ON ENTIRE FUNCTIONS THAT SHARE A VALUE WITH THEIR DERIVATIVES

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Abstract. Let \( f \) be a nonconstant entire function, \( a \) a finite complex number, \( k \) and \( m \) two distinct positive integers, and \( (k, m) \) the greatest common divisor of \( k \) and \( m \). If \( f, f^{(k)} \) and \( f^{(m)} \) share \( a \) CM, then
\[
f(z) = \frac{c}{d} + \sum_{j=1}^{q} C_j e^{\lambda_j z},
\]
where \( q \) is a positive integer with \( q \leq (k, m) \), \( c \) and \( C_j \) for \( 1 \leq j \leq q \) are nonzero constants, and \( \lambda_j \) for \( 1 \leq j \leq q \), are distinct nonzero constants satisfying \( (\lambda_j)^k = (\lambda_j)^m = c \), for \( a \neq 0 \), and \( (\lambda_j)^k = c, (\lambda_j)^m = d \), for \( a = 0 \), where \( d \) is a nonzero constant. This answers a question of Yang and Yi [14] for entire functions, and extends a result of Csillag [2].

1. Introduction and main results

Let \( f \) and \( g \) be two nonconstant meromorphic functions in the complex plane, and let \( a \) be a finite complex number. If \( f(z) - a \) and \( g(z) - a \) have the same zeros with the same multiplicities, then we say that \( f \) and \( g \) share \( a \) CM.

In 1986, Jank–Mues–Volkmann [8] proved

**Theorem A.** Let \( f \) be a nonconstant meromorphic function and \( a \) a nonzero finite complex number. If \( f, f' \) and \( f'' \) share \( a \) CM, then \( f \equiv f' \).

By Theorem A, the following question was posed.

**Question 1** (see [6], [7], [13], [14]). Let \( f \) be a nonconstant meromorphic function, \( a \) a nonzero finite complex number, and \( k, m \) two distinct positive integers. Suppose that \( f, f^{(k)} \) and \( f^{(m)} \) share \( a \) CM. Can we get \( f \equiv f^{(k)} \)?

The following example [12] shows that the answer to Question 1 is, in general, negative.

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Example 1. Let $k$, $m$ be positive integers satisfying $m > k + 1$, $b$ a nonzero constant such that $b^k = b^m \neq 1$ and $a = b^k$. Set

$$f(z) = e^{bz} + a - 1.$$  

Then $f$, $f^{(k)}$ and $f^{(m)}$ share a CM. But $f \neq f^{(k)}$.

In Example 1, $f$ is an entire function, and $f$, $f^{(k)}$ and $f^{(m)}$ share a CM. Although $f \neq f^{(k)}$, we have $f^{(k)} \equiv f^{(m)}$.

Naturally, we pose the following question.

Question 2. Let $f$ be a nonconstant meromorphic function, $a$ a nonzero finite complex number and $k$, $m$ two distinct positive integers. Suppose that $f$, $f^{(k)}$ and $f^{(m)}$ share a CM. Can we get $f^{(k)} \equiv f^{(m)}$?

In this paper, we give an affirmative answer to Question 2 for entire functions. In fact, we have proved the following more general result.

Theorem 1. Let $f$ be a nonconstant entire function, $a$ a finite complex number, $k$ and $m$ two distinct positive integers, and $(k, m)$ the greatest common divisor of $k$ and $m$. If $f$, $f^{(k)}$ and $f^{(m)}$ share a CM, then

$$f(z) = \left(1 - \frac{1}{c}\right)a + \sum_{j=1}^{q} C_j e^{\lambda_j z},$$

where $q$ is a positive integer with $q \leq (k, m)$, $c$ and $C_j$, $1 \leq j \leq q$, are nonzero constants, and $\lambda_j$, $1 \leq j \leq q$, are distinct nonzero constants satisfying

$$(\lambda_j)^k = (\lambda_j)^m = c, \quad \text{for } a \neq 0;$$

and

$$(\lambda_j)^k = c, \quad (\lambda_j)^m = d, \quad \text{for } a = 0,$$

where $d$ is a nonzero constant.

By Theorem 1, we can easily obtain the following results.

Corollary 2. Let $f$ be a nonconstant entire function, $a$ a nonzero finite complex number, and $k$, $m$ two distinct positive integers. Suppose that $f$, $f^{(k)}$ and $f^{(m)}$ share a CM. Then $f^{(k)} \equiv f^{(m)}$.

Corollary 2 gives an affirmative answer to Question 2 for entire functions.

Corollary 3 ([10, Theorem 1]). Let $f$ be a nonconstant entire function, $a$ a nonzero finite complex number and $k$ a positive integer. If $f$, $f^{(k)}$ and $f^{(k+1)}$ share a CM, then $f \equiv f'$.
Corollary 4 ([10, Theorem 2]). Let \( f \) be a nonconstant entire function, a nonzero finite complex number and \( k \geq 2 \) a positive integer. If \( f, f' \) and \( f^{(k)} \) share a CM, then
\[
(1.4) \quad f(z) = \left( 1 - \frac{1}{c} \right) a + Ce^{cz},
\]
where \( C \) and \( c \) are nonzero constants with \( c^{k-1} = 1 \).

Corollary 5 (Csillag [2], cf. [4, p. 67]). Let \( f \) be a nonconstant entire function, and \( k \) and \( m \) two distinct positive integers. If \( ff^{(k)}f^{(m)} \neq 0 \), then \( f = e^{Az+B} \), where \( A \) (\( \neq 0 \)) and \( B \) are constants.

Let \( f \) be a nonconstant meromorphic function in the complex plane. Throughout this paper, we use the basic results and notations of Nevanlinna theory (cf. [3], [4], [11], [14]). In particular, \( S(r, f) \) denotes any function satisfying
\[
S(r, f) = o\{T(r, f)\},
\]
as \( r \to +\infty \), possibly outside of a set of finite linear measure, where \( T(r, f) \) is Nevanlinna’s characteristic function.

As usual, the order \( \varrho(f) \) of \( f \) is defined as
\[
\varrho(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r}.
\]

2. Some lemmas

We will use \( P_d[f] \) to denote a differential polynomial in \( f \) of degree \( \leq d \) with constant coefficients which may be different at different occurrence. We denote the set of differential polynomials in \( f \) with constant coefficients by \( \mathcal{P}[f] \).

Lemma 1 (Clunie [1], cf. [4, p. 68]). Let \( f \) be a nonconstant meromorphic function, \( n \) be a positive integer, \( P[f] \) and \( Q[f] \) two differential polynomials in \( f \) with constant coefficients, and \( P[f] \neq 0 \). If the degree of \( P[f] \) is at most \( n \) and
\[
f^nQ[f] = P[f],
\]
then
\[
m(r, Q[f]) = S(r, f).
\]

Lemma 2 (cf. [9, p. 29–34]). Let \( f \) be a nonconstant entire function, \( n \) be a positive integer and \( a_j \), \( 0 \leq j \leq n \), meromorphic functions with \( a_n \neq 0 \). Suppose that
\[
(2.1) \quad a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 \equiv 0.
\]
Then
\[
(2.2) \quad T(r, f) \leq O\left(1 + \sum_{j=0}^{n} T(r, a_n)\right).
\]

The following result is an instant corollary of Lemma 2.
Lemma 3. Let $f$ be a nonconstant entire function, $n$ a positive integer and $a_j$, $0 \leq j \leq n$, meromorphic functions satisfying $T(r, a_j) = S(r, f)$. If

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \equiv 0,$$

then $a_j \equiv 0$ for $j = 0, 1, \ldots, n$.

Lemma 4 ([3, Lemma 3.12]). Let $f_j(z)$ $(\neq 0)$, $j = 1, 2, \ldots, n$, be $n$ meromorphic functions which are linearly independent such that

$$(2.3) \quad f_1(z) + f_2(z) + \cdots + f_n(z) \equiv 1.$$ 

Then for every $j$, $1 \leq j \leq n$,

$$(2.4) \quad T(r, f_j) \leq \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) + S(r),$$

where $D = W(f_1, f_2, \ldots, f_n)$ is the Wronskian, and $S(r)$ is a function which satisfies

$$S(r) = o\left(\max_{1 \leq k \leq n} T(r, f_k)\right)$$

as $r \to \infty$, possibly outside a set of finite linear measure.

Lemma 5 ([3, Lemma 5.1]). Let $a_j(z)$, $j = 0, 1, \ldots, n$, be entire and of finite order $\leq g$ $(< \infty)$. Let $g_j(z)$, $j = 1, \ldots, n$, be also entire such that each of the functions $g_i - g_j$, $i \neq j$, is a transcendental function or a polynomial of degree greater than $g$. If

$$(2.5) \quad \sum_{j=1}^{n} a_j(z)e^{g_j(z)} \equiv a_0(z),$$

then

$$(2.6) \quad a_j(z) \equiv 0, \quad j = 0, 1, \ldots, n.$$

Lemma 6. Let $f$ and $\alpha$ be nonconstant entire functions, $a$ a finite complex number and $k$ a positive integer. Suppose that

$$f^{(k)} = a + e^\alpha f.$$ 

Then for any positive integer $j$, $1 \leq j \leq k - 1$, we have

$$(2.8) \quad f^{(k+j)} = \gamma_0,j f + \gamma_1,j f' + \cdots + \gamma_{j,j} f^{(j)},$$
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and \( \gamma_{i,j} \) are entire functions satisfying

\[
\begin{pmatrix}
\gamma_{0,j} \\
\vdots \\
\gamma_{j,j}
\end{pmatrix}
= \begin{pmatrix}
A_{0,1,j}e^\alpha \\
\vdots \\
A_{j,1,j}e^\alpha
\end{pmatrix}
\]

where

\[
A_{i,1,j} = \frac{j!}{i!(j-i)!} e^{-\alpha}(e^\alpha)^{(j-i)}
\]

\[
= \frac{j!}{i!(j-i)!} ((\alpha')^{j-i} + P_{j-i-1}[\alpha']), \quad 0 \leq i \leq j,
\]

are differential polynomials in \( \alpha' \) with constant coefficients. In particular, \( A_{j,1,j} \equiv 1 \) for \( 1 \leq j \leq k-1 \). Here \( P_d[\alpha'] \equiv 0 \) for \( d \leq 0 \).

Proof. We prove this lemma by mathematical induction on \( j \). By (2.7), we have \( f^{(k+1)} = \alpha' e^\alpha f + e^\alpha f' \), so that (2.8)–(2.10) are true for \( j = 1 \). Now suppose that (2.8)–(2.10) are true for \( j \leq k-2 \). Thus by (2.8), we get

\[
f^{(k+j+1)} = \gamma_{0,j} f' + \gamma'_{1,j} f' + \cdots + \gamma'_{j,j} f^{(j)}
\]

\[+ \gamma_{0,j} f' + \cdots + \gamma_{j-1,j} f^{(j)} + \gamma_{j,j} f^{(j+1)}
\]

\[= \gamma_{0,j+1} f' + \gamma_{1,j+1} f' + \cdots + \gamma_{j-1,j+1} f^{(j)} + \gamma_{j,j+1} f^{(j+1)},
\]

where

\[
\gamma_{0,j+1} = \gamma'_{0,j},
\]

\[\gamma_{1,j+1} = \gamma'_{1,j} + \gamma_{0,j},
\]

\[\vdots
\]

\[
\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j},
\]

\[\vdots
\]

\[
\gamma_{j,j+1} = \gamma'_{j,j} + \gamma_{j-1,j},
\]

\[\gamma_{j+1,j+1} = \gamma_{j,j}.
\]

By (2.11)–(2.14), we know that (2.8)–(2.10) are true for \( j + 1 \). Thus (2.8)–(2.10) are true for \( j = 1, 2, \ldots, k-1 \). Lemma 6 is proved.

Lemma 7. Let \( f \) and \( \alpha \) be nonconstant entire functions, \( a \) a finite complex number and \( k \) a positive integer. Suppose that

\[
f^{(k)} = a + e^\alpha f.
\]
Then for any positive integer \( j \geq k \) \( j = sk + l, 0 \leq l \leq k - 1 \), we have
\[
(2.16) \quad f^{(k+j)} = \gamma_{-1,j} + \gamma_{0,j}f + \gamma_{1,j}f' + \cdots + \gamma_{k-1,j}f^{(k-1)},
\]
and \( \gamma_{i,j} \) are entire functions satisfying
\[
(2.17) \quad \begin{pmatrix}
\gamma_{-1,j} \\
\gamma_{0,j} \\
\vdots \\
\gamma_{l,j} \\
\gamma_{l+1,j} \\
\vdots \\
\gamma_{k-1,j}
\end{pmatrix} = \begin{pmatrix}
aA_{-1,1}e^\alpha + a \sum_{t=2}^{s} A_{-1,t,j}(e^\alpha)^t + aA_{0,1}e^\alpha + \sum_{t=2}^{s} A_{0,t,j}(e^\alpha)^t + A_{0,s+1,j}(e^\alpha)^{s+1} \\
A_{0,1}e^\alpha + \sum_{t=2}^{s} A_{0,t,j}(e^\alpha)^t + A_{0,s+1,j}(e^\alpha)^{s+1} \\
\vdots \\
A_{l,1}e^\alpha + \sum_{t=2}^{s} A_{l,t,j}(e^\alpha)^t + A_{l,s+1,j}(e^\alpha)^{s+1} \\
A_{l+1,1}e^\alpha + \sum_{t=2}^{s} A_{l+1,t,j}(e^\alpha)^t + A_{l+1,s+1,j}(e^\alpha)^{s+1} \\
\vdots \\
A_{k-1,1}e^\alpha + \sum_{t=2}^{s} A_{k-1,t,j}(e^\alpha)^t + A_{k-1,s+1,j}(e^\alpha)^{s+1}
\end{pmatrix},
\]
where \( A_{i,t,j} \in \mathcal{P}[\alpha'] \) satisfy
\[
(2.18) \quad \begin{pmatrix}
A_{-1,s,j} \\
A_{0,s+1,j} \\
\vdots \\
A_{l-1,s+1,j} \\
A_{l,s+1,j} \\
A_{l+1,s,j} \\
\vdots \\
A_{k-1,s,j}
\end{pmatrix} = \begin{pmatrix}
C_{-1,s,j}(\alpha') + P_{l-1}[\alpha'] \\
C_{0,s+1,j}(\alpha') + P_{l-1}[\alpha'] \\
\vdots \\
C_{l-1,s+1,j}(\alpha') + P_{l-1}[\alpha'] \\
C_{l,s+1,j}(\alpha') + P_{l-1}[\alpha'] \\
C_{l+1,s,j}(\alpha') + P_{l-1}[\alpha'] \\
\vdots \\
C_{k-1,s,j}(\alpha') + P_{l-1}[\alpha']
\end{pmatrix},
\]
and \( C_{i,s+1,j}, -1 \leq i \leq l - 1 \), and \( C_{i,s,j}, 1 \leq i \leq k - 1 \), are positive integers, and
\[
(2.19) \quad A_{i,1,j} = \frac{j!}{(j-i)!}e^{-\alpha}(e^\alpha)(j-i) = \frac{j!}{(j-i)!}((\alpha')^{j-i} + P_{j-i-1}[\alpha']).
\]
Here \( P_{d}[\alpha'] \equiv 0 \) for \( d \leq 0 \).

Proof. We prove this lemma by mathematical induction on \( j \). First we prove that (2.16)–(2.19) are true for \( j = k \). By Lemma 6, we have
\[
(2.20) \quad f^{(2k-1)} = \gamma_{0,k-1}f + \gamma_{1,k-1}f' + \cdots + \gamma_{k-1,k-1}f^{(k-1)}.
\]
This together with (2.15) yields
\[
(2.21) \quad f^{(2k)} = (f^{(2k-1)})' = \gamma_{0,k-1}f + \gamma_{1,k-1}f' + \cdots + \gamma_{k-1,k-1}f^{(k-1)}
\]
By Lemma 6, we get

\[(2.22) \quad \gamma_{-1,k} = a\gamma_{k-1,k-1} = ae^\alpha, \]
\[\gamma_{0,k} = \gamma_{0,k-1} + e^\alpha \gamma_{k-1,k-1} \]
\[(2.23) \quad = (e^\alpha)^{(k)} + (e^\alpha)^2, \]
\[\gamma_{i,k} = \gamma_{i,k-1} + \gamma_{i-1,k-1} \]
\[= \frac{(k-1)!}{i!(k-1-i)!} (e^\alpha)^{(k-i)} + \frac{(k-1)!}{(i-1)!(k-i)!} (e^\alpha)^{(k-i)} \]
\[(2.24) \quad + \frac{1}{i!(k-i)!} (e^\alpha)^{(k-i)}, \quad i = 1, \ldots, k-1. \]

Thus (2.16)–(2.19) are true for \( j = k \).

Now we assume that this lemma is true for a given \( j = sk + l \) with \( s \geq 1 \) and \( 0 \leq l \leq k-1 \). Next we show that this lemma is true for \( j + 1 \). First by (2.15) and (2.16), we get

\[f^{(k+j+1)} = \gamma_{-1,j} + \gamma_{0,j} + \gamma_{1,j} f' + \cdots + \gamma_{k-1,j} f^{(k-1)} + \gamma_{0,j} f' + \cdots + \gamma_{k-2,j} f^{(k-1)} + a\gamma_{k-1,j} + e^\alpha \gamma_{k-1,j} f.\]

It follows that (2.16) is true for \( j + 1 \) with

\[(2.25) \quad \gamma_{-1,j+1} = \gamma'_{-1,j} + a\gamma_{k-1,j}, \]
\[(2.26) \quad \gamma_{0,j+1} = \gamma'_{0,j} + e^\alpha \gamma_{k-1,j}, \]
\[(2.27) \quad \gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}, \quad i = 1, 2, \ldots, k-1. \]

Thus for \( l \leq k-2 \), by the assumptions,

\[(2.28) \quad \gamma_{-1,j+1} = \left( aA_{-1,1,j} e^\alpha + a \sum_{t=2}^{s-1} A_{-1,t,j} (e^\alpha)^t + aA_{-1,s,j} (e^\alpha)^s \right)'
\[+ a \left( A_{k-1,1,j} e^\alpha + \sum_{t=2}^{s-1} A_{k-1,t,j} (e^\alpha)^t + A_{k-1,s,j} (e^\alpha)^s \right)
\[= a(A'_{-1,1,j} + \alpha' A_{-1,1,j} + A_{k-1,1,j})e^\alpha
\[+ a \sum_{t=2}^{s-1} (A'_{-1,t,j} + t\alpha' A_{-1,t,j} + A_{k-1,t,j}) (e^\alpha)^t + aA_{-1,s,j+1} (e^\alpha)^s, \]
where \( A_{-1,s,j+1} = A'_{-1,s,j} + s\alpha'A_{-1,s,j} + A_{k-1,s,j} \),

\[
\gamma_{0,j+1} = \left( A_{0,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{0,t,j}(e^{\alpha})^t + A_{0,s+1,j}(e^{\alpha})^{s+1} \right)'
\]

\[
+ e^{\alpha} \left( A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^{s} \right)
\]

\[
= A_{0,1,j+1}e^{\alpha} + \sum_{t=2}^{s} (A'_{0,t,j} + t\alpha'A_{0,t,j} + A_{k-1,t-1,j})(e^{\alpha})^t
\]

\[
+ A_{0,s+1,j+1}(e^{\alpha})^{s+1},
\]

where \( A_{0,1,j+1} = A'_{0,1,j} + A_{0,1,j}' \), \( A_{0,s+1,j+1} = A'_{0,s+1,j} + (s+1)\alpha'A_{0,s+1,j} + A_{k-1,s,j} \), and for \( 1 \leq i \leq l \),

\[
\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}
\]

\[
= \left( A_{i,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i,t,j}(e^{\alpha})^t + A_{i,s+1,j}(e^{\alpha})^{s+1} \right)'
\]

\[
+ A_{i-1,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i-1,t,j}(e^{\alpha})^t + A_{i-1,s+1,j}(e^{\alpha})^{s+1}
\]

\[
= A_{i,1,j+1}e^{\alpha} + \sum_{t=2}^{s} (A'_{i,t,j} + t\alpha'A_{i,t,j} + A_{i-1,t,j})(e^{\alpha})^t
\]

\[
+ A_{i,s+1,j+1}(e^{\alpha})^{s+1},
\]

where \( A_{i,1,j+1} = A'_{i,1,j} + \alpha'A_{i,1,j} + A_{i-1,1,j} \), \( A_{i,s+1,j+1} = A'_{i,s+1,j} + (s+1)\alpha'A_{i,s+1,j} + A_{i-1,s+1,j} \), and for \( i = l+1 \),

\[
\gamma_{l+1,j+1} = \gamma'_{l+1,j} + \gamma_{l,j}
\]

\[
= \left( A_{l+1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{l+1,t,j}(e^{\alpha})^t + A_{l+1,s,j}(e^{\alpha})^{s} \right)'
\]

\[
+ A_{l,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{l,t,j}(e^{\alpha})^t + A_{l,s+1,j}(e^{\alpha})^{s+1}
\]

\[
= A_{l+1,1,j+1}e^{\alpha} + \sum_{t=2}^{s} (A'_{l+1,t,j} + t\alpha'A_{l+1,t,j} + A_{l,t,j})(e^{\alpha})^t
\]

\[
+ A_{l+1,s+1,j+1}(e^{\alpha})^{s+1},
\]

where \( A_{l+1,1,j+1} = A'_{l+1,1,j} + \alpha'A_{l+1,1,j} + A_{l,1,j} \), \( A_{l+1,s+1,j+1} = A_{l,s+1,j} \), and for \( l+2 \leq i \leq k-1 \),

\[
\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}
\]
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\[ = \left( A_{i,1,j} e^\alpha + \sum_{t=2}^{s-1} A_{i,t,j} (e^\alpha)^t + A_{i,s,j} (e^\alpha)^s \right) \]

\[ + A_{i-1,1,j} e^\alpha + \sum_{t=2}^{s-1} A_{i-1,t,j} (e^\alpha)^t + A_{i-1,s,j} (e^\alpha)^s \]

\[ = A_{i,1,j+1} e^\alpha + \sum_{t=2}^{s-1} (A'_{i,t,j} + t\alpha' A_{i,t,j} + A_{i-1,t,j}) (e^\alpha)^t + A_{i,s,j+1} (e^\alpha)^s, \]

where \( A_{i,1,j+1} = A'_{i,1,j} + \alpha'A_{i,1,j} + A_{i-1,1,j} \), \( A_{i,s,j+1} = A'_{i,s,j} + s\alpha' A_{i,s,j} + A_{i-1,s,j} \).

By (2.28)–(2.32), we know that (2.16)–(2.19) are true for \( j+1 \) when \( j = sk+l \) with \( 0 \leq l \leq k-2 \). Similarly, we can prove (2.16)–(2.19) are true for \( j+1 \) when \( j = sk+k-1 \). We omit the details here. Thus Lemma 7 is proved.

**Lemma 8.** Let

\[ \Delta_j = \begin{pmatrix} \gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1} \\ \gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1} \end{pmatrix}, \]

where \( \gamma_{i,j} \) are entire functions defined in Lemmas 6–7 (for \( 1 \leq j \leq k-1 \) and \( i > j \) set \( \gamma_{i,j} = 0 \)). Denote the determinant of \( \Delta_j \) by \( \det(\Delta_j) \). Then for \( j = sk+l \) (\( \geq 1 \)) with \( s \geq 0 \), \( 0 \leq l \leq k-1 \), we have

\[ \det(\Delta_j) = \left( (\alpha')^{kj} + P_{kj-1}[\alpha'] \right) (e^\alpha)^k \]

\[ + \sum_{t=k+1}^{(s+1)k+l-1} A_{t,j} (e^\alpha)^t + (-1)^{l(k-l)} (e^\alpha)^{(s+1)k+l}, \]

where \( A_{t,j} \in \mathcal{P}[\alpha'] \).

*Proof.* Obviously, by Lemmas 6–7, we have

\[ \det(\Delta_j) = \sum_{t=k}^\nu A_{t,j} (e^\alpha)^t \]

with \( \nu \geq k \) and \( A_{t,j} \in \mathcal{P}[\alpha'] \). Thus we need only to show that

\[ \nu = (s+1)k+l, \quad A_{\nu,j} = (-1)^{l(k-l)}, \]

and

\[ A_{k,j} = (\alpha')^{kj} + P_{kj-1}[\alpha']. \]


First we prove (2.36). By Lemmas 6–7, we have
\[ M_1 = \begin{pmatrix}
\gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1-l} \\
\gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1-l} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{l-1,j} & \gamma_{l-1,j+1} & \cdots & \gamma_{l-1,j+k-1-l}
\end{pmatrix}_{l \times (k-l)} = (\text{polynomials in } e^\alpha \text{ of degrees } \leq s + 1)_{l \times (k-l)}, \]

\[ M_2 = \begin{pmatrix}
\gamma_{0,j+k-l} & \gamma_{0,j+k-l+1} & \cdots & \gamma_{0,j+k-1} \\
\gamma_{1,j+k-l} & \gamma_{1,j+k-l+1} & \cdots & \gamma_{1,j+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{l-1,j+k-l} & \gamma_{l-1,j+k-l+1} & \cdots & \gamma_{l-1,j+k-1}
\end{pmatrix}_{l \times l} = \begin{pmatrix}
1 & A_{0,0,s+2,j+k-l+1} & \cdots & A_{0,0,s+2,j+k-1} \\
0 & 1 & \cdots & A_{1,0,s+2,j+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}_{l \times l} (e^\alpha)^{s+2} \]

+ (polynomials in \( e^\alpha \) of degrees \( \leq s + 1 \) \( l \times l \))

\[ M_3 = \begin{pmatrix}
\gamma_{l+1,j} & \gamma_{l+1,j+1} & \cdots & \gamma_{l+1,j+k-1-l} \\
\gamma_{l+1,j+1} & \gamma_{l+1,j+1} & \cdots & \gamma_{l+1,j+k-1-l} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1-l}
\end{pmatrix}_{(k-l) \times (k-l)} = \begin{pmatrix}
1 & A_{l,l+1,j+1} & \cdots & A_{l,l+1,j+k-1-l} \\
0 & 1 & \cdots & A_{l+1,l+1,j+k-1-l} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}_{(k-l) \times (k-l)} (e^\alpha)^{s+1} \]

+ (polynomials in \( e^\alpha \) of degrees \( \leq s \) \( (k-l) \times (k-l) \))

\[ M_4 = \begin{pmatrix}
\gamma_{l,j+k-l} & \gamma_{l,j+k-l+1} & \cdots & \gamma_{l,j+k-1} \\
\gamma_{l+1,j+k-l} & \gamma_{l+1,j+k-l+1} & \cdots & \gamma_{l+1,j+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{k-1,j+k-l} & \gamma_{k-1,j+k-l+1} & \cdots & \gamma_{k-1,j+k-1}
\end{pmatrix}_{(k-l) \times l} = \begin{pmatrix}
A_{l,l+1,s+1,j+k-l} & A_{l,l+1,s+1,j+k-l+1} & \cdots & A_{l,l+1,s+1,j+k-1} \\
A_{l+1,l+1,s+1,j+k-l} & A_{l+1,l+1,s+1,j+k-l+1} & \cdots & A_{l+1,l+1,s+1,j+k-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k-1,k-1,s+1,j+k-l} & A_{k-1,k-1,s+1,j+k-l+1} & \cdots & A_{k-1,k-1,s+1,j+k-1}
\end{pmatrix}_{(k-l) \times l} (e^\alpha)^{s+1} \]
matrices whose principal diagonal elements equal 1. Thus by (2.33) we get
\[ (\text{polynomials in } e^\alpha \text{ of degrees } \leq s)_{(k-l)\times l} \]
\[ = C_{(k-l)\times l}(e^\alpha)^{s+1} + (\text{polynomials in } e^\alpha \text{ of degrees } \leq s)_{(k-l)\times l}, \]
where \( A_{l\times l}, B_{(k-l)\times (k-l)}, C_{(k-l)\times l} \) are matrices whose elements are differential polynomials in \( \alpha' \). In particular, \( A_{l\times l} \) and \( B_{(k-l)\times (k-l)} \) are upper triangular matrices whose principal diagonal elements equal 1. Thus by (2.33) we get
\[
\det(\Delta_j) = \begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix}
\]
\[ = \begin{vmatrix} 0 & A \\ B & C \end{vmatrix} (e^\alpha)^{(s+1)k+l} + (\text{terms of degree } \leq (s+1)k+l-1)
\]
\[ = (-1)^{(l(k-l))} \det(A) \det(B)(e^\alpha)^{(s+1)k+l} + (\text{terms of degree } \leq (s+1)k+l-1)
\]
\[ = (-1)^{(l(k-l))}(e^\alpha)^{(s+1)k+l} + (\text{terms of degree } \leq (s+1)k+l-1), \]
where \( 0 = 0_{l\times (k-l)} \) is the zero matrix, \( A = A_{l\times l}, B = B_{(k-l)\times (k-l)}, C = C_{(k-l)\times l} \). This proves (2.36).

Next we prove (2.37). By Lemmas 6–7, we have
\[
A_{k,j} = \begin{vmatrix} A_{0,1,j} & A_{0,1,j+1} & \cdots & A_{0,1,j+k-1} \\ A_{1,1,j} & A_{1,1,j+1} & \cdots & A_{1,1,j+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k-1,1,j} & A_{k-1,1,j+1} & \cdots & A_{k-1,1,j+k-1} \end{vmatrix}
\]
\[ = \begin{vmatrix} 1 \\ \frac{j}{i} \\ \ldots \\ \frac{j}{k-1} \end{vmatrix} \begin{vmatrix} 1 \\ (j+1) \\ \ldots \\ (j+k-1) \end{vmatrix} \begin{vmatrix} 1 \\ \frac{1}{i} \\ \ldots \\ \frac{1}{k-1} \end{vmatrix} + (e^\alpha)^{kj} + P_{kj-1}[\alpha'],
\]
where
\[
\frac{j!}{i!(j-i)!}
\]
are the binomial coefficients. Since
\[
\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}
\]
is a polynomial in \( x \) of degree \( i \), by the calculating properties of determinant and the well-known Vandermonde’s determinant, we see that \( A_{k,j} = C(\alpha')^{kj} + P_{kj-1}[\alpha'] \), where \( C \) is a nonzero constant which is equal to
\[
\prod_{s=1}^{k-1} \frac{1}{s!} \prod_{1 \leq i < t \leq k} (t-i) = 1.
\]
This proves (2.37). Thus Lemma 8 is proved.
3. Proof of Theorem 1

By the assumptions, there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

\begin{align}
(3.1) \quad & \frac{f^{(k)}(z) - a}{f(z) - a} = e^{\alpha(z)}, \\
(3.2) \quad & \frac{f^{(m)}(z) - a}{f(z) - a} = e^{\beta(z)}.
\end{align}

Next we consider two cases.

Case 1. Either $\alpha$ or $\beta$ is a constant. Without loss of generality, we assume that $\alpha$ is a constant. Set $e^\alpha = c$. Then by (3.1), we get

\begin{align}
(3.3) \quad & f^{(k)} - cf = (1 - c)a.
\end{align}

Solving (3.3), we get

\begin{align}
(3.4) \quad & f(z) = \left(1 - \frac{1}{c}\right)a + \sum_{j=1}^{q} C_j e^{\lambda_j z},
\end{align}

where $q \ (\leq k)$ is a positive integer, and $C_j$, $\lambda_j$ are nonzero constants satisfying $(\lambda_j)^k = c$ and $\lambda_i \neq \lambda_j$, $i \neq j$. By (3.4) and (3.2), it follows that $\varrho(e^{\beta}) \leq 1$, where $\varrho$ ($e^{\beta}$) is the order of $e^{\beta}$, so that $e^{\beta} = de^{\mu z}$, where $d \ (\neq 0)$ and $\mu$ are constants. Thus by (3.2) and (3.4), we get

\begin{align}
(3.5) \quad & -a + \sum_{j=1}^{q} (\lambda_j)^m C_j e^{\lambda_j z} = -\frac{da}{c} e^{\mu z} + \sum_{j=1}^{q} C_j d e^{(\lambda_j + \mu) z}.
\end{align}

Applying Lemma 5 to (3.5), we deduce that $\mu = 0$ and $(\lambda_j)^m = d$. Further, if $a \neq 0$, then $c = d$.

By $(\lambda_j)^k = c$, $(\lambda_j)^m = d$ and the fact that $\lambda_j$, $1 \leq j \leq q$, are distinct, we know that $q \leq (k, m)$, where $(k, m)$ is the greatest common divisor of $k$ and $m$. In fact, by Euclidean division algorithm, there exist integers $k_0$ and $m_0$ such that $(k, m) = k_0 k + m_0 m$. Thus $(\lambda_j)^{(k, m)} = [(\lambda_j)^k]^{k_0} [(\lambda_j)^m]^{m_0} = c^{k_0} d^{m_0}$. Hence by the fact that $\lambda_j$, $1 \leq j \leq q$, are distinct, it follows that $q \leq (k, m)$.

Case 2. Both $\alpha$ and $\beta$ are not constants.

We will prove that this case cannot occur. Without loss of generality, we assume $k < m$. Let

\begin{align}
(3.6) \quad & F(z) = f(z) - a.
\end{align}
Then by (3.1) and (3.2), we have

\begin{align*}
F^{(k)} &= a + e^\alpha F, \\
F^{(m)} &= a + e^\beta F.
\end{align*}

Set

\begin{equation}
\phi = \frac{F^{(m)} - F^{(k)}}{F}.
\end{equation}

Then by (3.7) and (3.8), we get

\begin{equation}
\phi = e^\beta - e^\alpha.
\end{equation}

Next we consider two subcases.

Case 2.1: \( \phi \equiv 0 \). Then by (3.10), we get

\begin{equation}
e^\beta = e^\alpha.
\end{equation}

Thus by (3.1), (3.2) and (3.11), we get

\begin{equation}
f^{(m)} - f^{(k)} = 0.
\end{equation}

Solving (3.12), we get

\begin{equation}
f(z) = b(z) + \sum_{j=1}^s C_j e^{\lambda_j z},
\end{equation}

where \( b \) is a polynomial with \( \deg b \leq k - 1 \), \( s \leq m - k \) is a positive integer, and \( C_j, \lambda_j \) are nonzero constants with \( (\lambda_j)^{m-k} = 1 \) and \( \lambda_i \neq \lambda_j, i \neq j \). By (3.1) and (3.13), we know that \( g(e^\alpha) \leq 1 \). This together with that \( \alpha \) is nonconstant yields that \( e^\alpha = Ce^{cz} \), where \( C \) and \( c \) are nonzero constants. Thus by (3.1) and (3.13), we get

\begin{equation}
-a + \sum_{j=1}^s C_j(\lambda_j)^k e^{\lambda_j z} = C[b(z) - a]e^{cz} + \sum_{j=1}^s CC_j e^{(\lambda_j + c)z}.
\end{equation}

Applying Lemma 5 to (3.14), we get that \( c = 0 \), a contradiction.

Case 2.2: \( \phi \not\equiv 0 \). Then by the logarithmic derivative lemma, it follows from (3.9) that

\begin{equation}
m(r, \phi) = S(r, F).
\end{equation}
By (3.10), \( \phi \) is an entire function. Thus by (3.15), we get

\[
T(r, \phi) = S(r, F).
\]

Since \( \phi \neq 0 \), by (3.10), we get

\[
e^{\beta} \frac{\phi}{\phi} = 1 + e^{\alpha} \frac{\phi}{\phi}.
\]

Thus by (3.16), (3.17) and the second fundamental theorem we deduce that

\[
T\left( r, \frac{e^{\beta}}{\phi} \right) \leq N\left( r, \frac{e^{\beta}}{\phi} \right) + N\left( r, \frac{\phi}{e^{\beta}} \right) + N\left( r, \frac{1}{e^{\beta}} \frac{\phi}{\phi} - 1 \right) + S\left( r, \frac{e^{\beta}}{\phi} \right),
\]

\[
\leq N\left( r, \frac{e^{\beta}}{\phi} \right) + N\left( r, \frac{\phi}{e^{\beta}} \right) + N\left( r, \frac{1}{e^{\beta}} \right) + S\left( r, \frac{e^{\beta}}{\phi} \right)
\]

\[
\leq S(r, F) + S\left( r, \frac{e^{\beta}}{\phi} \right).
\]

This together with (3.16) yields that \( T(r, e^{\beta}) = S(r, F) \). It follows from (3.10) and (3.16) that \( T(r, e^{\alpha}) = T(r, e^{\beta} - \phi) = S(r, F) \). Thus we get

\[
T(r, e^{\alpha}) + T(r, e^{\beta}) = S(r, F).
\]

Now, for \( 0 \leq j \leq k - 1 \), set

\[
p_{i,j} = \gamma_{i,m-k+j}, \quad i = -1, 0, 1, \ldots, k - 1,
\]

where \( \gamma_{i,j} \) are defined as in Lemmas 6–8. Then by Lemmas 6–7, we have

\[
F^{(m+j)} = F^{(k+m-k+j)}
= p_{-1,j} + p_{0,j} F + p_{1,j} F' + \cdots + p_{k-1,j} F^{(k-1)}, \quad j = 0, 1, \ldots, k - 1.
\]

On the other hand, by (3.8) and Lemma 6, for \( 1 \leq j \leq k - 1 \), we have

\[
F^{(m+j)} = q_{0,j} e^{\beta} F + q_{1,j} e^{\beta} F' + \cdots + q_{j,j} e^{\beta} F^{(j)},
\]

where \( q_{i,j} \), \( i \leq j \), are differential polynomials in \( \beta' \) with constant coefficients. In particular, \( q_{j,j} \equiv 1 \) for \( j = 1, 2, \ldots, k - 1 \). Thus by (3.8), (3.21) and (3.22), we get

\[
(F, F', \ldots, F^{(k-1)}) (e^{\beta} Q - P) = \Gamma,
\]
where

\[
P = \begin{pmatrix}
p_{0,0} & p_{0,1} & \cdots & p_{0,k-1} \\
p_{1,0} & p_{1,1} & \cdots & p_{1,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k-1,0} & p_{k-1,1} & \cdots & p_{k-1,k-1}
\end{pmatrix},
\]

(3.24)

\[
Q = \begin{pmatrix}
1 & q_{0,1} & \cdots & q_{0,k-1} \\
0 & 1 & \cdots & q_{1,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix},
\]

(3.25)

\[
\Gamma = \left( p_{-1,0} - a, p_{-1,1}, \ldots, p_{-1,k-1} \right).
\]

By (3.23) and the theory of linear equations, we get

\[
\det(e^\beta Q - P)F = \det(T),
\]

(3.27)

where \( T \) is a matrix whose first line is \( \Gamma \) and the other lines are the same as those of \( e^\beta Q - P \).

Thus by (3.19) and (3.27), we know that

\[
\det(e^\beta Q - P) = 0.
\]

(3.28)

This yields that

\[
\det(e^\beta I - R) = 0,
\]

(3.29)

where \( I = I_{k \times k} \) is the \( k \)th unit matrix, \( R = Q^{-1}P \) and \( Q^{-1} \) is the inverse matrix of \( Q \). Obviously, the matrix \( Q^{-1} \) is also an upper triangular matrix whose elements are differential polynomial in \( \beta' \). By (3.29), we get

\[
(e^\beta)^k - a_1(e^\beta)^{k-1} + \cdots + (-1)^t a_t (e^\beta)^{k-t} + \cdots + (-1)^k a_k = 0,
\]

(3.30)

where \( a_t \) is the sum of all the principle minors of order \( t \) of \( R \). In particular, \( a_k = \det(R) = \det(P) \). Here, for a matrix

\[
A = (a_{i,j}) = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{pmatrix},
\]
and $t$ integers $1 \leq i_1 < i_2 < \cdots < i_t \leq n$, we call

\[
\begin{vmatrix}
    a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_t} \\
    a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_t} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{i_t,i_1} & a_{i_t,i_2} & \cdots & a_{i_t,i_t}
\end{vmatrix}
\]

a principle minor of order $t$ of $A$.

Obviously, by (3.24), (3.25) and the definition of $a_t$, $a_t$, $1 \leq t \leq k$, are polynomials in $e^\alpha$ whose coefficients are differential polynomials in $\alpha'$ and $\beta'$ with constant coefficients.

Next we consider the degrees of these polynomials $a_t$. Since $m > k$, there exist integers $s \geq 1$ and $0 \leq l \leq k - 1$ such that

\begin{equation}
(3.31) \quad m = sk + l.
\end{equation}

It is obvious that if $l = 0$ then $s > 1$. We claim that for $l \geq 1$,

\begin{equation}
(3.32) \quad \deg(a_t) \leq ts + l - 1, \quad t = 1, 2, \ldots, k - 1,
\end{equation}

and for $l = 0$,

\begin{equation}
(3.33) \quad \deg(a_t) \leq ts, \quad t = 1, 2, \ldots, k - 1.
\end{equation}

and

\begin{equation}
(3.34) \quad \deg(a_k) = m = ks + l.
\end{equation}

In order to prove (3.32)–(3.34), we first consider the degree of the elements of $R = (r_{i,j})$ which are polynomials in $e^\alpha$. By (3.20), we see that for $0 \leq j \leq k - 1 - l$, $p_{i,j} = \gamma_{i,(s-1)k+j+l}$, while for $k - l \leq j \leq k - 1$, $p_{i,j} = \gamma_{i,sk+j+l-k}$. Thus by Lemmas 6–7, for $0 \leq i, j \leq k - 1$,

\begin{equation}
(3.35) \quad \deg(p_{i,j}) \leq \begin{cases} 
    s & \text{if } 0 \leq j \leq k - 1 - l, 0 \leq i \leq j + l, \\
    s - 1 & \text{if } 0 \leq j \leq k - 1 - l, j + l + 1 \leq i \leq k - 1, \\
    s + 1 & \text{if } k - l \leq j \leq k - 1, 0 \leq i \leq j + l - k, \\
    s & \text{if } k - l \leq j \leq k - 1, j + l - k + 1 \leq i \leq k - 1.
\end{cases}
\end{equation}

By (3.25), we may assume that

\begin{equation}
(3.36) \quad Q^{-1} = \begin{pmatrix}
    1 & q_{0,1}^* & \cdots & q_{0,k-1}^* \\
    0 & 1 & \cdots & q_{1,k-1}^* \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
\end{pmatrix},
\end{equation}
where \( q_{i,j}^* \), \( 0 \leq i < j \leq k - 1 \), are differential polynomials in \( \beta' \) with constant coefficients. Thus by \( (r_{i,j}) = R = Q^{-1}P \), we get

\[
(3.37) \quad r_{i,j} = p_{i,j} + q_{i,i+1}^* p_{i+1,j} + q_{i,i+2}^* p_{i+2,j} + \cdots + q_{i,k-1}^* p_{k-1,j}.
\]

Thus by (3.35) and (3.37), we see that for \( 0 \leq i, j \leq k - 1 \),

\[
(3.38) \quad \text{deg}(r_{i,j}) \leq \begin{cases} 
 0 & \text{if } 0 \leq j \leq k - 1 - l, 0 \leq i \leq j + l, \\
 s - 1 & \text{if } 0 \leq j \leq k - 1 - l, j + l + 1 \leq i \leq k - 1, \\
 s + 1 & \text{if } k - l \leq j \leq k - 1, 0 \leq i \leq j + l - k, \\
 s & \text{if } k - l \leq j \leq k - 1, j + l - k + 1 \leq i \leq k - 1.
\end{cases}
\]

Now let

\[
L_{i_1,i_2,\ldots,i_t} = \begin{vmatrix}
  r_{i_1,i_1} & r_{i_1,i_2} & \cdots & r_{i_1,i_t} \\
  r_{i_2,i_1} & r_{i_2,i_2} & \cdots & r_{i_2,i_t} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{i_t,i_1} & r_{i_t,i_2} & \cdots & r_{i_t,i_t}
\end{vmatrix}
\]

be a principle minor of order \( t \leq k - 1 \) of \( R \), where \( 0 \leq i_1 < i_2 < \cdots < i_t \leq k - 1 \).

By (3.38), for the case of \( l = 0 \), the degrees of all \( r_{i,j} \) are at most \( s \), so that the degree of \( L_{i_1,i_2,\ldots,i_t} \) is at most \( ts \). It follows that the degree of \( a_t \) is at most \( ts \). This proves (3.33).

Next we consider the case of \( 1 \leq l \leq k - 1 \). By the definition of determinant, we have

\[
L_{i_1,i_2,\ldots,i_t} = \sum \delta_{j_1,j_2,\ldots,j_t} r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t},
\]

where the sum takes over all the permutations of \((i_1, i_2, \ldots, i_t)\), and \( \delta_{j_1,j_2,\ldots,j_t} = \pm 1 \) according to the permutation \((j_1, j_2, \ldots, j_t)\) of \((i_1, i_2, \ldots, i_t)\) is even or odd. Let

\[
L_t = r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t}.
\]

For \( t \leq l - 1 \), by (3.38), the degree of \( L_t \) is at most \( ts + l + 1 \).

For \( t \geq l \), if there exist \( x \leq l - 1 \) polynomials in \( r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_s,j_s} \) with degree \( s + 1 \), then by (3.38), the degree of \( L_t \) is at most \( x(s+1) + (t-x)s = ts + x \leq ts + l - 1 \). If there exist \( l \) polynomials in \( r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_s,j_s} \) with degree \( s + 1 \), then by (3.38), \( \{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\} \). It follows that there exists at least one of \( r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_s,j_s} \) whose degree is \( s - 1 \) (for otherwise, we must have \( \{l, \ldots, k-1\} \subset \{i_1, i_2, \ldots, i_t\} \). This together with \( \{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\} \) yields that \( t \geq k \), which contradicts \( t \leq k - 1 \). Hence the degree of \( L_t \) is at most \( l(s+1) + (s-1) + (t-l-1)s = ts + l - 1 \). It follows that \( \text{deg}(L_{i_1,i_2,\ldots,i_t}) \leq ts + l - 1 \). Thus (3.32) is proved.

Next we prove (3.34). In fact, it can be seen from (3.20), (3.24) and Lemma 8 that

\[
a_k = \det(P) = \det(\Delta_{m-k})
\]

\[
= \left( (\alpha')^k (m-k) + P_{k(m-k)-1}[\alpha'] \right) (e^\alpha)^k
\]

\[
+ \sum_{t=k+1}^{m-1} A_{t,m-k}(e^\alpha)^t + (-1)^l(k-l)(e^\alpha)^m.
\]

\[
(3.39)
\]
Thus we get (3.34).

By (3.32)–(3.34), we see that for \(1 \leq t \leq k - 1\), \(\deg(a_t) < \deg(a_k)\). Thus by (3.30) and Lemma 2, it follows that

\[
T(r, e^{\beta}) = O(T(r, e^{\alpha})) + S(r, e^{\beta}), \quad T(r, e^{\alpha}) = O(T(r, e^{\beta})) + S(r, e^{\alpha}).
\]

Hence we get

\[
(3.40) \quad S(r, e^{\alpha}) = S(r, e^{\beta}) = S(r) \quad \text{(say)}.
\]

Next we prove the following claims.

**Claim I.** For any rational number \(\theta = \nu/\mu\) with \(\nu \in \mathbb{Z}\) and \(\mu \in \mathbb{N}\),

\[
(3.41) \quad T(r, e^{\beta - \theta \alpha}) \neq S(r).
\]

Suppose on the contrary that there exists a rational number \(\theta = \nu/\mu\) such that

\[
(3.42) \quad T(r, e^{\beta - \theta \alpha}) = S(r).
\]

Let

\[
(3.43) \quad b(z) = e^{\beta - \theta \alpha}.
\]

Then \(b(z) \neq 0\) is entire and \(T(r, b) = S(r)\). By (3.43),

\[
(3.44) \quad e^{\beta} = b(z)e^{\theta \alpha} = b(z)(e^{\alpha/\mu})^\nu.
\]

On the other hand, by (3.20), (3.24) and Lemmas 6–7, we have

\[
(3.45) \quad P = (e^{\alpha})^{s+1}P_0 + (e^{\alpha})^sP_1 + \cdots + (e^{\alpha})P_s,
\]

where \(P_j\) are \(k \times k\) matrices whose elements are differential polynomials in \(\alpha'\). In particular, \(\det(P_s) \equiv A_{k,m-k}\), where \(A_{k,m-k}\) is defined by (2.38). By (3.28), (3.44) and (3.45), we get

\[
(3.46) \quad \det(b(e^{\alpha/\mu})^\nu Q - (e^{\alpha})^{s+1}P_0 - (e^{\alpha})^sP_1 - \cdots - (e^{\alpha})P_s) = 0.
\]

If \(\nu > \mu\), then by (3.46), we get

\[
(3.47) \quad \det(b(e^{\alpha/\mu})^{\nu-\mu} Q - (e^{\alpha})^s\mu P_0 - \cdots - (e^{\alpha/\mu})^{\mu}P_{s-1} - P_s) = 0.
\]

Since the left side of (3.47) is a polynomial in \(e^{\alpha/\mu}\) whose “constant” term is \(\det(-P_s) = (-1)^k A_{k,m-k}\), by Lemma 3, we get \(A_{k,m-k} = 0\). Thus by (2.10),
(2.19) and the fact that $\alpha$ is nonconstant, $\alpha'$ is nonconstant. For otherwise, let $\alpha' = c$. Then $c \neq 0$, and by (2.10), (2.19), we have

$$A_{i,1,j} = \binom{j}{i}(c)^{j-i},$$

so that by (2.38), it follows that $A_{k,m-k} = (c)^{k(m-k)} \neq 0$, which contradicts $A_{k,m-k} = 0$. Hence $\alpha'$ is nonconstant. Thus by (2.37) and Lemma 1, we deduce that $T(r, \alpha') = m(r, \alpha') = S(r, \alpha')$, a contradiction.

If $\nu < \mu$, then by (3.46), we get

$$\det(bQ) - (e^{\alpha/\mu})(s+1)^{\mu-\nu}P_0 - (e^{\alpha/\mu}s^{\mu-\nu}P_1 - \cdots - (e^{\alpha/\mu})^{\mu-\nu}P_s) = 0.$$ 

Using the same argument as that in case $\nu > \mu$, we deduce that $\det(bQ) = 0$. Thus by $\det(Q) = 1$, we get that $b = 0$, a contradiction.

If $\nu = \mu$, then $e^{\beta} = b(z)e^{\alpha}$. Thus by (3.32)–(3.34), we see that the left side of (3.30) is a polynomial in $e^{\alpha}$ whose leading term is $\varepsilon(e^{\alpha})^m$, where $\varepsilon = \pm 1$ is a constant. Thus applying Lemma 2 to (3.30), we get a contradiction: $T(r, e^{\alpha}) = S(r)$.

Hence Claim I is proved.

**Claim II.** We have

$$H = \sum_{t=1}^{k-1} (-1)^ta_t(e^{\beta})^{k-t} \equiv 0.$$ 

Suppose that $H \neq 0$. Then by the fact that $a_t$ are polynomials in $e^{\alpha}$, we can rewrite $H$ as

$$H = \sum_{(t,i) \in T \times I} a_{t,i}e^{(k-t)\beta + i\alpha},$$

where $T \subset \{1, \ldots, k-1\}$ and $I$ are finite index sets, $a_{t,i} \neq 0$ are differential polynomials in $\alpha'$ and $\beta'$ such that all the functions $a_{t,i}e^{(k-t)\beta + i\alpha}$, $(t, i) \in T \times I$ are linearly independent.

By (3.39), we rewrite $a_k$ as

$$(-1)^ka_k = \sum_{i \in J} a_{k,i}e^{i\alpha},$$

where $J \supset \{m\}$ is a finite index set, and $a_{k,i} (\neq 0)$, $i \in J$, are differential polynomials in $\alpha'$.

Hence by (3.30), (3.48)–(3.50), we get

$$e^{k\beta} + \sum_{(t,i) \in T \times I} a_{t,i}e^{(k-t)\beta + i\alpha} + \sum_{i \in J} a_{k,i}e^{i\alpha} = 0.$$
By (3.51), we get

\[(3.52) \quad \sum_{(t,i) \in T \times I} (-a_{t,i})e^{-(t+1)\alpha} + \sum_{i \in J} (-a_{k,i})e^{-(i+1)\alpha} = 1.\]

If the functions \((-a_{t,i})e^{-(t+1)\alpha}, (t,i) \in T \times I\) and \((-a_{k,i})e^{-(i+1)\alpha}, i \in J\) are linearly independent, then by Lemma 4 and the fact that \(m \in J\), we get

\[T(r, (-a_{k,m})e^{-(k+1)\alpha}) = S(r),\]

so that

\[T(r, e^{-(k+1)\alpha}) = S(r),\]

which contradicts Claim I.

Hence the functions \((-a_{t,i})e^{-(t+1)\alpha}, (t,i) \in T \times I\) and \((-a_{k,i})e^{-(i+1)\alpha}, i \in J\) are linearly dependent. That is, there exist constants \(C_{t,i}, (t,i) \in T \times I\) and \(C_{k,i}, i \in J\), at least one of them is not equal to 0, such that

\[\sum_{(t,i) \in T \times I} C_{t,i}a_{t,i}e^{-(t+1)\alpha} + \sum_{i \in J} C_{k,i}a_{k,i}e^{-(i+1)\alpha} = 0,\]

so that

\[(3.53) \quad \sum_{(t,i) \in T \times I} C_{t,i}a_{t,i}e^{(k-t)\beta+(i-1)\alpha} + \sum_{i \in J} C_{k,i}a_{k,i}e^{i\alpha} = 0.\]

By Lemma 3, at least one of \(C_{t,i}, (t,i) \in T \times I\) is not equal to 0. Set \(T_1 \times I_1 = \{(t,i) \in T \times I : C_{t,i} \neq 0\}\). Then \(T_1 \times I_1 \neq \emptyset\) (empty set). By the assumption that \(a_{t,i}e^{(k-t)\beta+(i-1)\alpha}, (t,i) \in T \times I\) are linearly independent, at least one of \(C_{k,i}, i \in J\) is not equal to 0. Set \(J_1 = \{i \in J : C_{k,i} \neq 0\}\). Then \(J_1 \neq \emptyset\). Let \(i_1 \in J_1\). Then by (3.53), we get

\[\sum_{(t,i) \in T_1 \times I_1} \frac{-C_{t,i}a_{t,i}}{C_{k,i}a_{k,i}}e^{(k-t)\beta+(i-1)\alpha} + \sum_{i \in J_1 \setminus \{i_1\}} \frac{-C_{k,i}a_{k,i}}{C_{k,i}a_{k,i}}e^{i\alpha} = 1.\]

If the functions

\[(3.54) \quad \frac{-C_{t,i}a_{t,i}}{C_{k,i}a_{k,i}}e^{(k-t)\beta+(i-1)\alpha}, (t,i) \in T_1 \times I_1 \quad \text{and} \quad \frac{-C_{k,i}a_{k,i}}{C_{k,i}a_{k,i}}e^{i\alpha}, i \in J_1 \setminus \{i_1\}\]

are linearly independent, then by Lemma 4, we get for \((t_0,i_0) \in T_1 \times I_1\),

\[T\left(r, \frac{C_{t_0,i_0}a_{t_0,i_0}}{C_{k,i_1}a_{k,i_1}}e^{(k-t_0)\beta+(i_0-1)\alpha}\right) = S(r),\]
so that
\[ T(r, e^{(k-t_0)\beta + (i_0-i_1)\alpha}) = S(r), \]
which again contradicts Claim I. Thus the functions showed in (3.54) are linearly dependent. Thus there exist constants \( D_{t,i}, (t, i) \in T_1 \times I_1 \) and \( D_{k,i}, i \in J_1 \setminus \{i_1\} \), at least one of them is not equal to 0, such that
\[
\sum_{(t,i) \in T_1 \times I_1} \frac{D_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}} e^{(k-t)\beta + (i-i_1)\alpha} + \sum_{i \in J_1 \setminus \{i_1\}} \frac{D_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}} e^{(i-i_1)\alpha} = 0,
\]
so that
\[
\sum_{(t,i) \in T_1 \times I_1} D_{t,i}a_{t,i} + \sum_{i \in J_1 \setminus \{i_1\}} D_{k,i}a_{k,i} e^{i\alpha} = 0. \tag{3.55}
\]
By Lemma 3, we see that at least one of \( D_{t,i}, (t, i) \in T \times I \) is not equal to 0, so that \( T_2 \times I_2 = \{(t, i) \in T_1 \times I_1 : D_{t,i} \neq 0\} \neq \emptyset \). By the assumption that \( a_{t,i}e^{(k-t)\beta + i\alpha}, (t,i) \in T \times I \) are linearly independent, at least one of \( D_{k,i}, i \in J \setminus \{i_1\} \) is not equal to 0, so that \( J_2 = \{i \in J_1 \setminus \{i_1\} : D_{k,i} \neq 0\} \neq \emptyset \). Let \( i_2 \in J_2 \). Then using an argument similar to that in the above step, there exist constants \( E_{t,i}, (t, i) \in T_2 \times I_2 \) and \( E_{k,i}, i \in J_2 \setminus \{i_2\} \), at least one of them is not equal to 0, such that
\[
\sum_{(t,i) \in T_2 \times I_2} E_{t,i}a_{t,i} e^{(k-t)\beta + i\alpha} + \sum_{i \in J_2 \setminus \{i_2\}} E_{k,i}a_{k,i} e^{i\alpha} = 0.
\]
Step by step, it follows that \( J \) is an infinite set. It is impossible. Hence we have proved Claim II.

Next we continue to prove Theorem 1. By (3.30), (3.50) and Claim II, we get
\[
e^{k\beta} + \sum_{i \in J} a_{k,i} e^{i\alpha} = 0,
\]
so that
\[
\sum_{i \in J} (-a_{k,i}) e^{i\alpha - k\beta} = 1. \tag{3.56}
\]
If the functions \( (-a_{k,i}) e^{i\alpha - k\beta}, i \in J \), are linearly independent, then by Lemma 3, we get
\[
T(r, a_{k,m} e^{m\alpha - k\beta}) = S(r),
\]
so that
\[
T(r, e^{m\alpha - k\beta}) = S(r).
\]
This contradicts Claim I.
Hence the functions \((-a_{k,i})e^{i\alpha-k\beta}, i \in J\), are linearly dependent. Thus there exist constants \(C_i, i \in J\), at least one of them is not equal to 0, such that
\[
\sum_{i \in J} C_i a_{k,i} e^{i\alpha-k\beta} = 0,
\]
so that
\[
\sum_{i \in J} C_i a_{k,i} e^{i\alpha} = 0.
\]
This contradicts Lemma 3.

The proof of Theorem 1 is complete.

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References


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