HALF-TWISTS AND EQUATIONS IN GENUS 2

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Abstract. The uniformization problem is to find equations for the algebraic curve associated to a given hyperbolic surface. If one can describe corresponding group actions both on the spaces of algebraic curves and hyperbolic surfaces, the whole orbits can be uniformized at the same time. We study here the action of a group generated by half-twists on the space of hyperbolic surfaces of genus 2 with a non-trivial involution and describe the corresponding action on the equations for the corresponding algebraic curves.

1. Introduction

The uniformization theorem of Poincaré and Koebe allows to assert that any compact connected Riemann surface of genus $g > 1$ is conformally equivalent to a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group, i.e. a discrete subgroup of $\text{PSL}_2(\mathbb{R})$. On the other hand, a Riemann surface is also an algebraic curve defined by an equation. The classical uniformization problem is to relate explicitly the two descriptions.

In this context, a new approach to tackle this question was initiated in [6], and developed in [2], [8] and [1].

It consists first in working inside families of surfaces with the idea that surfaces tiled with the same pattern by the same type of polygon must have equations of the same form. Then, in defining groups actions on those families. The two groups, one acting on the hyperbolic surfaces and the other on the algebraic curves, are not necessarily the same but there exists a correspondence between their actions. With this approach, whole orbits can be uniformized at the same time.

In this article, we generalize the $D_5$-action on the space of real genus 2 $M$-curves with a real involution $\mathcal{M}^{(2,3,0)}_\mathbb{R}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ described by Buser and Silhol in [6] to the complex family $F_2$ of those Riemann surfaces having a non-trivial involution.

There are, nevertheless, two major differences in the way we tackle the problem in this paper. The first difference is in the method: we use a quotient, namely the Riemann sphere ramified over 5 points, while the authors considered coverings to define the $D_5$-action in [6]. Secondly, the complex situation is less rigid than the
real one, in particular in [6] the hyperbolic description was guided by the natural
choice of a pants decomposition and Fenchel–Nielsen coordinates given by the real
structures. Here, we have to work with marked Riemann surfaces in the Teich-
müller space for the hyperbolic description of our group action while the algebraic
one is made with unmarked Riemann surfaces in the moduli space. In particular,
the two groups are different. The first one, \( G_Q \), is a group of transformations of a
special type of hyperbolic quadrilateral and can be identified with the Teichmüller
modular group of the sphere with 5 points removed. The second is the symmetric
group \( S_5 \), giving the 5 points on the quotient a symmetric role, and naturally
appears as a quotient of \( G_Q \). These two actions correspond and we give here this
correspondence in terms of equations and of generators for the Fuchsian groups
(see Theorem 4.6 and Table 3).

The fact that the two groups are different means that \( G_Q \) intersects the
Teichmüller modular group of genus two surfaces and thus allows to interpret this
difference in terms of Dehn twists.

But more interestingly and surprisingly, the action of the whole group \( G_Q \)
can be interpreted in terms of half-twists (see Theorem 5.1). Thus, by merging
Theorems 4.6 and 5.1, we obtain the main result of this paper which can be
expressed as the following.

**Theorem.** Let \( S \) be a genus 2 Riemann surface having a non-trivial involution \( \varphi \).

Then, on the one hand, the Fuchsian group has a set of generators of the form

\[
(e_1e_3)^2, e_3e_2e_1, e_1e_3e_2, e_3e_4e_1, e_1e_3e_4,
\]

where \( e_i \in \text{PSL}_2(\mathbb{R}), i = 1, \ldots, 4 \), with \( \text{tr}(e_i) = 0 \), and \( \text{tr}(\prod_{i=1}^4 e_i) = 0 \), and on
the other hand the underlying algebraic curve has a normalized equation of the form

\[
y^2 = (x^2 - 1)(x^2 - a)(x^2 - b).
\]

For such a surface \( S \), the half-twists along certain geodesics lead to surfaces
having a non-trivial involution. Sets of generators for the Fuchsian groups and
normalized equations for these surfaces can be explicitly deduced from those of \( S \).

We have, in particular, the correspondence between half-twists and changes
of equation given in Table 1.

The action of \( G_Q = \langle \eta_1, \ldots, \eta_4 \rangle \) for \( \eta_i, i = 1, \ldots, 4 \) as in Table 1, induces an
action of \( S_5 \) on the underlying algebraic curves.
2. Notation and preliminaries

We recall briefly some classical definitions and notation.

**Definition 2.1.** Let $S$ be a Riemann surface of genus 2, and $\tau$ be the hyperelliptic involution on $S$.

An automorphism $\varphi \in \text{Aut}(S)$, with $\varphi \neq \tau$ is said to be *non-trivial*.

For any hyperelliptic Riemann surface $S$, the hyperelliptic involution $\tau$ is in the center of $\text{Aut}(S)$. The *reduced automorphism group* of $S$ is then

$$\text{Aut}^r(S) = \text{Aut}(S)/\langle \tau \rangle.$$ 

The classification of Riemann surfaces of genus two in terms of their reduced automorphism group is due to Bolza ([4]). It is summarized Table 2 as well as the inclusions between families.

Except for $F_5$, every Riemann surface of genus two having a non-trivial automorphism has at least one non-trivial involution, and then belongs to $F_2$.

Let $S$ be a Riemann surface, $\tau$ the hyperelliptic involution and $\varphi$ a non-trivial involution on $S$. The involutions $\varphi$ and $\varphi \tau$ have two fixed points, say $p_1$ and $p_2$ for $\varphi$ and $q_1$ and $q_2$ for $\varphi \tau$, that satisfy

$$\varphi(q_1) = q_2, \quad \tau(p_1) = p_2.$$ 

**Lemma 2.2.** Let $S$, $\tau$, $\varphi$ as before. The covering $p_\varphi: S \to S/\langle \varphi, \tau \rangle \simeq \mathbb{P}^1(\mathbb{C})$ is ramified over 5 points among which 3 are the images of the Weierstrass points and the last two are $p = p_\varphi(p_1) = p_\varphi(p_2)$ and $q = p_\varphi(q_1) = p_\varphi(q_2)$. Those five points and the triple of them that lift to the set of Weierstrass points determine $S$ completely.

**Proof.** This follows directly from the fact that the surfaces of genus 2, are, as all the hyperelliptic algebraic curves, determined by their Weierstrass points. □
<table>
<thead>
<tr>
<th>Family</th>
<th>Aut</th>
<th>Classical form for the equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_2$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$y^2 = (x^2 - 1)(x^2 - a)(x^2 - b)$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$D_2$</td>
<td>$y^2 = (x^2 - 1)(x^2 - a)(x^2 - 1/a)$</td>
</tr>
<tr>
<td>$F_6$</td>
<td>$D_3$</td>
<td>$y^2 = x^6 - 2ax^3 + 1$</td>
</tr>
<tr>
<td>$F_{12}$</td>
<td>$D_6$</td>
<td>$y^2 = x^6 + 1$</td>
</tr>
<tr>
<td>$F_{24}$</td>
<td>$S_4$</td>
<td>$y^2 = x^4 - 1$</td>
</tr>
<tr>
<td>$F_5$</td>
<td>$\mathbb{Z}/5\mathbb{Z}$</td>
<td>$y^2 = x^5 - 1$</td>
</tr>
</tbody>
</table>

Corollary 2.3. Let $r_1, \ldots, r_5$ be five distinct points on $\mathbb{P}^1(\mathbb{C})$. There exist at most 10 different surfaces $S_j$, $\tau_j$, $\varphi_j$, $j = 1, \ldots, 10$, in the family $F_2$ such that the coverings $p_{\varphi_j}$ are ramified over the $r_i$'s.

Proof. Each $S_j$ correspond to the choice of a triple $\{r_k, r_l, r_m\}$ of points among the $r_i$'s that lift to the Weierstrass points of $S_j$. $\square$

3. Marked quadrilaterals

Let $S$, $\tau$, $\varphi$ as before. The hyperbolic structure on $S$ induces, via $p_\varphi$, a structure of hyperbolic sphere with five cone points of angle $\pi$ on the quotient $S/\langle \tau, \varphi \rangle$.

Such a surface can always be obtained by pasting the sides of a hyperbolic quadrilateral with interior angles adding up to $\pi$, as on Figure 1.

Figure 1.

This observation induces a particular presentation for the Fuchsian group of such a surface and motivates the following definitions.
Definition 3.1. An ordered system \( Q = (e_1, e_2, e_3, e_4), e_i \in \text{PSL}_2(\mathbb{R}) \), is a marked quadrilateral if it satisfies the following conditions:

(i) \( \text{tr}(e_i) = 0, \ i = 1, \ldots, 4. \)
(ii) \( \text{tr}(\prod_{i=1}^{4} e_i) = 0. \)
(iii) The \( e_i \)'s are positioned clockwise around the quadrilateral of vertices are the fixed points of \( e_1 e_2 e_3 e_4, e_2 e_3 e_4 e_1, e_3 e_4 e_1 e_2 \) and \( e_4 e_1 e_2 e_3 \) (see Figure 2).

Remark 3.2. Using trace relations, one can easily show that the quadrilateral of (iii) above is a convex domain, delimited by the axes of the hyperbolic transformations \( e_1 e_2 e_3, e_2 e_3 e_4, e_3 e_4 e_1, e_4 e_1 e_2 \). As it is uniquely determined we will also denote it by \( Q \).

Definition 3.3. We will denote by \( Q \) be the set of all marked quadrilaterals modulo the relation

\[
(e_1, e_2, e_3, e_4) \sim (e'_1, e'_2, e'_3, e'_4) \iff \exists \gamma \in \text{PSL}_2(\mathbb{R}), e'_i = \gamma e_i \gamma^{-1}, i = 1, \ldots, 4.
\]

Definition 3.4. Given a surface \( S_0 \) of signature \( (0; 2, 2, 2, 2) \), a quadrilateral fundamental domain for \( S_0 \) is a marked quadrilateral \( Q = (e_1, e_2, e_3, e_4) \) such that \( \Gamma_0(Q) = \langle e_1, e_2, e_3, e_4 \rangle \) is a Fuchsian group for \( S_0 \).

We will denote by \( Q_{S_0} \) the set of all quadrilateral fundamental domains for \( S_0 \) under the relation \( \sim \).

Conversely, given \( Q \in Q \), will denote by \( S_0(Q) \) the surface \( H/\Gamma_0(Q) = H/\langle e_1, e_2, e_3, e_4 \rangle \).

Remarks 3.5. 1. As the \( e_i \)'s are elliptic transformations of order 2, they completely determine their fixed points. So we may also denote by \( e_i \) the fixed point of \( e_i \) (notably on figures).

2. As a marked quadrilateral is defined up to direct isometry and separates into two triangles, it is also characterized by the following set of five lengths:
   - the lengths \( l_i \) of the \( i \)-th sides given by:
     \[
     \cosh\left(\frac{1}{2}l_i\right) = \frac{1}{2} |\text{tr}(e_{i+1} e_i e_{i+2} e_{i+3})|,
     \]
   - the length \( l \) of the first diagonal given by
     \[
     \cosh\left(\frac{1}{2}l\right) = \frac{1}{2} |\text{tr}(e_1 e_2 e_3 e_4 e_1 e_2)| = \frac{1}{2} |(\text{tr}(e_1 e_2))^2 + (\text{tr}(e_3 e_4))^2 - 2|.
     \]
3.1. Transformations of a marked quadrilateral

Definition 3.6. (1) We define the following transformations on Q:
(i) the circular permutation:

\[ \sigma_0: (e_1, e_2, e_3, e_4) \rightarrow (e_2, e_3, e_4, e_1), \]

(ii) \( \sigma_1 \) (see Figure 2):

\[ \sigma_1(e_1, e_2, e_3, e_4) \rightarrow (e_3, e_2, e_3 e_4 e_1 e_2, e_3 e_4 e_3). \]

(2) We denote by \( G_Q \), the group

\[ G_Q = \langle \sigma_0, \sigma_1 \rangle \]

Figure 2. The transformation \( \sigma_1 \).

Remarks 3.7. Let \( S_0 \) be a hyperbolic surface of genus 0 with five cone points of angle \( \pi \). Then \( \sigma_0 \) and \( \sigma_1 \) preserve \( Q_{S_0} \).

The transformations \( \sigma_0 \) and \( \sigma_1 \) are of different nature. While \( \sigma_0 \) only operates on the marking of \( Q \) but leaves the unmarked quadrilateral unchanged, \( \sigma_1 \) is mainly devoted to changing the choice of the point among the five cone points on the sphere \( S_0 \) which correspond to the vertices of \( Q \).

For further use, we introduce the following transformations in \( G_Q \):

Note that \( \sigma_2 \) and \( \sigma_3 \) are of infinite order and that they do not have fixed points in \( G_Q \).

Proposition 3.8. Let \( S_0 \) be a hyperbolic surface of signature \((0; 2, 2, 2, 2)\). \( G_Q \) acts transitively on \( Q_{S_0} \).
Proof. Let $Q$ and $Q'$ be two quadrilaterals of $Q_{S_0}$. We build a sequence of quadrilaterals of $Q_{S_0}$ using elements of $G_Q$.

Let $a_1, \ldots, a_4$ be the four geodesic arcs in $S_0$ corresponding to the sides of $Q$. The $a_i$’s are oriented such that they all have the same source.

Start, if necessary, with a transformation of the form $\sigma_1 \sigma_0^k$ leading to a quadrilateral $Q_1$, such that the vertices of $Q$ and $Q_1$ correspond to the same point of $S_0$.

For each $a_i$, consider the number $k_{i,Q_1}$ of connected components of $a_i \cap Q_1$. Let $a_{1,1}, \ldots, a_{1,k_{1,Q_1}}$ be the corresponding connected component.

We cut $S_0$ along the sides of $Q_1$.

We treat the $a_i$’s in the order given by the marking.

For $a_1$:
If $k_{1,Q_1} = 0$, then $a_1$ corresponds to one of the sides of $Q_1$, and we go to $a_2$.
If $k_{1,Q_1} \neq 0$, we give the arcs $a_{1,1}, \ldots, a_{1,k_{1,Q_1}}$ the $a_1$’s orientation.

Using a transformation of the form $\sigma_0^k$, we get a quadrilateral $Q_2$ such that the source of $a_{1,1}$ in $Q_2$ is at the intersection of the first and the fourth sides. Its end point is then necessarily on the second or the third side of $Q_2$. Using $\sigma_2$ in the first case and $\sigma_3$ in the second one, we build a quadrilateral $Q_3$ such that the number $k_{1,Q_3}$ of connected components of $a_1 \cap Q_3$ is strictly smaller than $k_{1,Q_2} = k_{1,Q_1}$. Assume that one of the $a_{1,k}$’s, say $a_{1,k_0}$ penetrates into the triangle of sides $a, b, c$, where $a$ is a part of $a_{1,1}$, $b$ is a part of the side of $Q_3$ which is not a side of $Q_3$, and $c$ is a part of the side of $Q_2$ which is not a side of $Q_3$. Then, the arc $a_{1,k_0}$ must leave this triangle through $b$ or $c$, the $a_{1,k}$’s being disjoint. Thus, the arc is at the same time cut and pasted once. Therefore the number of
connected components does not increase. As this situation is possible only in the case \( k_0 > 1 \), we have \( k_{1, Q_3} < k_{1, Q_2} \).

We treat similarly the path \( a_2, a_3, a_4 \), each construction respecting the preceding ones. □

Lemma 3.9. Given a generic surface \( S_0 \) of signature \((0; 2, 2, 2, 2, 2)\), \( G_Q \) operates without fixed points on \( Q_{S_0} \).

Proof. Let \( Q \in Q_{S_0} \) and \( \sigma \in G_Q \) such that \( \sigma \cdot Q = Q \). Then \( \sigma \) induces an isometry on \( S_0 \). □

3.2. Identification of \( G_Q \) with the modular group \( \Gamma_{0,5} \). The group \( G_Q \) acts on sets of four generators of Fuchsian groups of a sphere with five punctures. It is then naturally linked with the modular group of a sphere with five points removed, \( \Gamma_{0,5} \). More precisely, given a transformation \( \sigma \in G_Q \), one can associate the isotopy class of \( h \) to \( \varphi \), where \( h \) is a homeomorphism of the sphere preserving the set of five points and corresponding to the deformation from any quadrilateral \( Q \) to \( \sigma(Q) \) (the quadrilateral being simply connected) mapping the interior onto the interior and the \( i \)-th side onto the \( i \)-th side.

Define the following transformations of \( G_Q \):

\[
\eta_1: (e_1, e_2, e_3, e_4) \mapsto (e_1, e_3 e_2 e_3, e_4), \\
\eta_2: (e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, e_4 e_3 e_4), \\
\eta_3: (e_1, e_2, e_3, e_4) \mapsto (e_1 e_4 e_1, e_2, e_3, e_1), \\
\eta_4: (e_1, e_2, e_3, e_4) \mapsto (e_2 e_3 e_4 e_1, e_2, e_3, e_4).
\]

Then, with the topological interpretation below, if \( r_1, \ldots, r_4 \) are the points of \( S_0(Q) \) corresponding to the middle of the sides and \( r_5 \) is the point corresponding to the vertices, each \( \eta_i \) corresponds to an homeomorphism \( \varphi_i \) where \( \varphi_i \) is the identity outside a disk \( D_i \) enclosing \( r_{i+1} \) and \( r_{i+2} \) (subscript modulo 5) and \( \varphi_i \) exchanges \( r_{i+1} \) and \( r_{i+2} \). According to J.S. Birman (see [3, Theorem 4.5, p. 164 and Remark, p. 165]), this means that the set of geometric transformations \( \{\eta_1, \eta_2, \eta_3, \eta_4\} \) is a set of generators for \( \Gamma_{0,5} \) with the following full list of relations:

\[
\eta_i \eta_j = \eta_j \eta_i, \quad |i - j| \geq 2, \\
\eta_i \eta_{i+1} \eta_i = \eta_{i+1} \eta_i \eta_{i+1}, \\
\eta_1 \eta_2 \eta_3 \eta_2^2 \eta_3 \eta_2 \eta_1 = 1, \\
(\eta_1 \eta_2 \eta_3 \eta_4)^5 = 1.
\]

We remark without expanding the computations that the following correspondence between the \( \sigma_i \)'s and the \( \eta_i \)'s allows one to find a full list of relations for
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\[ \Gamma_{0,5} \] with the minimal set of generators \( \{ \sigma_0, \sigma_1 \} \):

\[ \eta_1 = \sigma_0 (\sigma_1^2 \sigma_0^3)^3 \sigma_0^3, \]
\[ \eta_2 = (\sigma_1^2 \sigma_0^3)^3, \quad \sigma_0 = \eta_4 \eta_3 \eta_2 \eta_1, \]
\[ \eta_3 = \sigma_0^3 (\sigma_1^2 \sigma_0^3)^3 \sigma_0, \quad \sigma_1 = \eta_2^{-1} \eta_3^{-1} \eta_4^{-2} \eta_2^{-1} \eta_4, \]
\[ \eta_4 = \sigma_0^2 \sigma_3 \sigma_2 \sigma_0^3 \sigma_1. \]

4. The genus two coverings

4.1. The surface \( S_Q \). Given \( Q \in \mathbb{Q} \), we construct a genus 2 cover \( S_Q \) of the sphere \( S_0(Q) \) as shown in Figure 3. In other words, we construct a set of generators for a Fuchsian group \( \Gamma_Q \) of signature \( (2,0) \), namely \( \Gamma_Q = \langle (e_1 e_3)^2, e_3 e_2 e_1, e_1 e_3 e_2, e_3 e_4 e_1, e_1 e_3 e_4 \rangle \), from the generators \( (e_1, e_2, e_3, e_4) \) of the Fuchsian group \( \Gamma_0(Q) \) of \( S_0(Q) \). Note that the Weierstrass points of the surface \( S_Q \) correspond to the conjugacy classes in \( \Gamma_Q \) of the centers of the elliptic transformations

\[ e_2, e_1 e_4 e_1, e_4, e_1 e_2 e_1, e_1 e_2 e_3 e_4, e_2 e_3 e_4 e_1 \]

(and to the middles of the sides labelled 2, 3, 4, 5, 7, 8, 9, 10 and the vertices of the polygon on Figure 3).

![Figure 3](image_url)

Note also that \( S_Q \) has two non-trivial involutions, the fixed points of the first being the conjugacy classes of \( e_1 \), and \( e_3 e_1 e_3 \), those of the second being \( e_3 \) and \( e_1 e_3 e_1 \).

By analogy with Lemma 2.2, we have:
Proposition 4.1. Let $S$ be a surface of $F_2$ with a non-trivial involution $\varphi$ and $S_0 = S/\langle \varphi, \tau \rangle$. Choose $Q \in Q_{S_0}$ such that the images $p$ and $q$ of the fixed points of $\varphi$ and $\varphi \tau$ via $p_\varphi$ are on the first and the third side of $Q$.

Then $S$ is isometric to $S_Q$.

Proposition 4.2. The map $Q \ni Q \mapsto S_Q$ is an injection of $Q$ into the Teichmüller space of Riemann surfaces of genus 2, $\mathcal{T}_2$.

Proof. We choose $Q_0 = (e_0^0, e_2^0, e_3^0, e_4^0) \in Q$ as a quadrilateral of reference. As a quadrilateral is simply connected, for any $Q \in Q$, there exists, up to isotopy, a unique homeomorphism $\eta_Q: Q_0 \rightarrow Q$ such that $\eta_Q \circ e_i^0 = e_i$.

This condition ensures that $\eta_Q$ is extendable to a homeomorphism $\tilde{\eta}_Q$ such that

$$
\begin{array}{c}
Q_0 \xrightarrow{\eta_Q} Q_0 \\
S_{Q_0} \xrightarrow{\tilde{\eta}_Q} S_Q
\end{array}
$$

is commutative.

The couple $(S_Q, \tilde{\eta}_Q)$ is then a marked Riemann surface.

The injectivity follows from the construction of $\tilde{\eta}_Q$. \qed

We will denote

$$F_2(Q) = \{S_Q, Q \in Q\} \subset \mathcal{T}_2.$$

Corollary 4.3. The induced action of $G_Q$ on $F_2(Q)$ is generically fixed point free.

Remark 4.4. While $F_2$ is a subspace of the space of isometry classes of Riemann surfaces of genus 2, $M_2$, $F_2(Q)$ is a subspace of the Teichmüller space of genus 2, $\mathcal{T}_2$. It is well known that the moduli space $M_2$ is a quotient of $\mathcal{T}_2$ by the modular group, generated by Dehn twists. We will be concerned by this point of view in Section 5.

4.2. Equations for surfaces in $F_2(Q)$—induced action on $F_2$. The classical normalization of the equations for the surfaces of $F_2$ under the form $y^2 = P(x)$ is the one given in Table 2. More precisely, given a surface $S \in F_2$, we choose the involutions $\varphi$ and $\varphi \tau$ so that they lift $x \mapsto -x$. We also impose that one of the Weierstrass points has coordinates $(1, 0)$.

This choice is equivalent to the choice of a (global) coordinate $x$ on the quotient $S/\langle \varphi, \tau \rangle$ such that

1. the images of the fixed points of $\varphi$ and $\varphi \tau$ via $p_\varphi$ are mapped onto 0 and $\infty$,
2. the image of a pair of the Weierstrass points exchanged by $\varphi$ and $\varphi \tau$ is mapped via $p_\varphi$ onto 1.
Given such an $x$, the remaining two points of $S/\langle \varphi, \tau \rangle$ are mapped upon $a$ and $b$ and

$$y^2 = (x^2 - a)(x^2 - 1)(x^2 - b)$$

is an equation of $S$.

Note that conditions (1) and (2) do not determine precisely the choice of the coordinate $x$, since $x/a$, $x/b$, $1/x$, $a/x$, and $b/x$ would also fulfill them.

As we want to describe the action of $G_Q$ on $F_2(Q)$ in terms of equations, we will, given a surface $S_Q$, make the choice of $x$ precise by taking into account the geometry of $Q$ as follows.

Let $Q \in Q$ and let $S_0 = S_0(Q)$ the genus 0 surface obtained by gluing the sides of $Q$. We then choose a coordinate $x_Q$ depending on the position of the cone points of $S_0$ on $Q$. We denote by $r_{1,Q}, \ldots, r_{5,Q}$ these points in the order given by the marking of $Q$ (in other words, if $Q = (e_1, e_2, e_3, e_4)$, then $r_{i,Q}$, $1 \leq i \leq 4$ is the conjugacy class of $e_i$ in $\Gamma_0(Q) = \langle e_1, e_2, e_3, e_4 \rangle$, and $r_{5,Q}$ is the conjugacy class of $e_1 e_2 e_3 e_4$).

More precisely, we choose $x_Q$ such that:

$$x_Q(r_{1,Q}) = 0, \quad x_Q(r_{3,Q}) = \infty, \quad x_Q(r_{5,Q}) = 1.$$  

We then call the couple

$$(a, b) = (x_Q(r_{2,Q}), x_Q(r_{4,Q}))$$

*normalized equation parameters for* $S_Q$.

Let now $Q \in Q$, and $\sigma \in G_Q$. As $S_0(Q)$ and $S_0(\sigma(Q))$ are isometric, we have

$$\{r_{1,Q}, \ldots, r_{5,Q}\} = \{r_{1,\sigma(Q)}, \ldots, r_{5,\sigma(Q)}\}.$$
This means that $\sigma$ acts as a permutation on the set of the cone points of $S_0$. More precisely, we associate to $\sigma$ the permutation $\bar{\sigma} \in S_5$ defined for all $Q \in Q$ and $i \in \{1, \ldots, 5\}$ by

$$r_{i,\sigma(Q)} = r_{\bar{\sigma}^{-1}(i),Q}.$$

The map $\sigma \mapsto \bar{\sigma}$ is a group homeomorphism as we have for all $Q \in Q$ and $i \in \{1, \ldots, 5\}$

$$r_{i,\sigma'(Q)} = r_{\bar{\sigma}'^{-1}(i),\sigma(Q)} = r_{\bar{\sigma}^{-1}\bar{\sigma}^{-1}(i),\sigma(Q)} = r_{\bar{\sigma}[\sigma^{-1}](i),\sigma(Q)}.$$

The image is the subgroup of $S_5$ generated by the images of the generators $\sigma_0$ and $\sigma_1$, for which we have

$$\sigma_0(r_1, Q, r_2, Q, r_3, Q, r_4, Q, r_5, Q) = (r_2, Q, r_3, Q, r_4, Q, r_1, Q, r_5, Q);$$

thus $\bar{\sigma}_0 = (4, 3, 2, 1)$, 

$$\sigma_1(r_1, Q, r_2, Q, r_3, Q, r_4, Q, r_5, Q) = (r_3, Q, r_2, Q, r_5, Q, r_4, Q, r_1, Q);$$

thus $\bar{\sigma}_1 = (1, 5, 3)$.

The map is surjective, since $(4, 3, 2, 1)$ and $(1, 5, 3)$ together generate $S_5$. Its kernel is the subgroup $H_Q$ of those transformations such that for all $Q \in Q$ and $i \in \{1, \ldots, 5\}$

$$r_{i, Q} = r_{i, \sigma(Q)}.$$

The normalized equation parameters for $S_Q$ depend only on the position of the cone points on $Q$. Then, for $\sigma \in G_Q$, it is $\bar{\sigma} \in S_5$ rather than $\sigma$ that acts on them. This action is as follows: the coordinates $x_Q$ and $x_{\sigma(Q)}$ on $S_0(Q) = S_0(\sigma(Q))$ are exchanged by the unique transformation $A_{\sigma, Q}$ of $\mathbb{P}^1$ mapping $x_Q(r_{1, \sigma(Q)})$ onto 0, $x_Q(r_{3, \sigma(Q)})$ onto $\infty$ and $x_Q(r_{1, \sigma(Q)})$ onto 1, i.e., $A_{\sigma, Q}$ is defined by:

$$A_{\sigma, Q}(z) = \begin{pmatrix} z - x_Q(r_{1, \sigma(Q)}) & x_Q(r_{5, \sigma(Q)}) - x_Q(r_{3, \sigma(Q)}) \\ z - x_Q(r_{3, \sigma(Q)}) & x_Q(r_{5, \sigma(Q)}) - x_Q(r_{1, \sigma(Q)}) \end{pmatrix}$$

and we have

$$\bar{\sigma}.(x_Q(r_2, Q), x_Q(r_4, Q)) = (x_Q(Q)(r_2, \sigma(Q)), x_Q(Q)(r_4, \sigma(Q)))$$

$$=(A_{\sigma, Q}(x_Q(r_2, \sigma(Q))), A_{\sigma, Q}(x_Q(r_4, \sigma(Q))))$$

$$=(A_{\sigma, Q}(x_Q(r_{\sigma^{-1}(2), Q})), A_{\sigma, Q}(x_Q(r_{\sigma^{-1}(4), Q}))).$$

For the generators $\sigma_0$ and $\sigma_1$ of $G_Q$, and a couple $(a, b)$ of normalized equation parameters, we have

$$\bar{\sigma}_0 = (4, 3, 2, 1), \quad \bar{\sigma}_1 = (1, 5, 3),$$

$$A_{\sigma_0}(z) = \frac{z - a}{z - b}, \quad A_{\sigma_1}(z) = \frac{1}{1 - z},$$

$$\bar{\sigma}_0.(a, b) = \left( \frac{1 - b}{1 - a}, \frac{a(1 - b)}{b(1 - a)} \right), \quad \bar{\sigma}_1.(a, b) = \left( \frac{1}{1 - a}, \frac{1}{1 - b} \right).$$

We have thus proved:
Theorem 4.6. The action of $G_Q$ on $F_2(Q)$ ($\subset \mathcal{S}_2$) induces an action of the symmetric group $S_5$ on $F_2$ given in terms of parameters of equations by the generators

$$\bar{\sigma}_0: (a, b) \mapsto \left( \frac{1 - b}{1 - a}, \frac{a(1 - b)}{b(1 - a)} \right), \quad \bar{\sigma}_1: (a, b) \mapsto \left( \frac{1}{1 - a}, \frac{1 - b}{1 - b} \right).$$

According to Corollary 2.3, given a generic surface of $F_2$, its quotient under its automorphism group has ten different genus two covers in $F_2$. A representative of each of these isomorphy classes is given in Table 3.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\bar{\sigma}$</th>
<th>$\sigma.(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Id</td>
<td>Id</td>
<td>$\sigma_1$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>(4, 3, 2, 1)</td>
<td>$\sigma_0 \sigma_1 \sigma_0$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>(1, 5, 3)</td>
<td>$\sigma_1 \sigma_0$</td>
</tr>
<tr>
<td>$\sigma_1^2$</td>
<td>(2, 3, 5)</td>
<td>$\sigma_1 \sigma_0^2$</td>
</tr>
<tr>
<td>$\sigma_1 \sigma_0$</td>
<td>(1, 4)</td>
<td>$\sigma_1 \sigma_0^3$</td>
</tr>
<tr>
<td>$(\sigma_1 \sigma_0)^2$</td>
<td>(2, 3)</td>
<td>$(\sigma_1 \sigma_0)^3$</td>
</tr>
<tr>
<td>$\sigma_0 \sigma_1$</td>
<td>(1, 2)</td>
<td>$(\sigma_0 \sigma_1)^3$</td>
</tr>
<tr>
<td>$\sigma_2 = \sigma_0 \sigma_1$</td>
<td>(1, 4, 5)</td>
<td>$(\sigma_0 \sigma_1)^3 \sigma_0^2$</td>
</tr>
<tr>
<td>$\sigma_3 = (\sigma_0 \sigma_1)^3$</td>
<td>(3, 4)</td>
<td>[(a(b - 1))(a - b)/(a(b - 1))]</td>
</tr>
</tbody>
</table>

Table 3.

Remarks 4.7. (1) While the action of $G_Q$ on $F_2(Q)$ is generically fixed point free, the induced action of $S_5$ on $F_2$ is such that there exists a subgroup of $S_5$, namely $S_{(1, 3)} \times S_{(2, 4, 5)}$, that preserves isometry classes.

(2) Call $K_Q$ the group of transformations $\sigma$ such that $\sigma \in S_{(1, 3)} \times S_{(2, 4, 5)}$. It is neither normal in $G_Q$ nor contains a normal subgroup of $G_Q$. Then $S_5$ is the smallest group whose action on the set of genus two Riemann surfaces ramified over 5 points of $\mathbb{P}^1$ is transitive.

Table 4 gives a representative in $K_Q$ for each element of $S_{1, 3} \times S_{2, 4, 5}$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\bar{\sigma}$</th>
<th>$\sigma.(a, b)$</th>
<th>$\sigma$</th>
<th>$\bar{\sigma}$</th>
<th>$\sigma.(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Id</td>
<td>Id</td>
<td>$(a, b)$</td>
<td>$\sigma_0 \sigma_1 \sigma_0$</td>
<td>(1, 3)</td>
<td>$(a, b/a)$</td>
</tr>
<tr>
<td>$\sigma_2 \sigma_0 \sigma_2 \sigma_0^{-1} \sigma_2$</td>
<td>(2, 4)</td>
<td>$(b, a)$</td>
<td>$\sigma_2 \sigma_0 \sigma_1 \sigma_0$</td>
<td>(2, 5)</td>
<td>$(a/b, 1/b)$</td>
</tr>
<tr>
<td>$\sigma_0^2 \sigma_0 \sigma_2 \sigma_0^{-1}$</td>
<td>(1, 3)</td>
<td>$(1/b, 1/a)$</td>
<td>$\sigma_2 \sigma_0 \sigma_1 \sigma_0$</td>
<td>(4, 5)</td>
<td>$(1/a, b/a)$</td>
</tr>
<tr>
<td>$\sigma_0 \sigma_1$</td>
<td>(2, 5, 4)</td>
<td>$(b/a, 1/b)$</td>
<td>$\sigma_0 \sigma_2 \sigma_0 \sigma_1 \sigma_0$</td>
<td>(1, 3)</td>
<td>$(a/b, a)$</td>
</tr>
<tr>
<td>$\sigma_2 \sigma_0 \sigma_1 \sigma_0$</td>
<td>(2, 4, 5)</td>
<td>$(b/a, 1/a)$</td>
<td>$\sigma_2 \sigma_0 \sigma_1 \sigma_0^2 \sigma_0$</td>
<td>(1, 3)</td>
<td>$(a/b, a)$</td>
</tr>
</tbody>
</table>

Table 4.
Remark 4.8. In [6] the authors identify the space $\mathcal{P}_2$ of pairs of pants having the lengths of two boundary geodesics equal, with the space $\mathcal{M}_{R}^{(2,3,0)}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ of the real genus 2 curves with 3 real components and whose real automorphisms group contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely, given a pair of pants $P$ of $\mathcal{P}_2$, they consider on the one hand the surface $S_P$ obtained by gluing 2 copies of $P$ with twist 0 on each component, and on the other hand the real algebraic curve whose real components correspond to the boundary components of $P$. The surface $S_P$ clearly belong to $F_2$ as the isometry exchanging the two boundary components of same lengths on $P$ can be extended to an isometry $\varphi$ of $S_P$. The involution $\varphi$ can be normalized in $(x, y) \mapsto (-x, y)$ and the algebraic curve has equation $y^2 = (x^2 - a)(x^2 - 1)(x^2 - b)$, $0 < a < 1 < b < 1$.

The authors then define (see [6, 5.17]) a $D_5$-action on $\mathcal{P}_2$ both in terms of the lengths $(l_1, l_2)$ (where a pair of pants has one boundary component of length $2l_1$, and two of lengths $2l_2$) and of the parameters $(a, b)$ of real equation above. We will show that our Theorem 4.6 is in fact a generalization of this result.

Let $P$ be the pair of pants given by length $(l_1, l_2)$. We denote, as in [6], by $\hat{l}_1$ the length of the common perpendicular to the two boundary components of length $2l_2$, and by $\hat{l}_2$ that of the common perpendicular arcs to the boundary of length $2l_1$ and each of those of length $2l_2$. Those arcs cut $P$ into two copies of a right-angled hyperbolic hexagon given by the lengths $(l_1, \hat{l}_2, l_2, \hat{l}_1, l_2, \hat{l}_2)$ in cyclic order. Each copy of this hexagon can be cut into two isometric mirror pentagons along the common perpendicular to the sides of length $l_1$ and $\hat{l}_1$ (see Figure 5).

The hyperbolic surface $S_P$ is isometric to the surface $S_{Q_P}$ where $Q_P$ is the quadrilateral obtained from $P$ as on Figure 5.

![Figure 5](image-url)

It is easily shown, using, for example, trigonometric formulas in triangles and trace relations (given in [5]), that $Q_P$ is given by the following trace relations:
These are exactly the generators for \( P(4,8) \), one can build from in fact the coordinate equation parameters for surfaces with a half-twist. The next section is devoted to showing that the action obtained by transporting the right-angled hexagon as in Figure 5 and thus the pair of pants \( P \) is then given by the lengths

\[
(l_1, l_2) = (\arccosh(\frac{1}{2} |tr(e_3 e_4)|^2 - 1), \arccosh(|tr(e_3 e_4 e_1)|)).
\]

From the algebraic point of view, the fact that the boundary components of \( P \) correspond to the real components of the algebraic curve

\[
y^2 = (x^2 - 1)(x^2 - a)(x^2 - b), \quad 0 < a < 1 < b
\]

ensures that the coordinate induced on the quotient \( S_P/\langle \varphi, \tau \rangle \) via \((x, y) \mapsto x^2\) is in fact the coordinate \( x_{Q_P} \) defined by (4.5). In particular \((a, b)\) are the normalized equation parameters for \( S_{Q_P} \).

Let us now consider the transformations of \( G_Q \):

\[
\begin{align*}
g_1 &: (e_1, e_2, e_3, e_4) \mapsto (e_2 e_3 e_4 e_1, e_2, e_1, e_1 e_3 e_1), \\
g_2 &= \sigma_0^2 : (e_1, e_2, e_3, e_4) \mapsto (e_3, e_4, e_1, e_2).
\end{align*}
\]

Straightforward computations show that \( g_1 \) and \( g_2 \) preserve relations (4.8), that \( g_1 \) is of order 5, and that the subgroup \( \langle g_1, g_2 \rangle \) of \( G_Q \) is isomorphic to the dihedral group \( D_5 \).

Using the above correspondence between this description and that given in [6], and the expression of \( \tilde{g}_1 \) and \( \tilde{g}_2 \) in terms of normalized equation parameters, we get

\[
\begin{align*}
g_1(l_1, l_2) &= (2h_1, \hat{l}_2), \quad \tilde{g}_1(a, b) = \left( \frac{b (1 - a)}{b - a}, \frac{b}{b - a} \right), \\
g_2(l_1, l_2) &= (\hat{l}_1, \hat{l}_2), \quad \tilde{g}_2(a, b) = \left( \frac{1}{b}, \frac{1}{b - a} \right).
\end{align*}
\]

These are exactly the generators for \( D_5 \) as given in [6].

Finally, we note that in [6], each orbit under \( D_5 \) consisted of five a priori different isometry classes (i.e., complex isomorphy classes) while the actions of \( G_Q \) and \( S_5 \) involve 10 different surfaces. The remaining five surfaces were in fact obtained by transporting the \( D_5 \)-action from \( \mathcal{P}_2 \) onto a space of Riemann surfaces with a half-twist. The next section is devoted to showing that the action of \( G_Q \) can in fact be completely interpreted in term of half-twists.
Examples 4.9. We give here some exact examples. The first two surfaces are isometric, and thus complex isomorphic as algebraic curves, to surfaces that can be found in [6]. However, for both examples the reduced automorphism groups contain at least two different involutions. For each of them, the involutions considered here and in [6] are such that the genus 0 quotient are not isometric. This in particular implies that the genus two curves described here are not real isomorphic to those in [6] and that the transformed surfaces are not isometric to those under the $D_5$-action in [6].

I have never found Example 3 in the literature.

1. Let $Q_0$ be the totally regular quadrilateral, i.e., such that $\sigma_0(Q_0) = Q_0$, defined in terms of length of its sides and first diagonal by:

$$\cosh(l_i) = 3 + 2\sqrt{2}, \quad \cosh(l) = 4\sqrt{2} + 5.$$ 

Then the pair $(a, b)$ of normalized equation parameters for $S_{Q_0}$ must satisfy

$$\sigma_0(a, b) = (a, b), \quad \text{i.e.,} \quad a = -i \text{ and } b = i \text{ or } a = i \text{ and } b = -i,$$

and thus $y^2 = (x^2 - 1)(x^4 + 1)$ is an equation for $S_{Q_0}$. As an algebraic curve, $S_{Q_0}$ is complex (but not real) isomorphic to that of equation $y^2 = (x^2 - 1)(x^2 - (3 + 2\sqrt{2}))(x^2 - (3 - 2\sqrt{2})).$

2. Let $Q_1$ be the quadrilateral such that $Q_1 = \sigma_1(Q_1)$, then the surface $S_{Q_1}$ must satisfy

$$\sigma_1(a, b) = (a, b) \quad \text{i.e.,} \quad \{a, b\} = \left\{\frac{1}{2}(1 + i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})\right\}.$$ 

3. Let $Q_2$ be the quadrilateral given by (in terms of hyperbolic cosine of length of the sides and of the first diagonal)

$$(3 + 2\sqrt{3}, 3 + 2\sqrt{3}, 3 + 2\sqrt{3}, 3 + 2\sqrt{3}) = 3 + 2\sqrt{3}.$$ 

Note that $Q_2$ is obtained by gluing two copies of a triangle with interior angles $(\frac{1}{6}\pi, \frac{1}{6}\pi, \frac{1}{6}\pi)$ we obtain $Q_2$. This implies that $D_4 \subseteq \text{Aut}(S_{Q_2})$ (and thus belongs to the family $F_4$ of Table 2) and that the surface $S_{Q_2}/D_4$ has an automorphism of order 3. We then have

$$b = \frac{1}{a}, \quad \frac{a + \frac{1}{a}}{2} = \pm i\sqrt{3}.$$ 


From easy but technical considerations (see [1] for more details) on real structures and on the position of the unit circle on the quotient, one can also deduce that in fact \( \frac{1}{2}(a + 1/a) = i \sqrt{3} \), and that the normalized equation parameters are:

\[
(a, b) = \left( a, \frac{1}{a} \right) = (i(\sqrt{3} - 2), i(\sqrt{3} + 2)).
\]

As the surface belongs to \( F_1 \), one can find only six a priori non-isometric transformed surfaces under the actions of \( G_Q \) and \( S_5 \). We give them in terms of length of the quadrilaterals and in normalized equation parameters in Table 5.

Note that the two last surfaces have the real structure induced by \( x \mapsto 1/\bar{x} \). Note also the two first ones have the real structures induced by \( x \mapsto i(\bar{x} - i)/(\bar{x} + i) \) (I wish to thank R. Silhol for this last remark). They are thus isomorphic as they are conjugated.

One can easily verify that the third and the fourth have no real structures. Exact examples without real structure are very rare in the literature.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\sigma & \cosh(l_1) & \cosh(l_2) & \cosh(l_3) & \cosh(l_4) & \cosh(l) \\
\hline
\text{Id} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} \\
\sigma_0 & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 11 + 6 \sqrt{3} \\
\sigma_1 & 7 + 4 \sqrt{3} & 1 + \sqrt{3} & 3 + 2 \sqrt{3} & 5 + 3 \sqrt{3} & 9 + 5 \sqrt{3} \\
\sigma_2 \sigma_0 & 3 + 2 \sqrt{3} & 1 + \sqrt{3} & 7 + 4 \sqrt{3} & 5 + 3 \sqrt{3} & 5 + 3 \sqrt{3} \\
\sigma_2 & 11 + 6 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} \\
(\sigma_1 \sigma_0)^3 & 51 + 30 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 3 + 2 \sqrt{3} & 21 + 12 \sqrt{3} \\
\hline
\end{array}
\]

Table 5.

5. Twist and half-twists

Recall that \( K_Q \) is the subgroup of transformations \( \sigma \) in \( G_Q \) such that \( \bar{\sigma} \) as in Section 4.2 belongs to \( S_{1,3} \times S_{2,4,5} \). As for any quadrilateral \( Q \) the surfaces \( S_Q \) and \( S_{\sigma(Q)} \) are isometric but correspond to a priori different points in Teichmüller space \( \mathcal{T}_2 \), they are linked by an element of the modular group.

On the other hand, \( K_Q \subseteq G_Q \), and the main result of this section is that \( G_Q \) is in fact generated by half-twists in the following sense. Let \( Q = (e_1, e_2, e_3, e_4) \in Q \). An oriented geodesic of the surface \( S_Q \) is represented by a word \( m(e_1, e_2, e_3, e_4) \) in
the letters \( e_1, e_2, e_3, \) and \( e_4 \) up to conjugacy in \( \Gamma_Q \). The word \( m \) then represents a free homotopy class for surfaces in \( \{ S_Q, Q \in \mathbb{Q} \} \).

Let \( \sigma \in G_Q \). We will say that \( \sigma \) corresponds to a half-twist (or a Dehn twist) along a geodesic \( m \) if given any \( Q = (e_1, e_2, e_3, e_4) \), one can go from \( S_Q \) to \( S_{\sigma(Q)} \) by a half-twist (or a Dehn twist) along the geodesic \( m(e_1, e_2, e_3, e_4) \) of \( S_Q \). Of course, as on the one hand the elements of \( G_Q \) preserve \( F_2(Q) \) and on the other hand only the twists along the geodesics stable under \( \text{aut}(S) \) preserve \( F_2(Q) \), \( m(e_i) \) has to be conjugated in the Fuchsian group \( \Gamma_Q \) to \( e_1m(e_i)e_1 \) or \( e_1m(e_i)^{-1}e_1 \). We will also consider simultaneous Dehn twists along \( m(e_i) \) and \( e_1m(e_i)e_1 \), in the case where the two corresponding geodesics are disjoint.

### 5.1. Half-twists

We show that the generators for \( G_Q \) given in Section 3.2 act on \( F(Q) \) as half-twists.

**Theorem 5.1.** The transformation \( \eta_1: (e_1, e_2, e_3, e_4) \longrightarrow (e_1, e_2, e_3, e_2e_3e_4) \) is a half-twist along the geodesic represented by the word \( (e_3e_2)^2 = w^4(e_1, e_2, e_3, e_4) \).

The transformation \( \eta_2: (e_1, e_2, e_3, e_4) \longrightarrow (e_1, e_2, e_4, e_3e_4e_1) \) is a half-twist along the geodesic represented by the word \( (e_4e_3)^2 = w^2(e_1, e_2, e_3, e_4) \).

The transformation \( \eta_3: (e_1, e_2, e_3, e_4) \longrightarrow (e_1e_4e_1, e_2, e_3, e_1) \) is a half-twist along the geodesic represented by the word \( (e_1e_4)^2 = w^3(e_1, e_2, e_3, e_4) \).

The transformation \( \eta_4: (e_1, e_2, e_3, e_4) \longrightarrow (e_2e_3e_4, e_1, e_2, e_3, e_4) \) is a half-twist along the geodesic represented by the word \( (e_2e_3e_4)^2 = w^4(e_1, e_2, e_3, e_4) \).

The group \( G_Q = \langle \eta_1, \eta_2, \eta_3, \eta_4 \rangle \) is thus generated by half-twists.

**Proof.** We treat the four cases with the same argument. Namely, for any \( Q \), we exhibit pants decompositions \( \mathcal{P}_i \) and \( \mathcal{P}_i' \) on \( S_Q \) and \( S_{\eta_i(Q)} \), respectively, such that:

1. The four pairs of pants obtained by cutting \( S_Q \) along \( \mathcal{P}_i \) and \( S_{\eta_i(Q)} \) along \( \mathcal{P}_i' \) are isometric.

2. The three geodesics of \( \mathcal{P}_i \) on \( S_Q \) and that of \( \mathcal{P}_i' \) on \( S_{\eta_i(Q)} \) correspond to the same decomposition according to the marking: if \( Q = (e_1, e_2, e_3, e_4) \) and \( \eta_i(Q) = (e_1^i, e_2^i, e_3^i, e_4^i) \), there exist three words in four letters \( w^i = w_1^i, w_2^i, w_3^i \) such that

\[
\mathcal{P}_i = (w_1^i(e_j), w_2^i(e_j), w_3^i(e_j)), \quad \mathcal{P}_i' = (w_1^i(e_j^i), w_2^i(e_j^i), w_3^i(e_j^i)).
\]

Conditions (1) imply in particular that the length parts of the Fenchel–Nielsen coordinates of \( S_Q \) and \( S_{\eta_i(Q)} \) associated to the pants decompositions \( \mathcal{P}_i \) and \( \mathcal{P}_i' \), respectively, are the same. With condition (2), this means that the transformation \( \eta_i \) corresponds to a change of these Fenchel–Nielsen parameters that affects only their twist part. In other words, \( \eta_i \) corresponds to a product of twist deformations (not necessarily Dehn twists) along the geodesics of the decomposition \( \mathcal{P}_i \) of the marked Riemann surface \( S_Q \).
It remains to show that the values of the twist parameters are 0 for \( w_2 \) and \( w_3 \) and \( \frac{1}{2} \) for \( w_4 \).

The first part is achieved by considering geodesics crossing \( w_2 \) and \( w_3 \) and observing that their word expression is unchanged by \( \eta \).

To show that the twist is \( \frac{1}{2} \) on \( w_1 \), we use the underlying algebraic curves observing that \( \eta^2 \) belongs to \( K_Q \) while \( \eta \) does not.

We take:

- for \( \eta_1 \): \( w_1^1(e_i) = (e_3e_2)^2 \), \( w_1^2(e_i) = e_1e_2e_3 \), \( w_1^3(e_i) = e_2e_3e_1 \),
- for \( \eta_2 \): \( w_1^1(e_i) = (e_4e_3)^2 \), \( w_1^2(e_i) = e_3e_4e_1 \), \( w_1^3(e_i) = e_1e_3e_4 \),
- for \( \eta_3 \): \( w_1^1(e_i) = (e_1e_4)^2 \), \( w_1^2(e_i) = e_3e_1e_4 \), \( w_1^3(e_i) = e_1e_4e_3 \),
- for \( \eta_4 \): \( w_1^1(e_i) = (e_2e_3e_4)^2 \), \( w_1^2(e_i) = e_2e_3e_4e_3 \), \( w_1^3(e_i) = e_3e_2e_3e_4 \).

Now consider the geodesic represented by the word \((e_2e_3e_4)^2\). It crosses the geodesics represented by \( w_2^1(e_i) \) and \( w_2^3(e_i) \) (or \( w_2^2(e_i) \) and \( w_3^2(e_i) \)) but not the one represented by \( w_2^1(e_i) \) (or \( w_2^3(e_i) \)) and its word expression is invariant by occurrences of \( \eta_1 \) (or \( \eta_2 \)). In particular, its length is unchanged. According to A. Douady in [7, Exposé 7], this means that the twist parameter along the geodesic represented by \( w_2^1(e_i) \) and \( w_2^3(e_i) \) when applying \( \eta_1 \) (or \( w_2^2(e_i) \) and \( w_3^2(e_i) \) when applying \( \eta_2 \)) is zero.

The same arguments for the geodesics represented by \((e_2e_1e_4)^2\) for \( \eta_3 \) and \((e_2e_2)^2\) for \( \eta_4 \) shows the nullity of the twist parameter on \( w_3^2(e_i) \) and \( w_3^3(e_i) \) when applying \( \eta_3 \) and on \( w_2^2(e_i) \) and \( w_3^3(e_i) \) when applying \( \eta_4 \).

**Remark 5.2.** Theorems 4.6 and 5.1 together allow to see some half-twists and Dehn twists on the equation of the associated algebraic curve.

**Remark 5.3.** As working on groups with representation by generators and relations is not an easy thing, we do not have a precise idea of what is possible in terms of twists with elements of \( G_Q \).

Note for example that simultaneous half-twists along disjoint geodesics exchanged by the non-trivial involutions of surfaces \( S_Q \) do not correspond to elements of \( G_Q \) in general. We give an example that illustrates this fact. Let \( P \in \mathcal{P}_2 \), and \( S_P \) the genus two surface constructed from \( P \) as in (4.8). According to [6], for a generic \( P \), the transforms of \( S_P \) under \( G_Q \) either have a real structure with 2 components or a real structure with 3 components. Now consider the surface \( S_P \) obtained from \( S_P \) by making simultaneous half-twists along the two real components exchanged by the involution of \( S_P \). Then, one can show that for a generic \( P \), \( S_P \) has no real structure with more than one component, and thus does not belong to the transforms of \( S_P \). See [1] for more details.

### 5.2. Dehn twists along the sides of the quadrilateral

There are several motivations to not consider only Dehn twists as double half-twists in our situation. First, as mentionned in Remark 5.3, simultaneous half-twists are not
allowed in general while we will see below an example of a correspondence between
an element of $G_Q$ and simultaneous Dehn twists.

Another motivation to deal with Dehn twists separately is their difference in
nature with half-twists: the Dehn twists are defined on the topological surface $T_2$
of genus two, while half-twists only make sense on the (marked) Riemann surfaces.
In particular, the composition of different transformations corresponding to half-
twists can only be considered as successive operations on successively different
Riemann surfaces. On the other hand, it makes sense to compose Dehn twists
even along geodesics which are not disjoint on the topological surface $T_2$.

We will show that in a way the group law in the mapping class group and in
the subgroup $K_Q$ and $G_Q$ are reversed.

Let $Q = (e_1, e_2, e_3, e_4) \in Q$, we choose $S_Q$ as a model for $T_2$. As $S_Q$ is a
hyperbolic surface, each homotopy class of a closed path $c$ on $T_2$ is represented by a
unique geodesic of $S_Q$, i.e., the conjugacy class of a word $w$ in the letters $e_1, e_2, e_3$
and $e_4$. We will denote by $\tau_{w(e_i)}$ the change of the marking corresponding to the
twist along $c$ on $T_2$.

We have then

**Proposition 5.4.** Let $\tau = \tau_{w_1(e_i)} \circ \cdots \circ \tau_{w_{k_1}(e_i)}$ and $\tau' = \tau_{m_1'(e_i)} \circ \cdots \circ \tau_{m_{k_2}'(e_i)}$
be two products of Dehn twists along geodesics of $S_Q$, for any $Q = (e_1, e_2, e_3, e_4) \in Q$.

Assume that there exist transformations $\sigma$ and $\sigma'$ in $K_Q$ such that for any
$Q = (e_1, e_2, e_3, e_4) \in Q$,

$$S_{\sigma(Q)} = \tau(S_Q) \quad \text{and} \quad S_{\sigma'(Q)} = \tau'(S_Q).$$

Then

(i) $\tau \circ \tau'(S_Q) = S_{\sigma' \circ \sigma(Q)}$.

(ii) For $\tilde{\sigma} \in K_Q$, and $\tau_{\tilde{\sigma}} = \tau_{w_1(\tilde{\sigma}(e_i))} \circ \cdots \circ \tau_{w_{k_1}(\tilde{\sigma}(e_i))}$, we have

$$\tau_{\tilde{\sigma}}(S_Q) = S_{\tilde{\sigma}^{-1} \circ \tilde{\sigma}(Q)}.$$

**Proof.** It is well known that if $\alpha$ is an homeomorphism of a surface $T_2$, and $c$
is a homotopy class of a simple closed path on $T_2$, then $\tau_{\alpha(c)} = \alpha \tau_c \alpha^{-1}$ (see for
example [3]).

Point (i) is a direct consequence of this fact. We have

$$\tau \circ \tau'(S_Q) = \tau \circ \left( \tau_{w_1(e_i)} \circ \cdots \circ \tau_{w_{k_1}(e_i)} \right)(S_Q)$$
$$= \tau \circ \left( \tau^{-1} \tau_{m_1'(e_i)} \circ \cdots \circ \tau^{-1} \tau_{m_{k_2}'(e_i)} \right)(S_Q)$$
$$= (\tau_{\tau(m_1'(e_i))} \circ \cdots \circ \tau_{\tau(m_{k_2}'(e_i))}) \circ \tau(S_Q)$$
$$= (\tau_{\tau(m_1'(e_i))} \circ \cdots \circ \tau_{\tau(m_{k_2}'(e_i))})(S_{\sigma(Q)})$$
$$= (\tau_{\tau(m_1'(\sigma(e_i)))} \circ \cdots \circ \tau_{\tau(m_{k_2}'(\sigma(e_i)))})(S_{\sigma(Q)}) = \tau'(S_{\sigma(Q)}) = S_{\sigma' \circ \sigma(Q)}.$$
(ii) As \( \tilde{\sigma} \) belongs to \( K_Q \), there exists \( \tilde{\tau} \in \Gamma_2 \) such that \( \tilde{\tau}(S_Q) = S_{\tilde{\sigma}(Q)} \) and 
\( \tilde{\tau}(w_k(e_i)) = \tilde{\tau}(w_k(\tilde{\sigma}(e_i))) \), and the result follows from (i). \( \square \)

We end by showing in an example how, given a Dehn twist or a product of Dehn twists along geodesics exchanged by the involutions of \( S_Q \), \( Q = (e_1, e_2, e_3, e_4) \), one can recover “by hand” the corresponding transformation of \( G_Q \).

Consider the topological model on the right-hand side of Figure 6 for \( S_Q \).

Let \( \tau_2 = \tau_{(e_3 e_4 e_1)} \circ \tau_{(e_1 e_3 e_4)} = \tau_{(e_1 e_3 e_4)} \circ \tau_{(e_3 e_4 e_1)} \). We obtain the correspondence between \( \tau_2 \) and the transformation \( \sigma = \sigma_0 \sigma_2 \sigma_3 \sigma_0^3 \sigma_1 \sigma_0 \) using the techniques developed in Section 3 as follows. We first determine the homotopy classes of the images under \( \tau_2 \) of the geodesics of \( S_Q \) corresponding to the sides of the different copies of \( Q \). The corresponding geodesics are mapped, in the quotient \( S_0(Q) = S_0(\sigma(Q)) \), onto the sides of the quadrilateral fundamental domain \( \sigma(Q) \).

Then, if one cuts the quotient along the sides of \( Q \), the sides of \( \sigma(Q) \) appear as geodesic arcs on \( Q \) (see Figure 7).
The transformation $\sigma$ is then built using the technique indicated in the proof of Proposition 3.8.

Using Theorem 5.1, the twist $\tau_1$ along $(e_2e_3e_4)^2$ corresponds to $\eta_4^2$ and Lemma 5.4(ii) we get the following correspondences for the Dehn twists along geodesics corresponding to the sides of the different copies of $Q$ in $S_Q$:

$\tau_1 = \tau_{(e_2e_3e_4)^2}$ corresponds to $\eta_4^2$, $\eta_4^2: (e_1, e_2, e_3, e_4) \mapsto ((e_2e_3e_4)^2, e_2, e_3, e_4)$,

$\tau_2 = \tau_{(e_3e_4e_1)} \circ \tau_{(e_1e_3e_4)} = \tau_{(e_1e_3e_4)} \circ \tau_{(e_3e_4e_1)}$

... correspond to $\sigma: (e_1, e_2, e_3, e_4) \mapsto (e_1, e_3e_4e_1e_2, e_2, e_3, e_4)$,

$\tau_3 = \tau_{(e_4e_1e_2)}^2 \tau_{(e_1e_2)}^2 \eta_4^2 \eta_4^2 \sigma_0^2 \sigma_0^2$, $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, (e_4e_1e_2)^2e_3, e_4)$,

$\tau_4 = \tau_{(e_1e_4e_3)} \circ \tau_{(e_3e_1e_2)} = \tau_{(e_3e_1e_2)} \circ \tau_{(e_1e_2e_3)} = \tau_{(e_1e_2)}^2 \sigma_0^2 (e_1)$, corresponds to $\sigma_0^2 \sigma_0^2 \sigma_0^2 \sigma_0^2: (e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, (e_3e_4e_1e_2e_3)^2e_3, e_4)$.

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References


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