ON FACTORIZATIONS OF ENTIRE FUNCTIONS
OF BOUNDED TYPE

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Abstract. We prove that if $f$ is a transcendental entire function and the set of all finite
singularities of its inverse function $f^{-1}$ is bounded, then $f(z) + P(z)$ is prime for any nonconstant
polynomial $P(z)$, unless $f(z)$ and $P(z)$ has a nonlinear common right factor. Particularly, it is
shown that $f(z) + az$ is prime for any constant $a \neq 0$.

1. Introduction

A transcendental meromorphic function $F$ is said to be prime (pseudo-prime)
if, and only if, whenever $F = f(g)$ for some meromorphic functions $f$ and $g$, either $f$ or $g$ must be bilinear (rational); $F$ is called left-prime (right-prime) if every
factorization of $F$ implies that $f$ is bilinear whenever $g$ is transcendental ($g$ is
linear if $f$ is transcendental). It is easily seen $F$ is prime if and only if $F$ is left-
prime as well as right-prime. We refer the readers to [3] or [4] for an introduction
to the factorization theory of entire and meromorphic functions.

A point $a$ is called a singularity of $f^{-1}$ (the inverse function of $f$), if $a$ is
either a critical value or asymptotic value of $f$. We denote by $\text{sing}(f^{-1})$ the set
of all finite singularities of $f^{-1}$, i.e.

$$\text{sing}(f^{-1}) = \{z \in \mathbb{C} : z \text{ is a singularity of } f^{-1}\}.$$

We denote by $B$ the class of all entire functions $f$ such that $\text{sing}(f^{-1})$ is bounded
and by $S$ the class of all entire functions $f$ such that $\text{sing}(f^{-1})$ is finite. If $f \in B$
($f \in S$), we say $f$ is of bounded (finite) type.

In 1981, Noda [8] proved the following result.

**Theorem A.** Let $f(z)$ be a transcendental entire function. Then the set

$$NP(f) = \{a \mid a \in \mathbb{C}, \ f(z) + az \text{ is not prime} \}$$

is at most countable

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As a further study on the cardinality of $NP(f)$, which is denoted by $|NP(f)|$, Ozawa and Sawada [9] posed the following interesting question:

**Question.** Is there any $f$ for which the exceptional set $NP(f)$ in Theorem A is really infinitely countable? Or what is the maximal cardinal number of the exceptional set $NP(f)$?

**Theorem B** (Ozawa and Sawada [9]). Let $G(w)$ be an entire function satisfying

$$M(R, G(w)) \leq \exp(KR)$$

for $R \geq R_0 > 0$ and for some constant $K > 0$. Then either $G(e^z) + az$ or $G(e^z) + bz$ is prime if $ab(a - b) \neq 0$.

This shows that the cardinality of $NP(G(e^z))$ is at most 2 if $M(R, G(w)) \leq \exp(KR)$ for $R \geq R_0 > 0$ and for some constant $K > 0$. As a study of the above question, Liao–Yang [6] proved the following result.

**Theorem C.** Let $f$ be a transcendental entire function of finite order in $S$. Then for any constant $a \neq 0$, $f(z) + az$ is prime, i.e. $|NP(f)| \leq 1$.


**Theorem D.** Let $P$, $Q$ be nonconstant polynomials, $\alpha \in B$, $h$ a periodic entire function of order one and mean type, $G(z) = P \circ h \circ \alpha(z)$. If $F(z) = G^n(z) + Q(z)$ has a factorization $F(z) = f(g(z))$, then $g(z)$ must be a common right factor of $\alpha(z)$ and $Q(z)$.

**Remark 1.** The original statement of Theorem D only requires that $h$ is of order one. Here we would like to point out that $h$ should be at most order one of mean type, as it is needed in the proof of Theorem D, Lemma 5 in [13]. However, $f$ in Lemma 5 should be an entire function of exponential type, i.e. $f$ has order less than one or order one and mean type; see p. 27 in [4].

**Remark 2.** Let $G$ be defined in Theorem D. Then $G^n(z) + az$ is prime for any constant $a \neq 0$.

As a continuation of the study of our previous work [6], we are able to extend Theorem C to a large class of functions, namely, functions of bounded type. The following is our main result.

**Theorem.** Let $f$ be a transcendental entire function in $B$, then for any nonconstant polynomial $P(z)$, $f(z) + P(z)$ is prime unless $f(z)$ and $P(z)$ has a nonlinear common right factor.
2. Some lemmas

Lemma 1 (Rippon and Stallard [11]). Let $f$ be a meromorphic function with a bounded set of all finite critical and asymptotic values. Then there exists $K > 0$ such that if $|z| > K$ and $|f(z)| > K$, then

$$|f'(z)| \geq \frac{|f(z)| \log |f(z)|}{16\pi |z|}.$$ 

Lemma 2 ([5]). Let $f$ be a transcendental entire function, and $0 < \delta < \frac{1}{4}$. Suppose that at the point $z$ with $|z| = r$ the inequality

$$|f(z)| > M(r, f)\nu(r, f)^{(1/4)+\delta}$$

holds. Then there exists a set $F$ in $\mathbb{R}^+$ and of finite logarithmic measure, i.e.,

$$\int_F \frac{dt}{t} < +\infty$$

such that

$$f^{(m)}(z) = \left(\frac{\nu(r, f)}{z}\right)^m (1 + o(1)) f(z)$$

holds whenever $m$ is a fixed nonnegative integer and $r \notin F$.

Lemma 3 (Baker and Singh [1], also see [2]). Let $f$ and $g$ be two entire functions. Then

$$\text{sing}((f \circ g)^{-1}) \subset \text{sing}(f^{-1}) \cup f(\text{sing}(g^{-1})).$$

Lemma 4 (Polya [10]). Let $f$ and $g$ be two transcendental entire functions. Then

$$\lim_{r \to \infty} \frac{M(r, f \circ g)}{M(r, g)} = \infty.$$ 

Lemma 5. Let $f$ be a transcendental entire function. Then

$$M(r, f') \leq M(r, f)^2$$

for a sufficiently large $r$.

Remark 3. This follows easily from a result of Valiron ([12]):

$$\lim_{r \to \infty} \frac{\log M(r, f')}{\log M(r, f)} = 1.$$
3. Proof of the theorem

Let \( F(z) = f(z) + P(z) \), \( P(z) \) is a nonconstant polynomial. We first prove that \( F \) is pseudo-prime. Assume that

\[
F(z) = g(h(z)),
\]

where \( g \) is a transcendental meromorphic function with at most one pole and \( h \) is a transcendental entire function. Thus

\[
f(z) = g(h(z)) - P(z), \quad f'(z) = g'(h(z))h'(z) - P'(z).
\]

First we consider the case that \( g \) is a transcendental entire function, and then we discuss two situations.

**Case 1:** \( g' \) has at least two zeros. Then there exists a zero \( c \) of \( g' \) such that \( h(z) = c \) has infinitely many roots \( \{z_k\}_{k=1}^{\infty} \). Thus we have

\[
f(z_k) = -P(z_k) + g(c), \quad f'(z_k) = -P'(z_k).
\]

By Lemma 1, we would have

\[
|P'(z_k)| \geq \frac{|P(z_k) - g(c)| \log |P(z_k) - g(c)|}{16\pi |z_k|},
\]

which leads to a contradiction.

**Case 2:** \( g' \) has at most one zero. Thus

\[
g'(w) = (w - w_0)^n e^{\alpha(w)}, \quad f'(z) = (h(z) - w_0)^n e^{\alpha(h(z))}h'(z) - P'(z),
\]

where \( n \) is a non-negative integer. Let \( K(z) = e^{-\alpha(h(z))/(n+3)} \), and assume that \( \Gamma \) is a simple curve tending to infinity such that if \( z \in \Gamma \) and \( |z| = r \), then \( |K(z)| = M(r, K) \). By Lemmas 4 and 5, we have, if \( z \in \Gamma \) and \( |z| = r \) is sufficiently large,

\[
|g'(h(z))h'(z)| = \left| \left( h(z) - w_0 \right)^n e^{\alpha(h(z))}h'(z) \right| = \frac{\left| \left( h(z) - w_0 \right)^n h'(z) \right|}{M(r, K)^{n+3}} \leq \frac{1}{M(r, K)} \to 0.
\]

Let \( L(z) = -\alpha(h(z))/(n + 3) \) and \( A(r, L) = \max_{|z| = r} \Re L(z) \). Thus if \( z \in \Gamma \), \( |K(z)| = M(r, K) = e^{A(r, L)} \), \( \Re L(z) = A(r, L) \). By Hadamard’s three-circle theorem, we have, for \( r_1 < r_2 < r_3 \),

\[
A(r_2, L) \leq \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} A(r_3, L) + \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} A(r_1, L).
\]
For $z_0 \in \Gamma$, we have

$$|L'(z_0)| = \lim_{z \rightarrow z_0, \mu \in \Gamma} \frac{|L(z) - L(z_0)|}{|z - z_0|} \geq \lim_{z \rightarrow z_0, \mu \in \Gamma} \frac{|\text{Re} L(z) - \text{Re} L(z_0)|}{|z - z_0|}.$$  

Let $|z_0| = r_0$ and $|z| = r_0 + h, h > 0$, then as $z \rightarrow z_0, h \rightarrow 0$. Thus, by (5) and (6), we have, for sufficiently large $r_0$,

$$|L'(z_0)| \geq \lim_{z \rightarrow z_0, \mu \in \Gamma} \frac{A(r_0 + h, L) - A(r_0, L)}{|z - z_0|} = \lim_{z \rightarrow z_0, \mu \in \Gamma} \frac{h}{|z - z_0|} \frac{A(r_0 + h, L) - A(r_0, L)}{h}$$

$$\geq \lim_{h \rightarrow 0} \frac{\log(1 + h/r_0)}{\log r_0} \frac{(A(r_0, L) - A(1, L))}{h} = \frac{A(r_0, L) - A(1, L)}{r_0 \log r_o} > 1.$$  

Let $w = G(z) = e^{\alpha(h(z))/(n+3)} = e^{-L(z)}$. Thus 0 is an asymptotic value of $G$ and $\Gamma$ is the corresponding asymptotic curve, $\gamma = G(\Gamma)$ is a simple curve connecting $G(0)$ and 0. Let $B$ be the length of $\gamma$, which is a finite number. And $dw = e^{-L(z)L'(z)}dz$. By this, (4) and (7), if $z \in \Gamma$, we have

$$|g(h(z))| = \left| \int_{z_0 \text{ along } \Gamma}^z g'(h(z))h'(z)dz + g(h(z_0)) \right|$$

$$\leq \int_{z_0 \text{ along } \Gamma}^z |g'(h(z))h'(z)||dz| + |g(h(z_0))|$$

$$\leq \int_{w_0 \text{ along } \gamma}^w \frac{1}{|L'(z)|} |dw| + |g(h(z_0))|$$

$$\leq \int_{w_0 \text{ along } \gamma}^w |dw| + |g(h(z_0))|$$

$$\leq B + |g(h(z_0))|.$$  

Thus we can find a sequence of $\{z_k\}_{k=1}^\infty$ such that $z_k \rightarrow \infty$ as $k \rightarrow \infty$, and

$$f(z_k) \sim -P(z_k), \quad f'(z_k) \sim -P'(z_k).$$  

A contradiction follows from this and Lemma 1.
If \( g' \) has just one pole \( w_1 \), so does \( g \), then \( h(z) \) does not assume \( w_1 \), i.e., \( h(z) = e^{\beta(z)} + w_1 \). Moreover, if \( g' \) has a zero \( c \), then \( h(z) = c \) has infinitely many roots. One can derive a contradiction by arguing similarly as in Case 1. Hence \( g' \) has no zeros, i.e.,

\[
g'(w) = \frac{1}{(w - w_1)^n} e^{\alpha(w)},
\]

and

\[
g'(h(z))h'(z) = \beta'(z) \exp(\alpha(e^{\beta(z)} + w_1) + (1 - n)\beta(z)).
\]

By the same argument as that in Case 2 above, we can get a contradiction. Thus \( F(z) = f(z) + P(z) \) is pseudo-prime. Now we assume that \( F(z) \) has the following factorization:

\[
F(z) = f(z) + P(z) = Q(g(z)),
\]

where \( Q \) is rational, \( g \) is a transcendental meromorphic function. If \( Q \) is a polynomial, then \( g \) is entire. If \( Q \) has a pole \( w_1 \), then \( g(z) \) does not assume \( w_1 \). Thus \( h(z) = 1/(g(z) - w_1) \) is an entire function and \( F(z) = Q_1(h(z)) \), where \( Q_1 \) is a rational function. Without loss of generality, we may assume that \( g(z) \) is entire, and \( Q(w) \) has at most one pole. Now we discuss the following two sub-cases.

Subcase 1: \( Q \) has one pole, say \( w_0 \), i.e., \( Q(w) = Q_1(w)/(w - w_0)^n \), where \( Q_1(w) \) is a polynomial with degree \( m \) and \( Q_1(w_0) \neq 0 \). Then \( g(z) = w_0 + e^{h(z)} \), where \( h(z) \) is a nonconstant entire function. Thus we have

\[
f(z) = Q_1(w_0 + e^{h(z)}) e^{-nh(z)} - P(z)
= a_0 e^{-nh(z)} + a_1 e^{-(n-1)h(z)} + \cdots + a_m e^{(m-n)h(z)} - P(z),
\]

where \( a_0, a_1, \ldots, a_m \) are constants and \( a_m \neq 0 \), \( a_0 = Q_1(w_0) \neq 0 \). Thus

\[
f'(z) = (-na_0 e^{-nh(z)} - (n-1)a_1 e^{-(n-1)h(z)} + \cdots
+ (m-n)a_m e^{(m-n)h(z)}) h'(z) - P'(z)
= \left[ -na_0 - (n-1)a_1 e^{h(z)} + \cdots
+ (m-n)a_m e^{nh(z)} \right] e^{-nh(z)} h'(z) - P'(z)
= P_1(e^{h(z)}) e^{-nh(z)} h'(z) - P'(z),
\]

where \( P_1(w) \) is a polynomial and \( P_1(0) = -na_0 \neq 0 \). If \( P_1(w) \) is a nonconstant polynomial, then \( P_1(w) \) has a zero \( c \neq 0 \) and \( e^{h(z)} = c \) has infinitely many roots. Let \( \{z_k\}_{k=1}^{\infty} \) be zeros of \( e^{h(z)} - c \), then \( f'(z_k) = -P'(z_k) \) and

\[
f(z_k) = \frac{Q_1(w_0 + c)}{c^n} - P(z_k).
\]
Again, by Lemma 1, we have a contradiction. If \( P_1(w) \) is a constant polynomial, then

\[
f(z) = a_0 e^{-nh(z)} + a_m - P(z), \quad f'(z) = -na_0 e^{-nh(z)} h'(z) - P'(z).
\]

Let \( K(z) = e^{nh(z)} \) and \( |z'| = r, |K(z')| = M(r, K) \). Then by Lemma 2, we have, for \( r \notin F \),

\[
| - na_0 e^{-nh(z')} h'(z') | = \left| a_0 \frac{1}{K(z')} K'(z') \right| = |a_0| \frac{1}{M(r, K)} \frac{v(r, K)}{r} (1 + o(1)),
\]

\[
|a_0 e^{-nh(z')}| = \frac{|a_0|}{M(r, K)}.
\]

Noting \( \lim_{r \to \infty} (v(r, K)/M(r, K)) = 0 \) for a transcendental entire function \( K \), we can find a sequence of \( \{z_k\}_{k=1}^{+\infty} \) such that \( |f(z_k)| \sim |P(z_k)|, |f'(z_k)| \sim |P'(z_k)| \). A contradiction follows from this and Lemma 1.

**Subcase 2:** \( Q(w) \) has no pole, i.e., \( Q(w) \) is a polynomial with degree \( \geq 2 \). If \( Q'(w) \) has at least two distinct zeros, then there exists a zero \( w_1 \) of \( Q'(w) \) such that \( g(z) = w_1 \) has infinitely many zeros \( \{z_n\}_{n=1}^{+\infty} \). Then

\[
f'(z_n) = Q'(g(z_n)) - P'(z_n) = -P'(z_n), \quad f(z_n) = Q(w_1) + P(z_n).
\]

However, by Lemma 1,

\[
|f'(z_n)| \geq \frac{|f(z_n)| \log |f(z_n)|}{16\pi |z_n|},
\]

which will lead to a contradiction. Therefore, we only need to treat the case that \( Q'(w) \) has only one zero \( w_0 \). If \( g(z) = w_0 \) has infinitely many zeros, again a contradiction follows from Lemma 1. Hence, we have

\[
g(z) = w_0 + p_1(z) e^{h(z)} \quad \text{and} \quad Q'(z) = A(w - w_0)^{n-1},
\]

where \( p_1(z) \) is a polynomial, \( h(z) \) a nonconstant entire function. Thus

\[
Q(w) = \frac{A}{n} (w - w_0)^n + B,
\]

\[
f(z) = \frac{A}{n} p_1(z)^n e^{nh(z)} + B - P(z),
\]

\[
f'(z) = \frac{A}{n} \left( p_1'(z) + p_1(z) nh'(z) \right) e^{nh(z)} - P'(z).
\]
Set $K(z) = e^{-nh(z)}$ and let $|z'| = r$, $K'(z') = M(r, K)$. Then it follows from Lemma 2, for $r \notin F$, that

$$\left| \frac{A}{n} (p_1'(z') + p_1(z') nh'(z')) e^{nh(z')} \right| = \left| \frac{A}{n} \left( \frac{p_1'(z')}{K(z')} - \frac{p_1(z')}{K(z')} \frac{K'(z')}{K(z')} \right) \right| \leq \frac{cr^t}{M(r, K)} + \frac{dr^t \nu(r, K)}{M(r, K)},$$

where $c$, $d$ are positive constants, $t = \deg p_1 - 1$. Noting

$$\lim_{r \to \infty} \frac{r^t \nu(r, K)}{M(r, K)} = 0$$

for a transcendental entire function $K$, there exists a sequence of $\{z_n\}_{n=1}^{+\infty}$ such that

$$f(z_n) \sim -P(z_n), \quad f'(z_n) \sim -P(z_n).$$

Again by Lemma 1, we get a contradiction. Thus we have proved that $F(z) = f(z) + P(z)$ is left-prime. Next we show that $F$ is right-prime. Let

$$F(z) = g(q(z)),$$

where $g$ is a transcendental entire function and $q(z)$ a polynomial with degree $\geq 2$. Thus

$$f(z) = g(q(z)) - P(z)$$

and hence

$$f'(z) = g'(q(z)) q'(z) - P'(z).$$

First, we prove that $g'(w)$ has infinitely many zeros. In fact, if $g'(w)$ has only finitely many zeros, then $g'(w) = s(w)e^{h'(w)}$, where $s(w)$ is a polynomial and $h(w)$ is a nonconstant entire function. Let $K(z) = e^{-h(z)/3}$. There exists a curve $\Gamma$ tending to infinity such that if $z \in \Gamma$, then $|K(z)| = M(|z|, K)$. Noting that $K$ is a transcendental entire function, we have that $M(r, K) \geq r^{2m+2}$ for $r \geq r_0$, where $m = \deg s$. Let $w = G(z) = e^{h(z)/3}$ and $\lambda = G(\Gamma)$. Then $dw = \frac{1}{3} h'(z) e^{h(z)/3}$. If $h(z)$ is nonconstant polynomial, then there exists a positive constant $c$ such that $|h'(z)| \geq c$ for sufficiently large $|z| = r$. If $h(z)$ is transcendental, then $|\frac{1}{3} h'(z)| > 1$ for $z \in \Gamma$ and sufficiently large $|z| = r$, by (7). Hence, we have, for $z \in \Gamma$ and $|z| \geq r_0$,

$$|g'(z)| \leq \frac{1}{M(r, K)^2},$$

$$|g(z)| = \left| \int_{z_0}^{z} g'(z) \, dz + g(z_0) \right| \leq \left| \int_{w_0}^{w} \, dw \right| \leq A,$$
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where $w_0 = G(z_0)$, $w = G(z)$ and $A$ is a positive constant. Let $\gamma$ be a component of $q^{-1}(\Gamma)$, and denote $R = |q(z)|$ for $z \in \gamma$. Then for $z \in \gamma$, we have

$$|g(q(z))| \leq A, \quad |g'(z)q'(z)| \leq \frac{BR^{m+1}}{M(R,K)^2} \to 0, \quad \text{as } z \to \infty,$$

where $A$ and $B$ are constants. Hence, for $z \in \gamma$, we have

$$|f(z)| \sim |P(z)|, \quad |f'(z)| \sim |P'(z)|.$$

Again, by Lemma 1, the above estimates will lead to a contradiction as before. Thus $g'$ has infinitely many zeros. Now let $n = \deg q$ and $m = \deg P$. Next we will prove that $n | m$, i.e., there is a positive integer $r$ such that $m = nr$. Let \( \{w_k\}_{k=1}^{\infty} \) denote the zeros of $g'(w)$ and set

$$q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$ 

We consider the roots of the equation

$$q(z) = w_k,$$

which implies

$$a_n z^n (1 + o(1)) = w_k. \quad (8)$$

On the other hand, the roots of the above equation can be expressed as

$$z_k^{(j)} = \left| \frac{w_k}{a_n} \right|^{1/n} e^{i(2j\pi + \phi_k)/n} \left( 1 + o(1) \right),$$

where

$$\phi_k = \arg \left( \frac{w_k}{a_n} \right), \quad j = 0, 1, 2, \ldots, n - 1.$$

Thus

$$P(z_k^{(0)}) \sim A|w_k|^{m/n},$$

$$P(z_k^{(1)}) \sim e^{2m\pi i/n} A|w_k|^{m/n},$$

$$P'(z_k^{(0)}) \sim B|w_k|^{(m-1)/n},$$

$$P'(z_k^{(1)}) \sim e^{2(m-1)\pi i/n} B|w_k|^{(m-1)/n},$$

where $A$, $B$ are constants depending on $q(z)$ and $P(z)$ only. Thus we have sequences $\{w_k\}_{k=1}^{\infty}$, with $w_k \to \infty$ as $k \to \infty$, $\{z_k^{(0)}\}_{k=1}^{\infty}$ and $\{z_k^{(1)}\}_{k=1}^{\infty}$ such that
\[
q(z_k^{(0)}) = q(z_k^{(1)}) = w_k,
\]
\[
P(z_k^{(0)}) - P(z_k^{(1)}) \sim (1 - e^{2m\pi i/n})A|w_k|^{m/n},
\]
\[
f'(z_k^{(0)}) = -P'(z_k^{(0)}) \sim -B|w_k|^{(m-1)/n},
\]
\[
f'(z_k^{(1)}) = -P'(z_k^{(1)}) \sim -e^{2(m-1)\pi i/n}B|w_k|^{(m-1)/n},
\]
\[
f(z_k^{(0)}) = g(w_k) - P(z_k^{(0)}),
\]
\[
f(z_k^{(1)}) = g(w_k) - P(z_k^{(1)}),
\]
\[
f(z_k^{(1)}) - f(z_k^{(0)}) = P(z_k^{(0)}) - P(z_k^{(1)}).
\]

If \(n \nmid m\), then \(1 - e^{2m\pi i/n} \neq 0\). Now we discuss two subcases.

**Subcase 1:** \(\{f(z_k^{(0)})\}_{k=1}^{\infty}\) is bounded. We have, by (10)--(15),

\[
|f(z_k^{(1)})| \sim |(1 - e^{2m\pi i/n})A| |w_k|^{m/n}.
\]

By this and Lemma 1, we obtain that

\[
|B| |w_k|^{(m-1)/n} \sim |f'(z_k^{(1)})| \geq \frac{|f(z_k^{(1)})| \log |f(z_k^{(1)})|}{16\pi |z_k^{(1)}|}
\]

\[
\sim C|w_k|^{(m-1)/n} \log(|1 - e^{2m\pi i/n})A| |w_k|^{m/n}),
\]

where

\[
C = \frac{|(1 - e^{2m\pi i/n})A| |a_n|^{1/n}}{16\pi},
\]

which is a contradiction.

**Subcase 2:** \(\{f(z_k^{(0)})\}_{k=1}^{\infty}\) is unbounded. Then there exists a sub-sequence of \(\{f(z_k^{(0)})\}_{k=1}^{\infty}\) tending to infinity, which we may, without confusing, denote by the original sequence: \(\{f(z_k^{(0)})\}_{k=1}^{\infty}\). Thus by Lemma 1, we have

\[
|B| |w_k|^{(m-1)/n} \sim |f'(z_k^{(0)})| \geq \frac{|f(z_k^{(0)})| \log |f(z_k^{(0)})|}{16\pi |z_k^{(0)}|}
\]

\[
\sim \frac{|a_n|^{1/n}|f(z_k^{(0)})| \log |f(z_k^{(0)})|}{16\pi |w_k|^{1/n}}.
\]

Hence,

\[
|f(z_k^{(0)})| = o(|w_k^{(m/n)}|).
\]
Thus

\[ |f(z_k^{(1)})| \sim |(1 - e^{2m\pi i/n})A| |w_k^{m/n}|. \]

By arguing similarly as in Subcase 1, we will arrive at a contradiction. Hence \( n \mid m \). Finally, we will prove that \( g(z) \) is a common right factor of \( f(z) \) and \( P(z) \). If \( g(z) \) is not a right factor of \( P(z) \), then there exist polynomials \( Q \) and \( P_1 \) with \( 0 < \deg P_1 < n = \deg q \) such that

\[ P(z) = Q(q(z)) + P_1(z). \]

Thus

\[ G(z) = f(z) + P_1(z) = g(q(z)) - Q(q(z)) = g_1(q(z)), \]

where \( g_1(w) = g(w) - Q(w) \) is a transcendental entire function. By arguing similarly as in the subcase above, it follows that \( n \mid \deg P_1 \), which is a contradiction. Thus, \( P(z) = Q(q(z)) \) and \( f(z) = g(q(z)) - Q(q(z)) \). The conclusion follows.

4. Concluding remarks

**Corollary.** Let \( f \) be a transcendental entire function in \( B \), then for any constant \( a \neq 0 \), \( f(z) + az \) is prime.

**Remark 4.** This corollary shows that if \( f(z) - az \in B \) for some constant \( a \), then \( |NP(f)| \leq 1 \).

**Remark 5.** If \( h \) is a periodic entire function of order one and mean type, then \( h \in B \). Thus if \( G(z) \) is as stated in Theorem D, then \( G^n \in B \).

**Remark 6.** The condition \( f \in B \) in the above theorem and corollary is not removable. For example, \( f(z) = e^z e^{e^z} + e^z \), then \( f(z) = (we^w + w) \circ e^z \), and \( f(z) + z = (e^w + w) \circ (e^z + z) \). This example shows the cardinality of \( NP(f) \) may be greater than one if \( f \notin B \).

**Remark 7.** If \( f \) is an entire function such that \( \text{sing}(f^{-1}) \subset \mathbb{R} \), then, by Lemma 3, \( \sin(f(z)) \in B \) and \( \cos(f(z)) \in B \). Thus, for any constant \( a \neq 0 \), \( \sin(f(z)) + az \) and \( \cos(f(z)) + az \) are prime. It was mentioned in [2] that the Pólya–Laguerre class \( LP \) consists of all entire functions \( f \) which have a representation

\[ f(z) = \exp(-az^2 + bz + c)z^n \prod \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{z}{z_k}\right), \]

where \( a, b, c \in \mathbb{R} \), \( a \geq 0 \), \( n \in \mathbb{N}_0 \), \( z_k \in \mathbb{R} \setminus \{0\} \) for all \( k \in \mathbb{N} \), and \( \sum_{k=1}^{\infty} |z_k|^{-2} < \infty \). Furthermore, if \( f_1, f_2, \ldots, f_n \in LP \), and \( f = f_1 \circ f_2 \circ \cdots \circ f_n \), then \( \text{sing}(f^{-1}) \subset \mathbb{R} \). Thus, for example, \( \sin(f(z)) + az \) is prime for \( a \neq 0 \), when \( f \in LP \).
References


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