COMPOSITION OPERATORS IN HYPERBOLIC $Q$-CLASSES

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Abstract. Function theoretic characterizations are given of when a composition operator mapping from a weighted Dirichlet space $D_q$ into a holomorphic $Q_s$-space is bounded or compact. If $X^*$ stands for the hyperbolic class corresponding to the space $X$, it is shown that a composition operator mapping from $D_q$ into $Q_s$ is bounded if and only if it is bounded from $D_q^*$ into $Q_s^*$, provided $q \leq 0$ and $s \leq 1$.

1. Introduction and statements of results

Let $H(D)$ denote the space of all analytic functions in the open unit disc $D$ of the complex plane, and let $B(D)$ be the subset of $H(D)$ consisting of those $h \in H(D)$ for which $|h(z)| < 1$ for all $z \in D$. Every $\varphi \in B(D)$ induces a linear composition operator $C_\varphi(f) = f \circ \varphi$ from $H(D)$ or $B(D)$ into itself. For the general theory of composition operators in analytic function spaces, see [4] and [13].

A function $f \in H(D)$ belongs to the $\alpha$-Bloch space $B_\alpha$, $0 < \alpha < \infty$, if

$$\|f\|_{B_\alpha} = \sup_{z \in D} |f'(z)|(1 - |z|^2)^{\alpha} < \infty.$$ 

The little $\alpha$-Bloch space $B_{\alpha,0}$ consists of those $f \in H(D)$ for which $|f'(z)|(1 - |z|^2)^{\alpha} \to 0$ as $|z| \to 1$. Denoting $h^*(z) = |h'(z)|(1 - |h(z)|^2)^{\frac{\alpha}{2}}$, the hyperbolic derivative of $h \in B(D)$, the hyperbolic $\alpha$-Bloch classes $B^*_\alpha$ and $B^*_{\alpha,0}$ are defined as the sets of those $h \in B(D)$ for which

$$\|h\|_{B^*_\alpha} = \sup_{z \in D} h^*(z)(1 - |z|^2)^{\alpha} < \infty$$

and $\lim_{|z| \to 1} h^*(z)(1 - |z|^2)^{\alpha} = 0$, respectively. If $\alpha = 1$, it is simply denoted $B^* = B^*_1$ and $B^*_{1,0} = B^*_{1,0}$. Clearly $B^*_\alpha$ and $B^*_{\alpha,0}$ are not linear spaces. Moreover, the

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Schwarz–Pick lemma implies $S_\alpha = B(D)$ if $\alpha \geq 1$, and therefore the hyperbolic $\alpha$-Bloch classes are only considered when $0 < \alpha \leq 1$.

For $s > -1$, the weighted Dirichlet space $D_s$ (respectively weighted hyperbolic Dirichlet class $D^*_s$) consists of those $f \in H(D)$ (respectively $h \in H(D)$) for which

$$
\|f\|_{D_s} = \left( \int_D |f'(z)|^2 \left( \log \frac{1}{|z|} \right)^s dA(z) \right)^{1/2} < \infty
$$

(respectively

$$
\|h\|_{D^*_s} = \left( \int_D (h^*(z))^2 \left( \log \frac{1}{|z|} \right)^s dA(z) \right)^{1/2} < \infty),
$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $D$. The Schwarz–Pick lemma implies $D^*_s = B(D)$ for $s > 1$, and therefore the class $D^*_s$ is considered only when $-1 < s \leq 1$. In this range the class $D^*_s$ contains no inner functions by [14, Theorem 1].

Let the Green’s function of $D$ be defined as $g(z, a) = -\log |\varphi_a(z)|$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the automorphism of $D$ which interchanges the points zero and $a \in D$. For $0 \leq s < \infty$, the Möbius invariant subspace (respectively subclass) $Q_s$ (respectively $Q^*_s$) of $D_s$ (respectively $D^*_s$) consists of those $f \in H(D)$ (respectively $h \in H(D)$) for which

$$
\|f\|_{Q_s} = \left( \sup_{a \in D} \int_D |f'(z)|^2 g^*(z, a) dA(z) \right)^{1/2} < \infty
$$

(respectively

$$
\|h\|_{Q^*_s} = \left( \sup_{a \in D} \int_D (h^*(z))^2 g^*(z, a) dA(z) \right)^{1/2} < \infty).
$$

The space $Q_{s,0}$ (respectively class $Q^*_{s,0}$) consists of those $f \in H(D)$ (respectively $h \in H(D)$) for which the integral expression in (1.1) (respectively (1.2)) tends to zero as $|a| \to 1$. If $s = 0$, then $Q_0$ is the classical Dirichlet space $\mathcal{D} = \mathcal{D}_0$. If $s > 1$, then, by [2, Theorem 1], the spaces $Q_s$ and $Q_{s,0}$ coincide with the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$, respectively, and the class $Q^*_s$ reduces to $B(D)$ by the Schwarz–Pick lemma.

The following characterization of bounded composition operators mapping from $\mathcal{B}_\alpha$ into $Q_s$ can be found in [17, Theorem 2.2.1(i)].

**Theorem A.** Let $0 < \alpha < \infty$, $0 \leq s < \infty$ and $\varphi \in B(D)$. Then the following statements are equivalent:
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(1) $C_\varphi: \mathcal{B}_\alpha \to Q_s$ is bounded;
(2) $C_\varphi: \mathcal{B}_{\alpha,0} \to Q_s$ is bounded;
(3) $\sup_{a \in D} \int_{\mathcal{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2s}} g^s(z,a) \, dA(z) < \infty$;
(4) $\sup_{a \in D} \int_{\mathcal{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2s}} (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty$.

To be precise, the case $s = 0$ of Theorem A does not appear in [17, Theorem 2.2.1(i)], but it has been included above since the same proof works also in this case.

A composition operator $C_\varphi: \mathcal{B}_\alpha^* \to Q_s^*$ is said to be bounded if there exists a positive constant $C$ such that $\|C_\varphi(h)\|_{Q_s^*} \leq C\|h\|_{\mathcal{B}_\alpha^*}$ for all $h \in \mathcal{B}_\alpha^*$. Hereafter it is agreed the same meaning for the boundedness of $C_\varphi$ mapping from one hyperbolic class $X^*$ into another hyperbolic class $Y^*$. This definition is of the same spirit as the definition in [9] of a bounded composition operator mapping from one meromorphic function class into another. However, it would be of interest to find metrics in $\mathcal{B}_\alpha^*$ and $Q_s^*$ such that these classes would become complete metric spaces and the continuity of $C_\varphi$ would be equivalent to the natural definition of a bounded composition operator given above.

The first result of this paper extends Theorem A to the corresponding hyperbolic classes.

**Theorem 1.1.** Let $0 < \alpha \leq 1$, $0 \leq s \leq 1$ and $\varphi \in \mathcal{B}(D)$. Then the following statements are equivalent:

(1) $C_\varphi: \mathcal{B}_\alpha \to Q_s$ is bounded;
(2) $C_\varphi: \mathcal{B}_\alpha^* \to Q_s^*$ is bounded;
(3) $C_\varphi: \mathcal{B}_{\alpha,0}^* \to Q_s^*$ is bounded.

Keeping the consideration mostly out of meromorphic function classes, it is settled to point out a somewhat surprising phenomenon which occurs here. Namely, by Theorem A, Theorem 1.1 and the theorem in [9], a composition operator $C_\varphi$ mapping from $\mathcal{B}$ into $Q_s$ is bounded, if and only if, it is bounded from $\mathcal{B}^*$ into $Q_s^*$, if and only if, it is bounded from $\mathcal{N}$ into $Q_s^#$, where $\mathcal{N}$ denotes the class of normal functions and $Q_s^#$ is the meromorphic $Q_s$-class. See [2] and [9] for necessary definitions. Since the functions in $\mathcal{B}_\alpha$ are bounded if $0 < \alpha < 1$, it is easy to see, by using functions with Hadamard gaps as in the proof of Theorem 1.1, that this result remains also valid when the domain space and classes are $\mathcal{B}_\alpha$, $\mathcal{B}_\alpha^*$ and $\mathcal{N}_\alpha^#$, $0 < \alpha < 1$, respectively.

Two quantities $a$ and $b$ are said to be comparable, denoted by $a \simeq b$, if there exists a positive constant $C$ such that $C^{-1}a \leq b \leq Ca$.

**Example 1.2.** For $0 < \beta < 1$, define $\phi_\beta(z) = 1 - (1 - z)^\beta$. Then $\phi_\beta(z)$ is a conformal mapping which fixes the points zero and one, and maps $D$ onto
Moreover, if the following statements are equivalent

\(\text{(1)}\) \(\phi_\beta: \mathcal{B} \to \mathcal{B}_s^*\) is bounded if and only if \(s\) is positive. It is now shown

\(\text{(2)}\) \(\phi_\beta: Q_{s_1}^* \to Q_{s_2}^*, \ 0 < s_1 \leq 1,\) is bounded if and only if \(s_2\) is positive. Since

\(\text{(3)}\) \(Q_{s_1}^* \subset \mathcal{B}^*\) with \(\|h\|_{\mathcal{B}^*} \leq C\|h\|_{Q_{s_1}^*}\), where \(C\) is a positive constant, it suffices
to show that \(\phi_\beta: Q_{s_1}^* \to \mathcal{B}\) is not bounded. But this easily follows by the fact

\(\phi_\beta(1-z) \simeq |1-z|^{-1}\) in \(\mathcal{D}\), since Fatou’s lemma and (3.2) below yield \(\phi_\beta \in Q_{s_1}^*\) for
\(0 < s_1 \leq 1\).

If \(0 < \alpha < \infty, \ 0 < s < \infty\) and \(\varphi \in B(\mathcal{D})\), then \(\phi_\beta: Q_s \to \mathcal{B}_\alpha\) is bounded
if and only if \(\varphi \in \mathcal{B}_\alpha^*\) by [17, Theorem 2.2.1(iii)]. This result is extended for the
corresponding hyperbolic classes in the following theorem.

**Theorem 1.3.** Let \(0 < \alpha \leq 1, \ 0 \leq s \leq 1\) and \(\varphi \in B(\mathcal{D})\). Then the following statements
are equivalent:

1. \(\phi_\beta: \mathcal{B} \to \mathcal{B}_\alpha\) is bounded;
2. \(\phi_\beta: \mathcal{B}^* \to \mathcal{B}_\alpha^*\) is bounded;
3. \(\phi_\beta: Q_s \to \mathcal{B}_\alpha\) is bounded;
4. \(\varphi \in \mathcal{B}_\alpha^*\).

Theorem 1.4 generalizes [16, Theorem 4.1(i)] since the term \(|\varphi'(\varphi(z))|\) in
conditions (2) and (3) below can be replaced by \((1 - |a|^2)^\tau / |1 - \bar{a}\varphi(z)|^{1+\tau}, \ 0 < \tau < \infty,\) by Lemma B below (in Section 2).

**Theorem 1.4.** Let \(-1 < s_1 < \infty, \ 0 < s_2 < \infty\) and \(\varphi \in B(\mathcal{D})\). Then the following statements are equivalent:

1. \(\phi_\beta: \mathcal{D}_{s_1} \to Q_{s_2}\) is bounded;
2. \(\sup_{a,b \in \mathcal{D}} \int_{\mathcal{D}} |\varphi'_a(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b) dA(z) < \infty;
3. \(\sup_{a,b \in \mathcal{D}} \int_{\mathcal{D}} |\varphi'_a(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 (1 - |\varphi_b(z)|^2)^{s_2} dA(z) < \infty.

Moreover, if \(s_1 \leq 0\) and \(s_2 \leq 1\), then (1)–(3) are equivalent to

4. \(\phi_\beta: \mathcal{D}_{s_1}^* \to Q_{s_2}^*\) is bounded.

By Theorem 1.3, \(\phi_\beta: \mathcal{D} \to \mathcal{B}\) is bounded for all \(\varphi \in B(\mathcal{D})\). Moreover, since
\(Q_s = \mathcal{B}\) for \(s > 1\), Theorem 1.4 implies that \(\phi_\beta: \mathcal{D} \to \mathcal{B}\) is bounded if and only if

\(\text{(1.3)}\) \(\sup_{a,b \in \mathcal{D}} \int_{\mathcal{D}} |(\varphi_a \circ \varphi)'(z)|^2 g^s(z,b) dA(z) < \infty, \ 1 < s < \infty.

However, (1.3) is equivalent to

\(\sup_{a,z \in \mathcal{D}} |(\varphi_a \circ \varphi)'(z)| (1 - |z|^2) = \|\varphi\|_{\mathcal{D}^*} < \infty,\)

which is, of course, satisfied for all \(\varphi \in B(\mathcal{D})\).
Example 1.5. For $0 \leq p < \infty$, define $\psi_p(z) = (p + z)/(p + 1)$. Then $\psi_p$ is a conformal mapping which maps $D$ onto the disc centered at $p/(p + 1)$ with radius $1/(p + 1)$. Clearly,

$$
(1.4) \quad \|f \circ \varphi\|_{Q_{s_2}} \leq C\|f\|_D \leq C\|f\|_{Q^*}
$$

for all $f \in D$ and $\varphi \in B(D)$, thus, in particular, $C_{\psi_p} : D \to Q_{s_2}$ is bounded. A similar reasoning shows that $C_{\psi_p} \colon Q^* \to Q^*_{s_2}$ is also bounded. However, a geometric argument or a straightforward calculation based on the identity

$$
1 - \bar{a}\psi_p(z) = \frac{p(1 - \bar{a}) + 1 - az}{p + 1}
$$

shows that $|1 - a\psi_p(z)| \leq |1 - az|$ for all $z \in D$ and $a \in (0, 1)$, and therefore

$$
\sup_{a, b \in D} \int_D \left| \varphi_a'(\psi_p(z)) \right|^{2+s_1} |\psi_p'(z)|^2 (1 - |\varphi_b(z)|)^{s_2} \, dA(z)
$$

$$
\geq \lim_{a \to 1} \frac{(1 - a^2)^{2+s_1+s_2}}{(p + 1)^2} \int_D \left| 1 - a\psi_p(z) \right|^{2(2+s_1)} |1 - az|^{2s_2} \, dA(z)
$$

$$
\geq \lim_{a \to 1} \frac{(1 - a^2)^{2+s_1+s_2}}{(p + 1)^2} \int_D \left| 1 - az \right|^{2(2+s_1+s_2)} \, dA(z) \approx \lim_{a \to 1} (1 - a^2)^{-s_1},
$$

from which it follows by Theorem 1.4 that $C_{\psi_p} : D_{s_1} \to Q_{s_2}$ is not bounded if $s_1$ is positive.

Example 1.6. Let $0 < \beta < 1$, $-1 < s_1 < \infty$ and $0 \leq s_2 < \infty$, and consider the map $\phi_\beta(z) = 1 - (1 - z)^\beta$. It is proved that $C_{\phi_\beta}$ admits the same behavior as $C_{\psi_p}$, does in the sense that $C_{\phi_\beta} : D_{s_1} \to Q_{s_2}$ is bounded if and only if $-1 < s_1 \leq 0$. In view of (1.4) it suffices to show that $C_{\phi_\beta} : D_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$. To this end, choose $f_a(z) = (1 - z)^{-a}$, $0 < a < \infty$. Now, by [5, Lemma on p. 65], there is a positive constant $C_1$ such that

$$
\|f_a\|_{D_{s_1}}^2 = a^2 \int_D \frac{(1 - |z|^{2s_1})}{(1 - z)^{2(1+a)}} \, dA(z) \leq C_1 a^2 \int_0^1 \frac{r \, dr}{(1 - r)^{2a+1-s_1}},
$$

and therefore $f_a \in D_{s_1}$ for $0 < a < \frac{1}{2} s_1$. Moreover, denote $\omega_{a, \beta}(z) = (1 - z)^{-a\beta} = f_a \circ \phi_\beta$. Then, by [2, Proposition 1], there is a positive constant $C_2$ such that

$$
\|f_a \circ \phi_\beta\|_{Q_{s_2}} = \|\omega_{a, \beta}\|_{Q_{s_2}} \geq C_2 \|\omega_{a, \beta}\|_{Q^*} = \infty,
$$

and therefore $f \circ \phi_\beta \notin Q_{s_2}$. Thus $C_{\phi_\beta} : D_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$.

The following result generalizes [16, Theorem 4.1(ii)].
Theorem 1.7. Let $-1 < s_1 < \infty$, $0 \leq s_2 < \infty$ and $\varphi \in B(D)$. Then the following statements are equivalent:

1. $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is compact;
2. $\lim_{|a| \to 1} \sup_{b \in \mathcal{D}} \int_{\mathcal{D}} \left| \varphi'_a(\varphi(z)) \right|^{2+s_1} \left| \varphi'(z) \right|^{2} g^{s_2}(z, b) \, dA(z) = 0$;
3. $\lim_{|a| \to 1} \sup_{b \in \mathcal{D}} \int_{\mathcal{D}} \left| \varphi'_a(\varphi(z)) \right|^{2+s_1} \left| \varphi'(z) \right|^{2} (1 - \left| \varphi_b(z) \right|^2)^{s_2} \, dA(z) = 0$.

Since, by the general definition of a bounded (respectively compact) operator mapping from one Banach space into another, $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2,0}$ is bounded (respectively compact) if and only if $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded (respectively compact) and $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$, the operator $C_\varphi: \mathcal{D}_{s_1}^{*} \to Q_{s_2,0}^{*}$ is said to be bounded, if $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}^{*}$ is bounded and $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$.

Theorem 1.8. Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(D)$. Then $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2,0}$ is bounded if and only if $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded and the following two conditions are satisfied:

1. $\varphi \in Q_{s_2,0}$;
2. $\lim_{|a|, |b|, t \to 1} \int_{|\varphi(z)| \geq t} \left| \varphi'_a(\varphi(z)) \right|^{2+s_1} \left| \varphi'(z) \right|^2 g^{s_2}(z, b) \, dA(z) = 0$.

Similarly, if $s_1 \leq 0$ and $s_2 \leq 1$, then $C_\varphi: \mathcal{D}_{s_1}^{*} \to Q_{s_2,0}^{*}$ is bounded if and only if $C_\varphi: \mathcal{D}_{s_1}^{*} \to Q_{s_2}^{*}$ is bounded, $\varphi \in Q_{s_2,0}^{*}$ and (2) is satisfied.

It is easy to show that the conditions (1) and (2) in Theorem 1.8 together are equivalent to

3. $\lim_{|a|, |b| \to 1} \sup_{D} \int_{D} \left| \varphi'_a(\varphi(z)) \right|^{2+s_1} \left| \varphi'(z) \right|^2 g^{s_2}(z, b) \, dA(z) = 0$.

and hence the first part of Theorem 1.8 implies the following result.

Theorem 1.9. Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(D)$. Then $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2,0}$ is compact if and only if $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is compact and the condition (3) above is satisfied.

The remaining part of the paper is organized as follows. In Section 2, some background material and auxiliary results needed later on are recalled, and Section 3 contains the proofs of the results presented in this section.

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2. Background material

A positive Borel measure \( \mu \) on \( \mathbb{D} \) is a \textit{bounded} \( s \)-\textit{Carleson measure}, if

\[
\sup_I \frac{\mu(S(I))}{|I|^s} < \infty, \quad 0 < s < \infty,
\]

where \( |I| \) denotes the arc length of a subarc \( I \) of the boundary of

\[
S(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, \ |1 - |I|| \leq |z| \right\}
\]

is the Carleson box based on \( I \), and the supremum is taken over all subarcs \( I \) such that \( |I| \leq 1 \). Moreover, if

\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0, \quad 0 < s < \infty,
\]

then \( \mu \) is a \textit{compact} \( s \)-\textit{Carleson measure}. If \( s = 1 \), then a bounded (respectively compact) 1-Carleson measure is just a standard bounded (respectively compact) Carleson measure.

Some well-known and useful characterizations of bounded \( s \)-Carleson measures are gathered in the following lemma. For the proof, see [1, Theorem 13], [3, Lemma 2.1], [10, pp. 89–90], and [11, Proposition 2.1].

**Lemma B.** Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \), \( 1 < s < \infty \), \( 0 < r < 1 \) and \( 0 < \tau < \infty \). Then the following statements are equivalent:

1. \( K_1 = \sup_I \frac{\mu(S(I))}{|I|^s} < \infty \);
2. \( K_2 = \sup_{z \in \mathbb{D}} \frac{\mu(D(z, r))}{(1 - |z|^2)^s} < \infty \);
3. \( K_3 = \sup_{z \in \mathbb{D}} \int_D \left( \frac{(1 - |z|^2)^\tau}{|1 -\bar{z}w|^{1+s}} \right)^s d\mu(w) < \infty \).

Moreover, \( K_1 \), \( K_2 \) and \( K_3 \) are comparable.

Here \( D(a, r) = \{ z \in \mathbb{D} : |\varphi_a(z)| < r \} \) is the pseudo-hyperbolic disc of center \( a \in \mathbb{D} \) and radius \( 0 < r < 1 \). The pseudo-hyperbolic disc \( D(a, r) \) is an Euclidean disc centered at \( (1 - r^2)a/(1 - |a|^2r^2) \) with radius \( (1 - |a|^2)r/(1 - |a|^2r^2) \), see [8, p. 3].

The following change of variables formula by C.S. Stanton, [6] and [15], was apparently first used by J.H. Shapiro [12] in the study of composition operators, and it also plays a key role in some of the proofs in this paper.
Lemma C. Let \( g \) and \( u \) be positive measurable functions on \( D \), and let \( \varphi \in B(D) \). Then
\[
\int_D (g \circ \varphi)(z) |\varphi'(z)|^2 u(z) \, dA(z) = \int_D g(w) U(\varphi, w) \, dA(w),
\]
where
\[
U(\varphi, w) = \sum_{z \in \varphi^{-1}(w)} u(z), \quad w \in D \setminus \{\varphi(0)\}.
\]

If \( u(z) = (-\log |z|)^s \), then \( U(\varphi, w) \) is the generalized Nevanlinna counting function
\[
N_{\varphi, s}(w) = \sum_{z \in \varphi^{-1}(w)} \left( \log \frac{1}{|z|} \right)^s.
\]

For the study of compactness, the following well-known result is needed. See [4, Proposition 3.11] for a similar result.

Lemma D. Let \( -1 < s_1 < \infty \), \( 0 \leq s_2 < \infty \) and \( \varphi \in B(D) \). Then \( C_{\varphi}: D_{s_1} \to Q_{s_2} \) is compact if and only if for any bounded sequence \( \{f_n\} \) in \( D_{s_1} \) with \( f_n \to 0 \) uniformly on compact subsets of \( D \) as \( n \to \infty \), \( \|f_n \circ \varphi\|_{Q_{s_2}} \to 0 \) as \( n \to \infty \).

3. Proofs

Proof of Theorem 1.1. It is enough to prove the implications (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1) since (2) \( \Rightarrow \) (3) is clearly true.

Suppose \( C_{\varphi}: B_\alpha \to Q_s \) is bounded, that is, (1) is satisfied. If \( h \in B_\alpha^* \), then
\[
\int_D ((h \circ \varphi)^s(z))^2 g^s(z, a) \, dA(z) \leq \|h\|_{B_\alpha^*}^2 \int_D \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2s}} g^s(z, a) \, dA(z),
\]
where the last integral is uniformly bounded for all \( a \in D \) by Theorem A, and therefore \( C_{\varphi}: B_\alpha^* \to Q_s^* \) is bounded. Thus (1) implies (2) is proved.

To prove (3) \( \Rightarrow \) (1), let first \( \alpha = 1 \), and suppose that \( C_{\varphi}: B_0^* \to Q_s^* \) is bounded. If \( h_b(z) = bz \), \( b \in D \), then \( h^*(z) = |b|(1 - |bz|^2)^{-1} \) and \( h_b \in B_0^* \) for all \( b \in D \), and
\[
\sup_{a \in D} \int_D \frac{|b|^2 |\varphi'(z)|^2}{(1 - |b\varphi(z)|^2)^{2s}} g^s(z, a) \, dA(z) \leq C\|h\|_{B_0^*}^2 \leq C|b|^2
\]
for some positive \( C \) by the assumption. Taking limit as \( |b| \to 1 \), \( b \in D \), Fatou’s lemma with Theorem A implies that \( C_{\varphi}: B \to Q_s \) is bounded, that is, (1) with \( \alpha = 1 \) holds.
If $0 < \alpha < 1$, functions with Hadamard gaps may be used. Define
\[ g_n(z) = \sum_{k=0}^{\infty} 2^{k(\alpha-1)}(b_n z)^{2^k}, \]
where $\{b_n\} \subset \mathbb{D}$ and $|b_n| \to 1$, as $n \to \infty$. Then $g_n \in \mathcal{B}_{\alpha,0}$ by [18, Theorem 1]. Since $|g_n(z)| \leq \sum_{k=0}^{\infty} 2^{k(\alpha-1)}$, there is a positive constant $C$, depending only on $\alpha$, such that $h_n = C^{-1}g_n$ satisfies $|h_n(z)| \leq \frac{1}{2}$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, and therefore $h_n^*(z) \simeq |h_n(z)|$ in $\mathbb{D}$. Now one may argue as in [1, p. 133] and use Fatou’s lemma to conclude that the condition (3) with $0 < \alpha < 1$ in Theorem A is satisfied, and hence (1) with $0 < \alpha < 1$ holds. \(\blacksquare\)

**Remark.** To characterize bounded composition operators from $Q^*_{s_1}$ to $Q^*_{s_2}$ when $0 < s_1, s_2 < 1$ appears to be more complicated. However, Example 1.2 shows that there is a function $\varphi$ for which $C_{\varphi}: \mathcal{B}^* \to Q^*_{s_2}$ is bounded if and only $C_{\varphi}: Q^*_{s_1} \to Q^*_{s_2}$ is bounded, provided $0 < s_1 \leq 1$.

**Proof of Theorem 1.3.** It suffices to prove the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) since (2) $\Rightarrow$ (3) is clearly true and (1) is equivalent to (4) by [17, Theorem 2.2.1(iii)].

Suppose $C_{\varphi}: Q^*_{s_1} \to \mathcal{B}^*_{s_2}$ is bounded, that is, (3) holds, and let first $0 < s \leq 1$. If $\phi_{\beta, a}(z) = 1 - (1 - \bar{a}z)\beta$, where $0 < \beta < 1$ and $a \in \mathbb{D}$, then
\[
\phi_{\beta, a}(z) = \frac{\beta|a|}{1 - |1 - (1 - \bar{a}z)\beta|} \simeq \frac{\beta|a|}{1 - \bar{a}}, \quad z \in \mathbb{D}.
\]
By [7, Lemma 2.5], there is a positive constant $C_1$ such that
\[
\int_{\mathbb{D}} (\phi_{\beta, a}^*(z))^{2}(1 - |\varphi_b(z)|^2)^s dA(z) \simeq \beta^2 |a|^2 (1 - |b|^2)^s
\]
\[
\cdot \int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - b\bar{z}|^2 |1 - \bar{a}z|^s} dA(z)
\]
\[
\leq C_1^2 \beta^2 |a|^2 \frac{(1 - |b|^2)^s}{|1 - \bar{a}b|^s},
\]
and it follows that $\|\phi_{\beta, a}\|_{Q^*_{s}} \leq C_1 2^{s/2} \beta |a|$ for all $a \in \mathbb{D}$, $0 < \beta < 1$ and $0 < s \leq 1$. By the assumption there exists a positive constant $C_2$ such that
\[
\frac{\beta |a|}{2 |1 - \bar{a}\varphi(z)|} (1 - |z|^2)^{\alpha} \leq (\phi_{\beta, a} \circ \varphi)^*(z)(1 - |z|^2)^{\alpha} \leq C_2 \beta^2 |a| |Q^*_{s}| \leq C_2 C_1 2^{s/2} \beta |a|,
\]
and the assertion $\varphi \in \mathcal{B}^*_{s}$ follows by choosing $a = \varphi(z)$. The case $s = 0$ can be proved in a similar manner by choosing the test function $\varphi_a(z)/2$.

If $h \in \mathcal{B}^* = B(\mathbb{D})$, then
\[
(h \circ \varphi)^*(z)(1 - |z|^2)^{\alpha} \leq \|h\|_{\mathcal{B}^*} \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |z|^2)^{\alpha} \leq \|h\|_{\mathcal{B}^*} \|\varphi\|_{\mathcal{B}^*},
\]
and (4) $\Rightarrow$ (2) follows. \(\blacksquare\)
Proof of Theorem 1.4. To prove that (2) implies (1), the reasoning in the proof of [10, Theorem 2.2] is followed. By Lemma C and the subharmonicity of $|f'(z)|^2$, there is a positive constant $C_1$ such that

$$
\int_{\mathcal{D}} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) = \int_{\mathcal{D}} |f'(w)|^2 d\mu_{a, s_2}(w)
\leq C_1 \int_{\mathcal{D}} \left( \frac{1}{(1 - |w|^2)^2} \int_{D(w, 1/2)} |f'(z)|^2 dA(z) \right) d\mu_{a, s_2}(w),
$$

where $d\mu_{a, s_2}(w) = N_{\varphi \circ \varphi, a} (w) \, dA(w)$. Then the symmetry

$$
\chi_{D(z, r)}(w) = \chi_{D(w, r)}(z)
$$

of the characteristic functions of pseudohyperbolic discs and Fubini's theorem yield

$$
\int_{\mathcal{D}} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) \leq C_1 \int_{\mathcal{D}} |f'(z)|^2 \left( \int_{D(z, 1/2)} \frac{d\mu_{a, s_2}(w)}{(1 - |w|^2)^2} \right) dA(z).
$$

By Lemmas B and C, the assumption (2) is equivalent to

$$
\sup_{a \in \mathcal{D}} \int_{D(z, 1/2)} d\mu_{a, s_2}(w) \leq C_2 (1 - |z|^2)^{2+s_1}, \quad z \in \mathcal{D},
$$

for some positive constant $C_2$. Since $1 - |w| \simeq 1 - |z|$ for $w \in D(z, 1/2)$, it follows by (3.3) and (3.4) that $C_{\varphi}: Q_{s_1} \rightarrow Q_{s_2}$ is bounded.

Suppose then that $C_{\varphi}: Q_{s_1} \rightarrow Q_{s_2}$ is bounded. For $a \in \mathcal{D}$, define $f_a(z) = \int_0^z (\varphi_a'(w))^{1+s_1/2} \, dw$. Then, by Forelli–Rudin estimates [19, Lemma 4.2.2] there is a positive constant $C_1$, depending only on $s_1$, such that

$$
\|f_a\|^2_{Q_{s_1}} = (1 - |a|^2)^{1+s_1/2} \int_{\mathcal{D}} \frac{(1 - |z|^2)^{s_1}}{1 - \bar{a}z|^{2+s_1}} \, d\sigma(z) \leq C_1
$$

for all $a \in \mathcal{D}$, and thus the family $\{f_a : a \in \mathcal{D}\}$ is norm bounded uniformly in $\mathcal{D}_{s_1}$. Since $C_{\varphi}: Q_{s_1} \rightarrow Q_{s_2}$ is bounded, there is a positive constant $C_2$ such that

$$
\sup_{b \in \mathcal{D}} \int_{\mathcal{D}} |\varphi_b'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 (1 - |\varphi_b(z)|^2)^{s_2} \, d\sigma(z) = \|f_a \circ \varphi\|^2_{Q_{s_2}} \leq C_2 \|f_a\|^2_{Q_{s_1}} \leq C_1 C_2
$$

for all $a \in \mathcal{D}$, and the condition (2) follows.

Since (2) and (3) are clearly equivalent, it is now proceeded to consider the hyperbolic case. Suppose that $s_1 \leq 0$ and $s_2 \leq 1$. If (2) is satisfied, then the same
reasoning as in the first part of the proof shows that \( C_\varphi: D_1 \to Q_2 \) is bounded since also \((h^*)^2\) is a subharmonic function in \( D \).

Suppose then that \( C_\varphi: D_1 \to Q_2 \) is bounded. For \( a \in D \) and \( \frac{1}{2} < \gamma \leq 1 \) let

\[
(3.5) \quad f_{a,\gamma}(z) = \int_0^z (\varphi_a'(w))^\gamma \, dw = \frac{(1 - |a|^2)^\gamma}{a(1 - 2\gamma)}((1 - \bar{a}z)^{1-2\gamma} - 1), \quad a \in D \setminus \{0\},
\]

and

\[
(3.6) \quad h_{a,\gamma}(z) = \begin{cases} \frac{(2\gamma - 1)a}{6} f_{a,\gamma}(z), & a \in D \setminus \{0\}, \\ \frac{z}{2}, & a = 0. \end{cases}
\]

Then \( \|h_{a,\gamma}\|_\infty \leq \frac{1}{2} \) for all \( a \in D \), and therefore \( h_{a,\gamma}(z) \simeq |h_{a,\gamma}(z)| \) in \( D \). The reasoning in the proof of the implication \((1) \Rightarrow (2)\) with the functions

\[
h_{a,1+\gamma/2}(z) = \frac{(1+\gamma)a}{6} \int_0^z (\varphi_a'(w))^{1+\gamma/2} \, dw
\]

yields the assertion \((2)\). □

**Proof of Theorem 1.7.** Suppose first that \( C_\varphi: D_1 \to Q_2 \) is compact and consider the functions \( f_a(z) = \int_0^z (\varphi_a'(w))^{1+\gamma/2} \, dw. \) Since, by the proof of Theorem 1.4, there is a positive constant \( C \) such that \( \|f_a\|_{\mathcal{D}_1} \leq C \) for all \( a \in D \), and further \( f_a \to 0 \) uniformly on compact subsets as \( |a| \to 1 \), Lemma D gives \((2)\).

Suppose now that \((2)\) holds. Let \( \{f_n\} \subset D_1 \) such that \( \|f_n\|_{\mathcal{D}_1} \leq C_1 < \infty \) for all \( n \in \mathbb{N} \), and \( f_n \to 0 \) uniformly on compact subsets of \( D \). By Lemma D, it suffices to show that \( \|f_n \circ \varphi\|_{\mathcal{Q}_2} \to 0 \) as \( n \to \infty \). For \( 0 < \delta < 1 \), let \( \Delta(0, \delta) \) denote the Euclidean disc centered at the origin and of radius \( \delta \). A similar reasoning as in the proof of Theorem 1.4 with the fact \( 1 - |w|^2 \simeq 1 - |z|^2 \simeq |1 - \bar{z}w|, w \in D(z, 1/2) \), yields

\[
\|f_n \circ \varphi\|^2_{\mathcal{Q}_2} = \sup_{a \in D} \int_D |f_n'(w)|^2 \, d\mu_{a,2}(w)
\]

\[
\leq C_2 \int_D |f_n'(z)|^2 (1 - |z|^2)^{s_1} \left( \sup_{a \in D} \int_{D(z, 1/2)} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,2}(w) \right) \, dA(z)
\]

\[
\leq C_2 \int_{D \setminus \Delta(0, \delta)} |f_n'(z)|^2 (1 - |z|^2)^{s_1} \left( \sup_{a \in D} \int_D |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,2}(w) \right) \, dA(z)
\] 

\[
+ \frac{C_2 4^{1+s_1}}{(1 - \delta)^2} \|\varphi\|^2_{\mathcal{Q}_2} \int_{\Delta(0, \delta)} |f_n'(z)|^2 \, dA(z)
\]

\[
= I_1(\delta) + I_2(\delta),
\]

\[
I_1(\delta) = \int_D |f_n'(z)|^2 (1 - |z|^2)^{s_1} \left( \sup_{a \in D} \int_{D(z, 1/2)} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,2}(w) \right) \, dA(z)
\]

\[
I_2(\delta) = \frac{C_2 4^{1+s_1}}{(1 - \delta)^2} \|\varphi\|^2_{\mathcal{Q}_2} \int_{\Delta(0, \delta)} |f_n'(z)|^2 \, dA(z)
\]
where $C_2$ is a positive constant. For a given $\varepsilon > 0$, by the assumption (2) and Lemma C, there exists a $\delta_0 \in (0, 1)$ such that, for all $|z| > \delta_0$,

$$\sup_{a \in D} \int_D |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w) \leq \frac{\varepsilon^2}{2C_1^2 C_2},$$

and it follows that $I_1(\delta_0) < \varepsilon^2/2$. In view of Theorem 1.4, the assumption (2) implies that $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded, and hence $\varphi \in Q_{s_2}$. Since $f_n \to 0$ uniformly on compact subsets of $D$, in particular, in $\Delta(0, \delta_0)$, there exists an $N \in \mathbb{N}$ such that, for $n \geq N$,

$$\int_{\Delta(0,\delta_0)} |f'_n(z)|^2 dA(z) < \frac{\varepsilon^2(1-\delta_0)^2}{2C_2^2 4^{1+s_1} \|\varphi\|_{Q_{s_2}}^2},$$

and therefore $I_2(\delta_0) < \varepsilon^2/2$. Thus, as $n \geq N$, $\|f_n \circ \varphi\|_{Q_{s_2}} < \varepsilon$, and $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is compact by Lemma D.

Since (2) and (3) are clearly equivalent, the proof is complete. \(\square\)

**Proof of Theorem 1.8.** Suppose first that $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2,0}$ is bounded, that is, $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded and $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$. Then, by using the functions $f(z) = z \in \mathcal{D}_{s_1}$ and $f_a(z) = f_0(\varphi'(w))^{1+s_1/2} dw \in \mathcal{D}_{s_1}$, the inclusion $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$ implies that (1) and (2) are satisfied.

Suppose now that $C_\varphi: \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded, and the conditions (1) and (2) are satisfied. It suffices to show that $f \circ \varphi \in Q_{s_2,0}$ for $f \in \mathcal{D}_{s_1}$. A similar reasoning as in the proof of Theorem 1.7 yields

$$\int_D |(f \circ \varphi)'(z)|^2 g^{s_2}(z,a) dA(z) \leq C \int_{\Delta(0,t)} |f'(z)|^2 (1-|z|^2)^{s_1}$$

$$+ \left( \int_D |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w) \right) dA(z)$$

$$+ C \int_{\Delta(0,t)} |f'(z)|^2 \left( \int_{D(z,1/2)} \frac{d\mu_{a,s_2}(w)}{(1-|z|^2)^2} \right) dA(z)$$

$$= I_1(t) + I_2(t),$$

(3.7)

where $C$ is a positive constant and $0 < t < 1$. To deal with $I_1(t)$, write

$$\int_D |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w) = \int_{\Delta(0,t)} |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w)$$

$$+ \int_{\Delta(0,t)} |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w)$$

$$\leq \frac{2^{2+s_1}}{(1-t)^{2+s_1}} \int_D |\varphi'(u)|^2 g^{s_2}(u,a) d\sigma(u)$$

$$+ \int_{\Delta(0,t)} |\varphi'_z(w)|^{2+s_1} d\mu_{a,s_2}(w) = I_3(t) + I_4(t).$$
By the assumption (2), for a given \( \varepsilon > 0 \), there exists a \( \delta_1 \in (0, 1) \) such that

\[
I_4(t) < \frac{\varepsilon^2}{3C'\|f\|_{\mathcal{D}_{s_1}}^2}
\]

for all \(|a|, |z|, t \geq \delta_1\). Let \( t \geq \delta_1 \) be fixed. Since \( \varphi \in Q_{s_{2,0}} \) by the assumption (1), there exists a \( \delta_2 \in [\delta_1, 1) \) such that

\[
I_3(t) < \frac{\varepsilon^2(1-t)^{2+s_1}}{3 \cdot 2^{2+s_1} C'\|f\|_{\mathcal{D}_{s_1}}^2}
\]

for all \(|a| \geq \delta_2\). Since \(|z| \geq t\) in the term \( I_1(t) \), it follows by combining (3.7), (3.8) and (3.9) that \( I_1(t) \leq \frac{2\varepsilon^2}{3} \) for \(|a| \geq \delta_2\). Further, since \( \varphi \in Q_{s_{2,0}} \) by the assumption (1), there exists a \( \delta_3 \in [\delta_2, 1) \) such that

\[
I_2(t) \leq \frac{C\pi t^2}{(1-t^2)^2} \sup_{|z|=t} |f'(z)|^2 \int_\Delta |\varphi'(z)|^2 g^{s_2}(z, a) \, dA(z) < \frac{\varepsilon^2}{3}
\]

for all \(|a| \geq \delta_3\). Therefore one finally concludes

\[
\int_\Delta |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) \leq I_1(t) + I_2(t) < \frac{2\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2
\]

for all \(|a| \geq \delta_3\), that is, \( f \circ \varphi \in Q_{s_{2,0}} \) as one wished to prove. \( \Box \)

Proof of Theorem 1.9. As it was pointed out in Section 1, Theorem 1.9 follows by Theorem 1.8. \( \Box \)

References

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