ON A DISTANCE DEFINED BY THE LENGTH SPECTRUM ON TEICHMÜLLER SPACE

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Dedicated to the memory of Professor Nobuyuki Suita

Abstract. We consider a distance \( d_L \) on the Teichmüller space \( T(S_0) \) of a hyperbolic Riemann surface \( S_0 \). The distance is defined by the length spectrum of Riemann surfaces in \( T(S_0) \) and we call it the length spectrum metric on \( T(S_0) \). It is known that the distance \( d_L \) determines the same topology as that of the Teichmüller metric if \( S_0 \) is a topologically finite Riemann surface.

In this paper we show that there exists a Riemann surface \( S_0 \) of infinite type such that the length spectrum distance \( d_L \) on \( T(S_0) \) does not define the same topology as that of the Teichmüller distance. Also, we give a sufficient condition for these distances to have the same topology on \( T(S_0) \).

1. Introduction and results

On the Teichmüller space \( T(S_0) \) of a hyperbolic Riemann surface \( S_0 \), we have the Teichmüller distance \( d_T(\cdot, \cdot) \), which is a complete distance on \( T(S_0) \). In this paper, we study another distance \( d_L(\cdot, \cdot) \) which is defined by the length spectrum on Riemann surfaces in \( T(S_0) \). Li [4] discussed the distance \( d_L(\cdot, \cdot) \) on the Teichmüller space of a compact Riemann surface of genus \( g \geq 2 \) and showed that the distance \( d_L \) defines the same topology as that of the Teichmüller distance. Recently, Liu [5] showed that the same statement is true even if \( S_0 \) is a Riemann surface of topologically finite type, and he asked whether the statement holds for Riemann surface of infinite type. Our first result gives a negative answer to this question.

Theorem 1.1. There exist a Riemann surface \( S_0 \) of infinite type and a sequence \( \{p_n\}_{n=0}^\infty \) in \( T(S_0) \) such that

\[
d_L(p_n, p_0) \to 0, \quad n \to \infty,
\]

while

\[
d_T(p_n, p_0) \to \infty, \quad n \to \infty.
\]

From the proof of this theorem, we show the incompleteness of the length spectrum distance.

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Corollary 1.1. There exists a Riemann surface of infinite type such that the length spectrum distance $d_L$ is incomplete in the Teichmüller space.

Next, we give a sufficient condition for the length distance to define the same topology as that of the Teichmüller distance as follows.

Theorem 1.2. Let $S_0$ be a Riemann surface. Assume that there exists a pants decomposition $S_0 = \bigcup_{k=1}^{\infty} P_k$ of $S_0$ satisfying the following conditions.

1. Each connected component of $\partial P_k$ is either a puncture or a simple closed geodesic of $S_0$, $k = 1, 2, \ldots$.
2. There exists a constant $M > 0$ such that if $\gamma$ is a boundary curve of some $P_k$ then

$$0 < M^{-1} < l_{S_0}(\gamma) < M$$

holds.

Then $d_L$ defines the same topology as that of $d_T$ on the Teichmüller space $T(S_0)$ of $S_0$.

2. Preliminaries

Let $S_0$ be a hyperbolic Riemann surface. We consider a pair $(S, f)$ of a Riemann surface $S$ and a quasiconformal homeomorphism $f$ of $S_0$ onto $S$. Two such pairs $(S_j, f_j)$, $j = 1, 2$, are called equivalent if there exists a conformal mapping $h: S_1 \to S_2$ which is homotopic to $f_2 \circ f^{-1}_1$, and we denote the equivalence class of $(S, f)$ by $[S, f]$. The set of all equivalence classes $[S, f]$ is called the Teichmüller space of $S_0$: we denote it by $T(S_0)$.

The Teichmüller space $T(S_0)$ has a complete distance $d_T$ called the Teichmüller distance which is defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \inf_f \log K[f],$$

where the infimum is taken over all quasiconformal mappings $f: S_1 \to S_2$ homotopic to $f_2 \circ f^{-1}_1$ and $K[f]$ is the maximal dilatation of $f$.

We define another distance on $T(S_0)$ by length spectrum of Riemann surfaces. Let $\Sigma(S)$ be the set of closed geodesics on a hyperbolic Riemann surface $S$. For any two points $[S_j, f_j]$, $j = 1, 2$, in $T(S_0)$, we set

$$\varrho([S_1, f_1], [S_2, f_2]) = \sup_{c \in \Sigma(S_1)} \max \left\{ \frac{l_{S_1}(c)}{l_{S_2}(f_2 \circ f^{-1}_1(c))}, \frac{l_{S_2}(f_2 \circ f^{-1}_1(c))}{l_{S_1}(c)} \right\},$$

where $l_S(\alpha)$ is the hyperbolic length of a closed geodesic on $S$ freely homotopic to a closed curve $\alpha$. For two points $[S_j, f_j] \in T(S_0)$, $j = 1, 2$, we define a distance $d_L$ called the length spectrum distance by

$$d_L([S_1, f_1], [S_2, f_2]) = \log \varrho([S_1, f_1], [S_2, f_2]).$$

Wolpert ([7]) shows that $l_{S_2}(f(c)) \leq K[f] l_{S_1}(c)$ holds for every quasiconformal mapping $f: S_1 \to S_2$ and for every $c \in \Sigma(S_1)$. Thus, we have immediately:
Lemma 2.1. An inequality

\[ d_L(p, q) \leq d_T(p, q) \]

holds for every \( p, q \in T(S_0) \).

3. Proofs of Theorem 1.1 and Corollary 1.1

3.1. Proof of Theorem 1.1. First, we take monotone divergent sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) of positive numbers so that \( a_{n+1} = b_n \) and \( b_n/a_n > n \), \( n = 1, 2, \ldots \). For each \( n \), we take a right-angled hexagon \( H_n \) so that the lengths of three edges are \( a_n, b_n, t_n \) as Figure 1.

![Figure 1](image-url)

We take \( b_n \) so large that

\[ t_n > na_n \]

holds, where \( t_n \) is the height of \( H_n \), the distance from the bottom edge of length \( a_n \) to the top edge. Gluing two copies of \( H_n \), we obtain a pair of pants \( P_n \) as in Figure 2.

The hyperbolic lengths of the three boundary curves of \( P_n \) are \( 2a_n, 2b_n, 2b_n \). Note that the lengths of boundaries of \( P_n \) are long but the distances between any two boundaries are short. From these pairs of pants \( \{P_n\}_{n=1}^{\infty} \), we construct a Riemann surface \( S_0 \) as follows.

Step 1. For each \( k \in \mathbb{N} \), glue two copies of \( P_k \) along the boundaries of length \( 2a_k \). Then we obtain a Riemann surface of type \((0,4)\), say \( S_{k,1} \). Let \( \gamma_k \) denote the “core” curve in \( S_{k,1} \) with length \( 2a_k \) (see Figure 3).
Step 2. Make four copies of $P_{k+1}$ and glue each copy with $S_{k,1}$ along their boundary.
curves of length $2b_k = 2a_{k+1}$. The resulting Riemann surface $S_{k,2}$ is of type $(0,8)$.

Step 3. Continue the above construction inductively. Namely, make $2^{n+1}$ copies of $P_{k+n}$ and glue each copy with $S_{k,n-1}$ along their boundary curves of length $2b_{k+n-1} = 2a_{k+n}$. Then we obtain a Riemann surface $S_{k,n}$ of type $(0, 2^{n+1})$.

Step 4. Take $n(k) \in \mathbb{N}$ so large that the distance between $\gamma_k$ and $\partial S_{k,n(k)}$ is greater than $ka_k$. Put $S^{(k)} = S_{k,n(k)}$.

From the construction, we see the following:

**Observation.** For any points $p, q \in \gamma_k$, let $\gamma$ be a geodesic arc from $p$ to $q$. Then the hyperbolic length $l_{S_0}(\gamma)$ of $\gamma$ satisfies

$$l_{S_0}(\gamma) > ka_k,$$

if $\gamma \not\subseteq \gamma_k$.

Step 5. Make another copy $S^{(-k)}$ of $S^{(k)}$ for each $k \in \mathbb{N}$ and construct a Riemann surface $S_0$ of infinite type from $\{S^{(k)}\}_{k=-\infty}^{1} \cup \{S^{(k)}\}_{k=1}^{\infty}$ and another pair of pants with geodesic boundaries such that each $S^{(k)}$ is isometrically embedded in $S_0$.

Let $f_n$ be the positive Dehn twist for $\gamma_n$, $n \in \mathbb{N}$. Here, the “positive” Dehn twist means the Dehn twist with left turning. Set $p_n = [S_0, f_n]$ and $p_0 = [S_0, \text{id}]$. The following lemma shows that $\lim_{n \to \infty} d_T(p_n, p_0) = \infty$.

**Lemma 3.1.** Let $K_n$ be the maximal dilatation of the extremal quasiconformal mapping which is homotopic to $f_n$. Then, $\lim_{n \to \infty} K_n = \infty$. Thus, $\lim_{n \to \infty} d_T(p_n, p_0) = \infty$ as $n \to \infty$.

**Proof.** We consider a neighbourhood $U_n$ of $\gamma_n$ which is defined by

$$U_n = \{ z \in S_0 \mid d_{S_0}(z, \gamma_n) < \varepsilon \}$$

for some $\varepsilon > 0$, where $d_{S_0}(\cdot, \cdot)$ is the hyperbolic distance on $S_0$. We take $\varepsilon > 0$ small enough that $U_n$ is conformally equivalent to an annulus. We define the Dehn twist $f_n$ such that $f_n \mid U_n$ is the standard Dehn twist on the annulus and the identity on $S_0 \setminus U_n$. That is, $f_n \mid U_n$ is defined in terms of the polar coordinates in the annulus by

$$r \exp(i\theta) \mapsto r \exp\left\{i\left(\theta + 2\pi\frac{r-1}{R-1}\right)\right\},$$

if $U_n$ is equivalent to $\{ 1 < |z| < R \}$.

Let $\alpha_n$ be a simple closed geodesic in $S_{n,1} \subset S_0$ perpendicular to $\gamma_n$. We consider connected components of $\pi^{-1}(U_n)$, $\pi^{-1}(\gamma_n)$ and $\pi^{-1}(\alpha_n)$ on $H$, where $\pi: H \to S_0$ is a universal covering map. We may assume that the connected
component of \( \pi^{-1}(\gamma_n) \) is the imaginary axis and that of \( \pi^{-1}(\alpha_n) \) is \( \delta := \{|z| = 1\} \cap \mathbf{H} \), the unit circle in \( \mathbf{H} \). Let \( \tilde{U}_n \) denote the connected component of \( \pi^{-1}(U_n) \) containing the imaginary axis.

Let \( F_n: \mathbf{H} \to \mathbf{H} \) be a lift of an extremal quasiconformal mapping which is homotopic to \( f_n \). We may take \( F_n \) so that \( F_n(0) = 0 \), \( F_n(i) = i \) and \( F_n(\infty) = \infty \). It is well known that \( F_n \) can be extended to a homeomorphism of \( \overline{\mathbf{H}} \) and the boundary mapping \( F_n | \mathbf{R} \) depends only on the homotopy class of \( f_n \) up to \( \text{Aut}(\mathbf{H}) \).

Let \( z_1, z_2, \text{Re} \, z_1 < 0 < \text{Re} \, z_2 \), be the points of \( \delta \cap \partial \tilde{U}_n \). Since \( f_n \) is the positive Dehn twist, we see that \( F_n(z_1) = z_1 \) and \( F_n(z_2) = e^{2\alpha_n} z_2 \). Hence, \( F_n(\delta \cap \tilde{U}_n) \) is an arc connecting \( z_1 \) and \( e^{2\alpha_n} z_2 \) in \( \tilde{U}_n \). Applying the similar argument to a subarc of \( \delta \) in each component of \( \pi^{-1}(U_n) \), we see that

\[-1 < F_n(-1) < e^{2\alpha_n} < F_n(1).\]

In particular, \( \lim_{n \to \infty} F_n(1) = \infty \). Therefore, for the cross ratio

\([a, b, c, d] = (a - b)(c - d)(a - d)^{-1}(c - b)^{-1}\)

we have

\[[-1, 0, 1, \infty] = -1\]

and

\[\lambda_n = [F_n(-1), F_n(0), F_n(1), F_n(\infty)] = [F_n(-1), 0, F_n(1), \infty] = \frac{F_n(-1)}{F_n(1)}.\]

Thus, we have \( \lim_{n \to \infty} \lambda_n = 0 \). Therefore, the conformal modulus of a quadrilateral \( \mathbf{H} \) with vertices \( F_n(-1), F_n(0), F_n(1) \) and \( F_n(\infty) \) degenerates as \( n \to \infty \). Since the quasiconformal mapping \( F_n \) maps a quadrilateral \( \mathbf{H} \) with vertices \(-1, 0, 1 \) and \( \infty \) onto the quadrilateral \( \mathbf{H} \) with vertices \( F_n(-1), F_n(0), F_n(1) \) and \( F_n(\infty) \), we have \( K_n = K(F_n) \to +\infty \). \( \square \)

Next, we shall show that \( d_L(p_n, p_0) \to 0 \).

Let \( \alpha \) be a closed geodesic on \( S_0 \). If \( \alpha \cap \gamma_n = \emptyset \), then it is obvious that \( l_{S_0}(\alpha) = l_{S_0}(f_n(\alpha)) \).

Suppose that \( \#(\alpha \cap \gamma_n) = m > 0 \). Since each point of \( \alpha \cap \gamma_n \) makes a Dehn twist, we have

\[(3) \quad l_{S_0}(f_n(\alpha)) \leq l_{S_0}(\alpha) + ml_{S_0}(\gamma_n) = l_{S_0}(\alpha) + 2ma_n.\]

On the other hand, from (2) we have

\[(4) \quad l_{S_0}(\alpha) > mna_n.\]
Combining (3) and (4), we have

\[ \frac{l_{S_0}(f_n(\alpha))}{l_{S_0}(\alpha)} < 1 + \frac{1}{2n} \to 0, \]

as \( n \to \infty \). Since \( f_n^{-1} \) is also a Dehn twist, from the same argument as above, we have

\[ \frac{l_{S_0}(\alpha)}{l_{S_0}(f_n(\alpha))} < 1 + \frac{1}{2n}. \]

Therefore, we note that \( \lim_{n \to \infty} d_L(p_n, p_0) = 0 \) and complete the proof of Theorem 1.1.

### 3.2. Proof of Corollary 1.1

We use the same Riemann surface \( S_0 \) and the same quasiconformal mappings \( f_n \) as in the proof of Theorem 1.1. Set \( F_n = f_1 \circ f_2 \circ \cdots \circ f_n \) and \( q_n = [S_0, F_n] \). Then, we see that \( \lim_{n \to \infty} d_L(q_m, q_n) = 0 \) and \( \{q_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( T(S_0) \) with respect to the length spectrum distance \( d_L \). However, it does not converge to any point in \( T(S_0) \) because \( F_\infty = \prod_{n=1}^\infty f_n \) is not homotopic to a quasiconformal mapping. Indeed, by using the same argument as in the proof of Lemma 3.1, we see that the maximal dilatation of any homeomorphism homotopic to \( F_\infty \) is not finite. Hence, the distance \( d_L \) is not complete in \( T(S_0) \).

### 4. Proof of Theorem 1.2

It follows from Lemma 2.1 that \( d_L(p_n, p_0) \to 0 \) when \( d_T(p_n, p_0) \to 0 \). Thus, it suffices to show that \( d_T(p_n, p_0) \to 0 \) as \( d_L(p_n, p_0) \to 0 \).

By using Lemma 2.1 again, we see that the condition of Theorem 1.2 is quasiconformal invariant, that is, any Riemann surface which is quasiconformally equivalent to \( S_0 \) satisfies the condition of Theorem 1.2 for some constant. Hence we may assume that \( p_0 = [S_0, \text{id}] \).

Put \( p_n = [S_n, f_n] \in T(S_0) \) and assume that \( \lim_{n \to \infty} d_L(p_n, p_0) = 0 \). Let \( \mathcal{C}_P \) be the set of closed geodesics which are boundaries of some \( P_k \) in \( S_0 \). For each \( \alpha \in \mathcal{C}_P \), there exist a closed geodesic in \( S_n \) homotopic to \( f_n(\alpha) \). We denote the closed geodesic by \( [f_n(\alpha)] \). The set \( \{[f_n(\alpha)]\}_{\alpha \in \mathcal{C}_P} \) together with punctures of \( S_n \) gives a pants decomposition of \( S_n \). Let \( P_k^{(n)} \) denote a pair of pants in the pants decomposition of \( S_n \) such that each boundary component is a closed geodesic homotopic to a component of \( f_n(P_k) \) or a puncture of \( \partial f_n(P_k) \).

From the definition of \( d_L \), we have

\[ (d_L(p_n, p_0))^{-1}l_{S_0}(\alpha) \leq l_{S_n}([f_n(\alpha)]) \leq d_L(p_n, p_0)l_{S_0}(\alpha) \]

for any \( \alpha \in \mathcal{C}_P \). As \( M^{-1} \leq l_{S_0}(\alpha) \leq M \) for \( \alpha \in \mathcal{C}_P \), we see that \( \{l_{S_n}([f_n(\alpha)])\}_{n=1}^\infty \) converges to \( l_{S_0}(\alpha) \) uniformly on \( \mathcal{C}_P \), that is, for any \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \)
such that if $n \geq n_0$, then
\begin{equation}
|l_{S_n}([f_n(\alpha)]) - l_{S_0}(\alpha)| < \varepsilon
\end{equation}
holds for any $\alpha \in \mathcal{C}_P$.

It is known that the lengths of boundaries determine the moduli of the pair of pants. Hence, if $n$ is sufficiently large, then from (5) we verify that there exists a quasiconformal mapping $g_{k,n}$ of $P_k$ onto $P^{(n)}_k$ with small dilatation. However, we need to find quasiconformal mappings $g_n$ on the whole surface $S_0$ such that $\lim_{n \to \infty} K[g_n] = 1$. To obtain such mappings, we consider the “Fenchel–Nielsen coordinates” of the infinite-dimensional Teichmüller space $T(S_0)$. The classical Fenchel–Nielsen coordinates are defined in the Teichmüller space of Riemann surfaces of finite type. We define the coordinates in $T(S_0)$ by using an exhaustion of $S_0$.

We take a subregion $S_0^{(m)}$, $m = 1, 2, \ldots$, of $S_0$ satisfying the following conditions (after rearrangement of the numbers of $\{P_k\}_{k=1}^{\infty}$).

1. $S_0^{(m)} = \text{Int}\left(\bigcup_{k=1}^{k(m)} P_k\right)$ for some $k(m) \in \mathbb{N}$,
2. $S_0^{(1)} \subset S_0^{(2)} \subset \cdots \subset S_0^{(m)} \subset S_0^{(m+1)} \subset \cdots$, and
3. $S_0 = \bigcup_{m=1}^{\infty} S_0^{(m)}$.

Similarly, we take an exhaustion $\{S_n^{(m)}\}_{m=1}^{\infty}$ of $S_n$ from $\{P_k^{(n)}\}_{k=1}^{\infty}$.

Let $\tilde{S}_0^{(m)}$, respectively $\tilde{S}_n^{(m)}$ be the Nielsen extension of $S_0^{(m)}$, respectively $S_n^{(m)}$. Since $(f_n)_*(\pi_1(S_0^{(m)})) = \pi_1(S_n^{(m)})$, we see that there exists a quasiconformal mapping $f_{n,m}: \tilde{S}_0^{(m)} \to \tilde{S}_n^{(m)}$ such that
\begin{equation}
(f_{n,m})_* \circ (i_n^{(m)})_* = (i_n^{(m)})_* \circ (f_n)_* \mid \pi_1(S_0^{(m)})
\end{equation}
on $\pi_1(S_0^{(m)})$, where $i_0^{(m)}$ and $i_n^{(m)}$ are the natural inclusion maps from $S_0^{(m)}$ to $\tilde{S}_0^{(m)}$ and from $S_n^{(m)}$ to $\tilde{S}_n^{(m)}$, respectively. A pair $(\tilde{S}_n^{(m)}, f_{n,m})$ gives a point in $T(\tilde{S}_0^{(m)})$. Obviously, if $m' > m$, then
\begin{equation}
(f_{n,m'})_* \mid \pi_1(S_0^{(m)}) = (f_{n,m})_*. 
\end{equation}

Now, we consider the Fenchel–Nielsen coordinates (cf. [3]) on $T(\tilde{S}_0^{(m)})$ with respect to the pants decomposition given by $\{P_k^{(m)}\}_{k=1}^{k(m)}$. The Fenchel–Nielsen coordinates consist of length coordinates and twist coordinates. The length coordinates are the collection of lengths of boundaries of the pants decomposition and the twist coordinates are the collection of twist angles along the boundaries of the pants decomposition. From (6) or the construction of $\tilde{S}_n^{(m)}$, we see that if $\alpha \in S_0^{(m)}$ and $m' > m$, then
\begin{equation}
l_{\tilde{S}_n^{(m)}}([f_{n,m}(\alpha)]) = l_{\tilde{S}_n^{(m')}}([f_{n,m'}(\alpha)]) = l_{S_n}([f_n(\alpha)]).
\end{equation}
Thus, it follows from (5) that the length coordinates of \( \hat{p}_n^{(m)} = [\hat{S}_n^{(m)}, f_{n,m}] \) converge to those of \( \hat{p}_0^{(m)} = [\hat{S}_0^{(m)}, \text{id}] \) as \( n \to \infty \).

Similarly, it is also seen that the twist parameter along \([f_{m,n}(\alpha)]\) does not depend on \( m \) if \([f_n(\alpha)] \subset S_n^{(m)}\). Thus, we may denote the twist parameter along \([f_n(\alpha)], \alpha \in \mathcal{C}_P\), by \( \theta_n(\alpha) \).

**Lemma 4.1.** The sequence \( \{\theta_n(\alpha)\}_{n \in \mathbb{N}} \) converges to \( \theta_0(\alpha) \), the twist parameter along \( \alpha \) of \( p_0 \in T(S_0) \), uniformly on \( \mathcal{C}_P \). Namely, for any \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \), then

\[
|\theta_n(\alpha) - \theta_0(\alpha)| < \varepsilon
\]

holds for any \( \alpha \in \mathcal{C}_P \).

**Proof.** For any closed geodesic \( \alpha \in \mathcal{C}_P \), the following two cases occur (Figure 4):

(A) \( \alpha \) is a non-dividing curve and is contained in a subregion \( T(\alpha) \) of \( S_0 \) of genus one with one geodesic boundary curve in \( \mathcal{C}_P \).
(B) \( \alpha \) is a dividing curve and is contained in a planar subregion of \( S_0 \) bounded by four geodesic curves in \( \mathcal{C}_P \).

![Figure 4.](image-url)
We shall show the statement of the lemma in Case (A) since the proof is similar in Case (B).

Let $\alpha^*$ be the shortest simple closed geodesic in all closed curves intersecting $\alpha$ (Figure 5). From the second condition in Theorem 1.2 we verify that there exists a constant $C_1 = C_1(M) > 0$ depending only on $M$ such that

\[(7) \quad 0 < C_1^{-1} < l_{S_0}(\alpha^*) < C_1.\]

Indeed, since $l_{S_0}(\alpha) < M$, the collar theorem (cf. [2, Chapter 4]) guarantees that the geodesic $\alpha \subset S_0$ has a collar with some width depending only on $M$. Hence the existence of a positive lower bound of $l_{S_0}(\alpha^*)$ is obvious and the lower bound depends only on $M$. On the other hand, since the hyperbolic length of any boundary curve of any $P_k$ is less than $M$, the hyperbolic distance between any two boundary curves of $P_k$ is less than some constant $C_1$ depending only on $M$. Therefore, it is easy to see $l_{S_0}(\alpha^*) < C_1$.

Next, we show that there exists a constant $\Theta$ depending only on $M$ such that

\[(8) \quad |\theta_n(\alpha)| < \Theta\]

holds for any $\alpha \in \mathcal{C}_P$ and for any $n \in \mathbb{N}$.

Indeed, if such a constant does not exist, then there exist sequences $\{n_t\}_{t \in \mathbb{N}}$ in $\mathbb{N}$ and $\{\alpha_{n_t}\}_{t \in \mathbb{N}}$ in $\mathcal{C}_P$ such that

\[\lim_{t \to \infty} |\theta_{n_t}(\alpha_{n_t})| = +\infty.\]

Considering that $l_{S_{n_t}}(\alpha_{n_t}) > M^{-1} > 0$, we have that

\[\lim_{t \to \infty} l_{S_{n_t}}(f_{n_t}(\alpha_{n_t}^*)) = +\infty.\]
Therefore, from (7) we see that
\[ d_L(p_{n_t}, p_0) \geq \log \frac{l_{S_{n_t}}(f_{n_t}(\alpha_{n_t}^*))}{l_{S_0}(\alpha_{n_t}^*)} \rightarrow +\infty \]
as \( t \rightarrow \infty \), and we have a contradiction.

Now, we prove the lemma. Assume that \( \{\theta_n(\alpha)\}_{n \in \mathbb{N}} \) does not uniformly converge to \( \theta_0(\alpha) \). Then, there exist a constant \( \varepsilon_0 > 0 \), a sequence \( \{n_t\}_{t \in \mathbb{N}} \) in \( \mathbb{N} \) and \( \{\alpha_{n_t}\}_{t \in \mathbb{N}} \) in \( \mathcal{C}_P \) such that
\[ |\theta_{n_t}(\alpha_{n_t}) - \theta_0(\alpha_{n_t})| \geq \varepsilon_0 \]
holds for any \( n_t \). From this inequality and (8), we can find a simple closed geodesic \( \beta \) in \( S_0 \) such that
\[ |l_{S_{n_t}}(f_{n_t}(\beta)) - l_{S_0}(\beta)| > \delta \]
hold for some \( \delta > 0 \) and for all \( n_t \) (cf. [8]). Hence, we have
\[ \left| \frac{l_{S_{n_t}}(f_{n_t}(\beta))}{l_{S_0}(\beta)} - 1 \right| > \frac{\delta}{l_{S_0}(\beta)} > 0. \]
It contradicts \( \lim_{t \rightarrow \infty} d_L(p_{n_t}, p_0) = 0 \) and thus we complete the proof of the lemma. \( \square \)

Now, the proof of Theorem 1.2 is immediate. From the boundedness of the length and the angle parameters and from the uniform convergence of them, we can construct quasiconformal mappings from \( \mathcal{S}_0^{(m)} \) onto \( \mathcal{S}_n^{(m)} \) with small dilatations. More precisely, for any \( \varepsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) and quasiconformal mappings \( g_{n,m} \) from \( \mathcal{S}_0^{(m)} \) onto \( \mathcal{S}_n^{(m)} \), \( m = 1, 2, \ldots \), such that
(1) \( g_{n,m} \) is homotopic to \( f_{n,m} \).
(2) If \( n \geq n_0 \), then the maximal dilatation \( K[g_{n,m}] \) of \( g_{n,m} \) is less than \( (1 + \varepsilon) \).
By taking the limit of \( \{g_{n,m}\}_{m \in \mathbb{N}} \) as \( m \rightarrow \infty \), we have a quasiconformal mapping \( g_n \) from \( S_0 \) onto \( S_n \) with \( K[g_n] \leq 1 + \varepsilon \). From the construction, \( [S_n, g_n] = [S_n, f_n] = p_n \) and we conclude that \( \lim_{n \rightarrow \infty} d_T(p_n, p_0) = 0 \) as we desired.

References


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