particular we extend Korry’s result to the end point case $p = n/(n-2)$, where the weak-type spaces have to considered). The main argument behind this problem is the existence of a minimal space $L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ contained in all the r.i. spaces with the fixed point property.

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2. Background on rearrangement invariant spaces

Since we work in the context of rearrangement invariant spaces it will be convenient to start by reviewing some basic definitions about these spaces.

A rearrangement invariant space \( X = X(\mathbb{R}^n) \) (r.i. space) is a Banach function space on \( \mathbb{R}^n \) endowed with a norm \( \| \cdot \|_{X(\mathbb{R}^n)} \) such that
\[
\|f\|_{X(\mathbb{R}^n)} = \|g\|_{X(\mathbb{R}^n)}
\]
whenever \( f^* = g^* \). Here \( f^* \) stands for the non-increasing rearrangement of \( f \), i.e., the non-increasing, right-continuous function on \([0, \infty)\) equimeasurable with \( f \).

An r.i. space \( X(\mathbb{R}^n) \) has a representation as a function space on \( X(0, \infty) \) such that
\[
\|f\|_{X(\mathbb{R}^n)} = \|f^*\|_{X(0, \infty)}.
\]

Any r.i. space is characterized by its fundamental function
\[
\phi_X(s) = \|\chi_E\|_{X(\mathbb{R}^n)}
\]
(here \( E \) is any subset of \( \mathbb{R}^n \) with \( |E| = s \)) and the fundamental indices
\[
\beta_X = \inf_{s>1} \frac{\log M_X(s)}{\log s} \quad \text{and} \quad \beta_X = \sup_{s<1} \frac{\log M_X(s)}{\log s},
\]
where
\[
M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \quad s > 0.
\]
It is well known that
\[
0 \leq \beta_X \leq \beta_X \leq 1.
\]
(We refer the reader to [2] for further information about r.i. spaces.)

3. Main result

Before formulating our main result, it will be convenient to start with the following remarks (see [7]):

**Remark 3.1.** By Lebesgue’s differentiation theorem one easily obtains that
\[
|f(x)| \leq \mathcal{M} f(x) \quad \text{a.e.} \ x \in \mathbb{R}^n;
\]
thus \( f \) is a fixed point of \( \mathcal{M} \), if and only if \( f \) is positive and
\[
\frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy \leq f(x) \quad \text{a.e.} \ x \in \mathbb{R}^n,
\]
or equivalently \( f \) is a positive super-harmonic function (i.e. \( \Delta f \leq 0 \), where \( \Delta \) is the Laplacian operator).
Invariant spaces with fixed points

Remark 3.2. If \( f \) is a non-constant fixed point of \( \mathcal{M} \), and \( \varphi \geq 0 \) belongs to the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \), with \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \), then the function \( f_t(x) = (f * \varphi_t)(x) \), with \( \varphi_t(x) = t^{-n} \varphi(x/t) \) is also a non-constant fixed point of \( \mathcal{M} \) which belongs to \( \mathcal{C}^\infty(\mathbb{R}^n) \) (notice that using the Lebesgue differentiation theorem, there exists some \( t > 0 \) such that \( f_t \) is non-constant, since \( f \) is non-constant). In particular if \( X(\mathbb{R}^n) \) is an r.i. space and \( f \in X(\mathbb{R}^n) \) is a non-constant fixed point of \( \mathcal{M} \), since \( \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) we get that \( f_t \in X(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n) \) is a non-constant fixed point of \( \mathcal{M} \).

Remark 3.3. Using the theory of weighted inequalities for \( \mathcal{M} \) (see [5]), if \( \mathcal{M} f = f \), in particular \( f \in A_1 \) (the Muckenhoupt weight class), and hence \( f(x) \, dx \) defines a doubling measure. Hence, \( f \notin L^1(\mathbb{R}^n) \). Also, using the previous remark we see that if \( f \in L^p(\mathbb{R}^n) \) is a fixed point, then \( f \in L^q(\mathbb{R}^n) \), for all \( p \leq q \leq \infty \).

Definition 3.4. Given an r.i. space \( X(\mathbb{R}^n) \), we define

\[
D_{I_2}(X(\mathbb{R}^n)) = \{ f \in L^0(\mathbb{R}^n) : \|I_2 f\|_{X(\mathbb{R}^n)} < \infty \},
\]

where \( I_2 \) is the Riesz potential,

\[
(I_2 f)(x) = \int_{\mathbb{R}^n} |x-y|^{2-n} f(y) \, dy.
\]

It is not hard to see that the space \( D_{I_2}(X(\mathbb{R}^n)) \) is either trivial or is the largest r.i. space which is mapped by \( I_2 \) into \( X(\mathbb{R}^n) \), and is also related with the theory of the optimal Sobolev embeddings (see [4] and the references quoted therein).

Theorem 3.5. Let \( X(\mathbb{R}^n) \) be an r.i. space. The following statements are equivalent:

1. There is a non-constant fixed point \( f \in X(\mathbb{R}^n) \) of \( \mathcal{M} \).
2. \( n \geq 3 \) and \( |x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n) \).
3. \( n \geq 3 \) and \( x_0 1(t) + t^{2/1-n} \chi_{[1,\infty)}(t) \in X(0,\infty) \).
4. \( n \geq 3 \) and \( (L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \subset X(\mathbb{R}^n) \).
5. \( n \geq 3 \) and \( D_{I_2}(X(\mathbb{R}^n)) \neq \{0\} \).

Proof. (1) \( \Rightarrow \) (2) Since if \( n = 1 \) or \( n = 2 \), the only positive super-harmonic functions are the constant functions (see [8, Remark 1, p. 210]), necessarily \( n \geq 3 \). Moreover, it is proved in [7] that, if \( f \in \mathcal{C}^\infty(\mathbb{R}^n) \) is a non-constant fixed point of \( \mathcal{M} \), then

\[
f(x) \geq c |x|^{2-n} \chi_{\{x:|x|>1\}}(x).
\]

Since \( f \in X(\mathbb{R}^n) \), then \( |x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n) \).

(2) \( \Rightarrow \) (3) Since if \( |x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n) \), then

\[
F(x) = \chi_{\{x:|x|\leq 1\}}(x) + |x|^{2-n} \chi_{\{x:|x|>1\}}(x) \in X(\mathbb{R}^n).
\]

\[
\int_{\mathbb{R}^n} |x-y|^{2-n} f(y) \, dy.
\]
An easy computation shows that
\[ F^*(t) \simeq \chi_{[0,1]}(t) + t^{2/n-1} \chi_{[1,\infty)}(t). \]

(3) \implies (4) Since \( f \in \left(L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\right) \) if and only if
\[ \sup_{t>0} f^*(t) W(t) < \infty, \]
where \( W(t) = \max(1, t^{1-2/n}) \), we have that
\[ f^*(t) \leq \|f\|_{L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)} W^{-1}(t) \]
and since \( W^{-1}(t) = \chi_{[0,1]}(t) + t^{2/n-1} \chi_{[1,\infty)} \in \mathcal{X}(0, \infty) \) we have that
\[ \|f\|_{\mathcal{X}(\mathbb{R}^n)} = \|f^*\|_{\mathcal{X}(0, \infty)} \leq c \|f\|_{L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)} \]
with \( c = \|W^{-1}\|_{\mathcal{X}(0, \infty)}. \)

(4) \implies (5) Since (see [9] and [1])
\[ (I_2 f)^*(t) \leq c_1 \left( t^{2/n-1} \int_0^t f^*(s) \, ds + \int_t^\infty f^*(s) s^{2/n-1} \, ds \right) \leq c_2 (I_2 f^0)^*(t) \]
where \( f^0(x) = f^*(c_n |x|^n) \), \( c_n \) = measure of the unit ball in \( \mathbb{R}^n \). (Observe that \( (f^0)^* = f^* \).) Rewriting the middle term in the above inequalities, using Fubini’s theorem, we get
\[ (I_2 f)^*(t) \leq d_1 \left( \frac{n}{n-2} \int_t^\infty f^{**}(s) s^{2/n-1} \, ds \right) \leq d_2 (I_2 f^0)^*(t), \]
where \( f^{**}(t) = t^{-1} \int_0^t f^*(s) \, ds \). Thus, \( f \in D_{I_2}(X(\mathbb{R}^n)) \) if and only if
\[ (1) \quad \left\| \int_t^\infty f^{**}(s) s^{2/n-1} \, ds \right\|_{\mathcal{X}(0, \infty)} < \infty. \]
Since
\[ F(t) = \int_t^\infty \chi_{[0,1]}(s) s^{2/n-1} \, ds = c (\chi_{[0,1]}(t) + t^{2/n-1} \chi_{[1,\infty)}(t)) \]
is a decreasing function, and
\[ F^0(x) = F(c_n |x|^n) \simeq (\chi_{\{x:|x|\leq 1\}}(x) + |x|^{2-n} \chi_{\{x:|x|>1\}}(x)) \in L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \]
we get that \( \chi_{[0,1]}^0 \in D_{I_2}(X(\mathbb{R}^n)) \).
Another argument to prove this part is the following:
Since, if $n \geq 3$ (see [2, Theorem 4.18, p. 228])
\[ I_2: L^1(\mathbb{R}^n) \to L^{n/(n-2),\infty}(\mathbb{R}^n) \quad \text{and} \quad I_2: L^{n/2,1}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \]
is bounded, we have that
\[ I_2: (L^1(\mathbb{R}^n) \cap L^{n/2,1}(\mathbb{R}^n)) \to (L^{n/(n-2),\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \subset X(\mathbb{R}^n) \]
is bounded, and hence $L^1(\mathbb{R}^n) \cap L^{n/2,1}(\mathbb{R}^n) \subset D_{I_2}(X(\mathbb{R}^n))$.

(5) $\Rightarrow$ (1) Since $n \geq 3$, we can use the classical formula of potential theory (see [10, p. 126])
\[ -h = \triangle(I_2 h) \]
to conclude that there is a positive function $f = I_2 \chi_{[0,1]} \in X(\mathbb{R}^n)$. Then $0 \leq f_t = I_2(\chi_{[0,1]} \ast \varphi_t) \in X(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ and $\triangle f_t \leq 0$. □

We now consider particular examples, like the Lorentz spaces:

**Corollary 3.6.** Let $1 \leq p < \infty$, and assume $\Lambda^p(\mathbb{R}^n, w)$ is a Banach space (i.e., $w \in B_p$ if $1 < p < \infty$ or $p \in B_{1,\infty}$ if $p = 1$, see [3]). Then, there exists a non-constant function $f \in \Lambda^p(\mathbb{R}^n, w)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and
\[ \int_1^\infty \frac{w(t)}{t^{p(1-2/n)}} dt < \infty. \]
In particular, this condition always holds, for $p > 1$ and $n$ large enough.

**Proof.** The integrability condition follows by using the previous theorem. Now, if $w \in B_p$, then there exists an $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$, and hence it suffices to take $n > 2/\varepsilon$. Observe that if $w = 1$ and $p = 1$, then $\Lambda^1(\mathbb{R}^n, w) = L^1(\mathbb{R}^n)$, which does not have the fixed point property for any dimension $n$. □

**Corollary 3.7.** Let $1 \leq p, q \leq \infty$ (if $p = 1$ we only consider $q = 1$). Then, there exists a non-constant function $f \in L^{p,q}(\mathbb{R}^n)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and
\[ \left\{ \begin{array}{l} n/(n-2) < p \leq \infty \\ p = n/(n-2) \text{ and } q = \infty. \end{array} \right. \]

**Corollary 3.8.** (See [7]) Let $1 \leq p \leq \infty$. There exists a non-constant function $f \in L^p(\mathbb{R}^n)$ such that $\mathcal{M}(f) = f$ if and only if $n \geq 3$ and $n/(n-2) < p \leq \infty$.

It is interesting to know when given an r.i. space $X(\mathbb{R}^n)$, the space $D_{I_2}(X(\mathbb{R}^n))$ is not trivial, or equivalently
\[ (2) \quad D_{I_2}(X(\mathbb{R}^n)) := \left\{ f \in L^0([0,\infty)) : \left\| \int_0^\infty f**(s)s^{2/n-1} ds \right\|_{X(0,\infty)} < \infty \right\} \]
is not trivial. This will be done in terms of the fundamental indices of $X$. We start by computing the fundamental function of $D_{I_2}(X(\mathbb{R}^n))$. 
Lemma 3.9. Let \( X \) be an r.i. space on \( \mathbb{R}^n \), \( n \geq 3 \). Let \( Y \) be given by (2). Then
\[
\phi_Y(s) \simeq s^{n/2} \| P_{1-2/n} \chi_{[0,s]} \|_X
\]
where \( P_{1-2/n} f(t) = t^{2/n-1} \int_0^t f(s)s^{-2/n} \, ds \).

Proof. \( s^{n/2} P_{1-2/n} \chi_{[0,s]}(t) \simeq s^{n/2} \left( \chi_{[0,s]}(t) + \left( \frac{2}{t} \right)^{1-2/n} \chi_{(s,\infty)}(t) \right) \)
\[
\simeq \int_t^\infty \chi_{[0,s]}(r)r^{2/n-1} \, dr. \quad \square
\]

Theorem 3.10. Let \( X \) be an r.i. space on \( \mathbb{R}^n \), \( n \geq 3 \). Let \( Y \) be given by (2). Then
(1) If \( \beta_X < 1 - 2/n \), then \( Y \neq \{0\} \).
(2) If \( Y \neq \{0\} \) then \( \beta_X \leq 1 - 2/n \).

Proof. (1) Let \( \chi_r = \chi_{[0,r]} \). Then
\[
P_{1-2/n} \chi_r(t) = \int_0^1 \chi_r(\xi t) \frac{d\xi}{\xi^{n/2}} \leq c \sum_{k=0}^\infty 2^{-k(1-n/2)} \chi_{2^k r}(t).
\]
Thus
\[
\| P_{1-2/n} \chi_r \|_X \leq c \sum_{k=0}^\infty 2^{-k(1-n/2)} \phi_X(2^k r) \leq c \phi_X(r) \sum_{k=0}^\infty 2^{-k(1-n/2)} M_X(2^k).
\]
Let \( \varepsilon > 0 \) be such that \( \beta_X + \varepsilon < 1 - 2/n \). Then by the definition of \( \beta_X \) it follows readily that there is a constant \( c > 0 \) such that
\[
M_X(2^k) \leq c 2^{k(\beta_X + \varepsilon)},
\]
and hence
\[
\sum_{k=0}^\infty 2^{-k(1-n/2)} M_X(2^k) \leq \sum_{k=0}^\infty 2^{-k(1-n/2-\beta_X - \varepsilon)} < \infty,
\]
which implies that \( \chi_r \in Y \).
(2) Since \( Y \neq \{0\} \) if and only if \( \| P_{1-2/n} \chi_{[0,1]} \|_X < \infty \) and
\[
\sup_{t>0} (P_{1-2/n} \chi_{[0,1]})^{**}(t) \phi_X(t) \leq \| P_{1-2/n} \chi_{[0,1]} \|_X < \infty,
\]
and easy computations show that (3) implies that
\begin{equation}
1 \leq \sup_{t \geq 1} \frac{\phi_X(t)}{t^{1-2/n}} = c < \infty,
\end{equation}
then, by (4)
\begin{align*}
M_X(a) &= \max \left( \sup_{t \geq 1/a} \frac{\phi_X(ta)}{\phi_X(t)}, \sup_{t < 1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right) \\
&= \max \left( \sup_{t \geq 1/a} \frac{\phi_X(ta)}{(at)^{1-2/n}} \phi_X(t) \phi_X(t), \sup_{t < 1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right) \\
&\simeq \max \left( a^{1-2/n} \sup_{t \geq 1/a} \frac{t^{1-2/n}}{\phi_X(t)} \sup_{t < 1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right),
\end{align*}
Thus, if $a < 1$, using again (4) we get
\begin{equation*}
M_X(a) \geq a^{1-2/n} \sup_{t \geq 1/a} \frac{t^{1-2/n}}{\phi_X(t)} \geq a^{1-2/n}
\end{equation*}
which implies that
\begin{equation*}
\beta_X \leq 1 - 2/n. \quad \blacksquare
\end{equation*}

Let us see that the converse in the previous theorem is not true.

**Proposition 3.11.** There are rearrangement invariant spaces $X$ such that
\begin{enumerate}
\item $Y \neq \{0\}$ and $\beta_X \geq 1 - 2/n$.
\item $Y = \{0\}$ and $\beta_X < 1 - 2/n$.
\end{enumerate}

**Proof.** Let $\varphi(t) = t^a \chi_{[0,1]}(t) + t^b \chi_{[1,\infty)}(t)$, with $0 \leq a, b \leq 1$. Let
\begin{equation*}
X = \left\{ f \in L^0([0, \infty)) : \sup_{t > 0} f^{**}(t) \varphi(t) < \infty \right\}.
\end{equation*}
Since $\varphi$ is a quasi-concave function, we have that
\begin{equation*}
\varphi(t) = \phi_X(t)
\end{equation*}
and
\begin{equation*}
\beta_X = \min(a, b), \quad \bar{\beta}_X = \max(a, b).
\end{equation*}
On the other hand, the space $Y$ defined by (2) is not trivial if and only if
\begin{equation*}
b \leq 1 - 2/n.
\end{equation*}

Now, to prove (1) take $b \leq 1 - 2/n$ and $a \geq 1 - 2/n$. And to see (2) take $b > 1 - 2/n$ and $a \leq 1 - 2/n. \quad \blacksquare
Remark 3.12. If we consider
\[ X_0 = \left\{ f \in L^0([0, \infty)) : \sup_{t>0} f^{**}(t)t^{1-2/n}(1 + \log^+ t) < \infty \right\} \]
and
\[ X_1 = \left\{ f \in L^0([0, \infty)) : \sup_{t>0} f^{**}(t)\frac{t^{1-2/n}}{(1 + \log^+ t)} < \infty \right\} \]
then \( \beta_{X_i} = \beta_{X_i} = 1 - 2/n, \ Y_0 = \{0\} \) and \( Y_1 \neq \{0\} \).

Remark 3.13. It was proved in [7] that if we consider the strong maximal function (i.e., the maximal operator associated to centered intervals in \( \mathbb{R}^n \)), then there were no fixed points in any \( L^p(\mathbb{R}^n) \) space, regardless of the dimension. The same argument works to show that \( L^p(\mathbb{R}^n) \) cannot be replaced by any different r.i. space. Also, if we study this question for other kind of sets, like, e.g., Buseman–Feller differentiation bases (see [6]), then the only possible fixed points are the constant functions. This observation applies to any non-centered maximal operator (with respect to balls, cubes, etc.).

References


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