MÖBIUS MODULUS OF RING DOMAINS IN $\mathbb{R}^n$

Zair Ibragimov
University of Michigan, Department of Mathematics
Ann Arbor, MI 48109, U.S.A.; ibragim@umich.edu

Abstract. We introduce a new Möbius invariant modulus for ring domains $R$ in $\mathbb{R}^n$ which coincides with the usual modulus whenever $R$ is a Möbius annulus, i.e.,

$$f(R) = \{x \in \mathbb{R}^n : 1 < |x| < t\}$$

for some Möbius transformation $f$ of $\mathbb{R}^n$ and some $t > 1$. We obtain a sharp upper bound for the Möbius modulus of a ring $R$ which separates two pairs $\{a, b\}$ and $\{c, d\}$ of distinct points in $\mathbb{R}^n$. Our result proves a conjecture made by M. Vuorinen in 1992 [14].

1. Introduction

Notation. We denote by $\mathbb{R}^n$ the $n$-dimensional Euclidean space and by $\{e_1, e_2, \ldots, e_n\}$ its standard basis. The one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ is denoted by $\mathbb{R}^\infty$. The open and closed balls of radius $r > 0$ and centered at $x \in \mathbb{R}^n$ are denoted by $B^n(x, r)$ and $\overline{B}^n(x, r)$, respectively. $S^{n-1}(x, r)$ is a sphere of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. The closed segment between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is denoted by $[x, y]$. For $x \in \mathbb{R}^n$, $x \neq 0$, we set

$$[x, \infty] = \{tx : t \geq 1\} \cup \{\infty\}.$$

The group of all Möbius transformations of $\mathbb{R}^n$ is denoted by $\text{Möb}(\mathbb{R}^n)$.

A ring is a domain $R \subset \mathbb{R}^n$ whose complement is the union of two disjoint non-degenerate compact connected sets. A ring with complementary components $C_1$ and $C_2$ is denoted by $R(C_1, C_2)$. A ring $R(C_1, C_2)$ is said to separate the sets $E$ and $F$ if $E \subset C_1$ and $F \subset C_2$. Hence a ring $R(C_1, C_2)$ separates the complementary components of a ring $R(E, F)$ if $E \subset C_1$ and $F \subset C_2$.

If $R$ is a ring in $\mathbb{R}^2$, then $R$ can be mapped conformally onto a circular annulus

$$\{z \in \mathbb{C} : 1 < |z| < t\}$$

and the modulus of $R$ is defined to be $\log t$.

In $\mathbb{R}^n$, $n > 2$, by Liouville’s theorem the Möbius transformations are the only conformal mappings in $\mathbb{R}^n$. A ring $R$ is said to be a Möbius annulus if

$$f(R) = \{x \in \mathbb{R}^n : 1 < |x| < t\}$$

for some $f \in \text{Möb}(\mathbb{R}^n)$ and $t > 1$.

2000 Mathematics Subject Classification: Primary 30C35; Secondary 30C75.
The points \( f^{-1}(0) \) and \( f^{-1}(\infty) \) are called relative centers of the Möbius annulus \( R \). The modulus of such a ring \( R \) is defined as \( \log t \).

In general, the modulus of a ring \( R = R(C_1, C_2) \) is defined as follows. Let \( \Gamma \) be the family of all curves joining \( C_1 \) and \( C_2 \) in \( R \) and let \( F(\Gamma) \) be the set of all non-negative Borel functions

\[
g : \mathbb{R}^n \to \mathbb{R}^1 \quad \text{such that} \quad \int_\gamma g \, ds \geq 1
\]

for every locally rectifiable curve \( \gamma \in \Gamma \). Then the modulus of the curve family \( \Gamma \) is defined as

\[
M(\Gamma) = \inf_{g \in F(\Gamma)} \int_{\mathbb{R}^n} g^n \, dm.
\]

Observe that since \( C_1 \) and \( C_2 \) are non-degenerate, \( 0 < M(\Gamma) < \infty \) by [13, 11.5 and 11.10]. The modulus of \( R \) is defined as

\[
\text{mod } R = \left[ \frac{\omega_{n-1}}{M(\Gamma)} \right]^{1/(n-1)}.
\]

See, for instance, [1, 8.30]. Here \( \omega_{n-1} \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

The rings of Grötzsch and Teichmüller play an important role in the theory of quasiconformal mappings. The complementary components of Grötzsch ring \( R_G(s) \), \( s > 1 \), are \( \mathbb{B}^n(0, 1) \) and \( [se_1, \infty] \) while those of Teichmüller ring \( R_T(t) \), \( t > 0 \), are \( [-e_1, 0] \) and \( [te_1, \infty] \).

For the convenience of the reader we recall some properties of the modulus of Grötzsch and Teichmüller rings. The following functional relation holds.

\[
(1.1) \quad \text{mod } R_T(t) = 2 \text{mod } R_G(\sqrt{t + 1}).
\]

See [4] and [1, 8.32 and 8.37(1)]. The function

\[
(1.2) \quad \text{mod } R_T(t) - \log(t + 1)
\]

is a nondecreasing function in \((0, \infty)\) and

\[
(1.3) \quad \lim_{t \to \infty} \left( \text{mod } R_T(t) - \log(t + 1) \right) = \log \lambda_n^2 < \infty.
\]

Here \( \lambda_n \) is a constant which depends only on \( n \). See, for instance, [4] and [1, 8.38].

Our main focus in this paper is the following extremal problem of Teichmüller. Let \( a, b, c, d \) be distinct points in \( \mathbb{R}^n \). Among all the rings which separate the sets \( \{a, b\} \) and \( \{c, d\} \) it is required to find one with the largest modulus.

For \( n = 2 \), this problem was considered by O. Teichmüller [12] in 1938 and a complete solution was given by M. Schiffer [11] in 1946. In this case the points
Möbius modulus of ring domains in $\mathbb{R}^n$

$a, b, c, d$ can be normalized so that $a = -1, b = 1, c = \xi, d = -\xi$, where $\xi \in \mathbb{B}^2(0, 1)$ is a unique point with

$$
\frac{c-a}{c-b} \frac{d-b}{d-a} = \left( \frac{\xi + 1}{\xi - 1} \right)^2.
$$

If $\mathcal{R}(\xi) = R(C_1(\xi), C_2(\xi))$ is an extremal ring, i.e., a ring with the largest modulus, then

$$
\begin{align*}
\text{sh}_1(C_1(\xi)) &= C_1(\xi), & \text{sh}_1(C_2(\xi)) &= C_2(\xi) \\
\text{sh}_2(C_1(\xi)) &= C_2(\xi), & \text{sh}_2(C_2(\xi)) &= C_1(\xi),
\end{align*}
$$

where

$$
\text{sh}_1(z) = -z \quad \text{and} \quad \text{sh}_2(z) = \frac{\xi}{z}.
$$

In particular,

$$
0 \in C_1(\xi) \quad \text{and} \quad \infty \in C_2(\xi).
$$

See [8, pp. 199–200].

For $n > 2$, Teichmüller’s problem is solved only when the points $a, b, c, d$ lie on a circle or a line in this order. In this case the points $a, b, c, d$ can be normalized so that

$$
a = -e_1, \quad b = 0, \quad c = te_1, \quad d = \infty, \quad \text{where} \quad t = \frac{|b - c| |d - a|}{|b - a| |d - c|}.
$$

Then by means of a spherical symmetrization one shows that Teichmüller’s ring $R_T(t)$ is an extremal ring. See, for instance, [4], [10] and [1, Theorem 8.46].

When the points $a, b, c, d$ do not lie on a circle or a line in this order, the problem is still open. In general the points $a, b, c, d$ can be normalized so that

$$
a = 0, \quad b = e_1, \quad c = x, \quad d = \infty \quad \text{where} \quad x \in \mathbb{R}^n \quad \text{and} \quad |x| = \frac{|a - c| |b - d|}{|a - b| |c - d|}.
$$

M. Vuorinen has considered a ring $R_0$ whose complementary components are some circular arc joining the points 0 and $e_1$ and some ray emanating from the point $x$ [14]. This ring coincides (up to a Möbius transformation) with Teichmüller’s ring when the points $a, b, c, d$ lie on a circle or a line in this order. It follows from Theorem 3.16 [14] that there exists a Möbius annulus $A$ separating the complementary components of $R_0$ and such that

$$
\text{mod} A = \text{arccosh} (|x| + |x - e_1|).
$$

Due to the monotonicity of the modulus we then have

$$
\text{arccosh} (|x| + |x - e_1|) = \text{mod} A \leq \text{mod} R_0.
$$

In order to assure that $\text{arccosh} (|x| + |x - e_1|)$ is the sharpest lower bound for $\text{mod} R_0$ obtained in this manner, Vuorinen was led to the following conjecture.
Conjecture 1.5. For \( x \in \mathbb{R}^n \setminus [0, e_1] \),
\[
\max_A \text{mod } A = \arccosh (|x| + |x - e_1|),
\]
where the maximum is taken over all Möbius annuli \( A \) which separate the sets \( \{0, e_1\} \) and \( \{x, \infty\} \).

In this paper we consider a new measure for ring domains called the Möbius modulus. Our main result, Theorem 3.8 below, shows that Teichmüller’s problem has a complete solution when considered with respect to the Möbius modulus. As a corollary to Theorem 3.8 we settle the conjecture of Vuorinen. After this paper was submitted the referee pointed out that an alternative proof of the conjecture was also given in [3].

2. Some results on the cross-ratio

The main result of this section is Theorem 2.16. The cross-ratio of a quadruple \( a, b, c, d \) of points in \( \mathbb{R}^n \) with \( a \neq b \) and \( c \neq d \) is defined as follows. If \( a, b, c, d \in \mathbb{R}^n \), then

\[
|a, b, c, d| = \frac{|a - c| |b - d|}{|a - b| |c - d|}.
\]

Otherwise we omit the terms containing \( \infty \). For example,

\[
|a, b, c, \infty| = \frac{|a - c|}{|a - b|}.
\]

A homeomorphism \( f: \mathbb{R}^n \to \mathbb{R}^n \) belongs to Möb(\( \mathbb{R}^n \)) if and only if

\[
|f(a), f(b), f(c), f(d)| = |a, b, c, d|
\]

for all quadruples \( a, b, c, d \) in \( \mathbb{R}^n \). See [2, Theorem 3.2.7]. For a quadruple \( a, b, c, d \) in \( \mathbb{R}^n \) we put

\[
\sigma(a, b, c, d) = |a, b, c, d| + |b, a, c, d|.
\]

Hence

\[
\sigma(a, b, c, d) = \frac{|a - c| |b - d| + |a - d| |b - c|}{|a - b| |c - d|} \geq 1
\]

with equality if and only if the points \( a, c, b, d \) lie on a circle or a line in this order. A simple computation shows that

\[
|a, b, c, d| = \frac{\sigma(a, b, c, d) + 1}{\sigma(a, c, b, d) + 1} \quad \text{and} \quad \sigma(a, b, c, d) = \frac{|a, d, c, b| + 1}{|d, a, c, b|}.
\]

It follows from (2.3) and (2.6) that a homeomorphism \( f: \mathbb{R}^n \to \mathbb{R}^n \) belongs Möb(\( \mathbb{R}^n \)) if and only if

\[
\sigma(f(a), f(b), f(c), f(d)) = \sigma(a, b, c, d)
\]

for all quadruples \( a, b, c, d \) in \( \mathbb{R}^n \).

The following two lemmas are used in the proof of Theorem 2.16. The first lemma is an immediate consequence of Corollary 7.25 [1].
Lemma 2.8. Let \( a, b, c, d \in \mathbb{R}^n \) be distinct points. Let \( f \) be a Möbius transformation such that
\[
f(a) = -e_1, \quad f(b) = e_1, \quad f(c) = -w \quad \text{and} \quad f(d) = w
\]
for some \( w \in \mathbb{R}^n \) with \( |w| \geq 1 \). Then
\[
|w| = p + \sqrt{p^2 - 1}, \quad \text{where} \quad p = \sigma(a, b, c, d).
\]

Proof. Let \( g \in \text{Möb}(\mathbb{R}^n) \) be the inversion in \( S^{n-1}(0, 1) \). Then by applying Corollary 7.25 [1] to the composition \( g \circ f \) and using (2.6) we obtain
\[
\frac{1}{|w|} = \frac{|d, a, c, b|}{1 + |a, d, c, b| + \sqrt{(1 + |a, d, c, b|)^2 - |d, a, c, b|^2}} = \frac{1}{p + \sqrt{p^2 - 1}}
\]
as required. \( \Box \)

The next lemma is an extension of a special case of Lemma 2.12 [6].

Lemma 2.9. If \( a_1, a_2, a_3, a_4 \) are distinct points in \( \mathbb{R}^n \) with
\[
|a_1| = |a_2| = s < t = |a_3| = |a_4|,
\]
then
\[
s^2 + t^2 \leq \frac{2st}{2st} \leq \sigma(a_1, a_2, a_3, a_4).
\]

Equality holds if and only if
\[
a_1 + a_2 = a_3 + a_4 = 0.
\]

Proof. Define \( c_{ij} \) as the cosine of the angle \( \angle(a_i, 0a_j) \) and set
\[
g(u) = \frac{N(u)}{D}, \quad \text{where} \quad D = (1 - c_{12})^{1/2}(1 - c_{34})^{1/2}
\]
and
\[
N(u) = (u - c_{13})^{1/2}(u - c_{24})^{1/2} + (u - c_{14})^{1/2}(u - c_{23})^{1/2}.
\]
Note that \( D \leq 2 \) and that \( D = 2 \) if and only if (2.12) holds. Next
\[
\frac{d}{du}((u - c_{ij})^{1/2}(u - c_{kl})^{1/2}) = \frac{1}{2} \left( (u - c_{kl})^{1/2} + (u - c_{ij})^{1/2} \right) \geq 1,
\]
whence
\begin{equation}
(2.13) \quad g'(u) = \frac{N'(u)}{D} \geq 1 \quad \text{for } u > 1.
\end{equation}

Since
\[ g(1) = \sigma \left( a_1, a_2, \frac{s}{t}a_3, \frac{s}{t}a_4 \right) \geq 1, \]
we have
\begin{equation}
(2.14) \quad g(u) = \int_1^u g'(r) \, dr + g(1) \geq u \quad \text{for all } u \geq 1.
\end{equation}

In particular,
\begin{equation}
(2.15) \quad \sigma(a_1, a_2, a_3, a_4) = g \left( \frac{s^2 + t^2}{2st} \right) \geq \frac{s^2 + t^2}{2st}
\end{equation}
which completes the proof of (2.11).

Assume next that (2.11) holds with equality and set
\[ v = \frac{s^2 + t^2}{2st}. \]

Then (2.15) implies that \( g(v) = v \). Using the differentiability of \( g \) along with (2.14) we obtain
\[ g'(v) = \lim_{u \to v} \frac{g(v) - g(u)}{v - u} \leq \lim_{u \to v} \frac{v - u}{v - u} = 1 \]
and hence using (2.13) we conclude that \( g'(v) = 1 \). Since \( N'(v) \geq 2 \) and \( D \leq 2 \), the equality \( g'(v) = 1 \) implies \( D = 2 \) which, as noted above, implies (2.12) as required.

Finally, a simple computation shows that (2.12) implies that (2.11) holds with equality. \( \Box \)

**Theorem 2.16.** Let \( a, b, c, d \in \mathbb{R}^n \) be distinct points and \( p = \sigma(a, b, c, d) \). Then for all distinct pairs \( \{u, v\} \) of points in \( \mathbb{R}^n \setminus \{a, b, c, d\} \)
\begin{equation}
(2.17) \quad \min \{ |u, a, c, v|, |u, a, d, v|, |u, b, c, v|, |u, b, d, v| \} \leq p + \sqrt{p^2 - 1}.
\end{equation}
Moreover, if \( p > 1 \), then there exists a unique pair \( \{u, v\} \) for which the equality holds.
**Proof.** Let \( u, v \in \mathbb{R}^n \setminus \{a, b, c, d\} \) be distinct points. Since (2.17) is invariant under the elements of \( \text{Möb}(\mathbb{R}^n) \), we can assume that \( u = 0 \) and \( v = \infty \). Then

\[
\min \{ |u, a, c, v|, |u, a, d, v|, |u, b, c, v|, |u, b, d, v| \} = \frac{\min\{|c|, |d|\}}{\max\{|a|, |b|\}} = \frac{t}{s}.
\]

Since \( p + \sqrt{p^2 - 1} \geq 1 \), there is nothing to prove if \( s \geq t \). Hence we may assume that \( s < t \). Then (2.17) is equivalent to

\[
(2.18) \quad \frac{s^2 + t^2}{2st} \leq \sigma(a, b, c, d).
\]

Let

\[
S^{n-1}(v_1, r_1) \subset B^n(0, s) \quad \text{and} \quad S^{n-1}(v_2, r_2) \subset \mathbb{R}^n \setminus B^n(0, t)
\]

be spheres such that \( a, b \in S^{n-1}(v_1, r_1) \) and \( c, d \in S^{n-1}(v_2, r_2) \). Then we have \( R_1 \subset R_2 \), where

\[
R_1 = R(\overline{B}^n(0, s), \mathbb{R}^n \setminus B^n(0, t))
\]

and

\[
R_2 = R(\overline{B}^n(v_1, r_1), \mathbb{R}^n \setminus B^n(v_2, r_2)).
\]

In particular,

\[
(2.19) \quad \text{mod } R_1 \leq \text{mod } R_2.
\]

Choose \( h \in \text{Möb}(\mathbb{R}^n) \) that maps \( S^{n-1}(v_1, r_1) \) and \( S^{n-1}(v_2, r_2) \) onto concentric spheres \( S^{n-1}(0, s') \) and \( S^{n-1}(0, t') \), respectively. Then

\[
0 < |h(a)| = |h(b)| = s' < t' = |h(c)| = |h(d)|
\]

and we have

\[
(2.20) \quad \log \frac{t}{s} = \text{mod } R_1 \leq \text{mod } R_2 = \log \frac{t'}{s'}.
\]

Using (2.11) we now have

\[
\frac{s^2 + t^2}{2st} \leq \frac{s'^2 + t'^2}{2s't'} \leq \sigma(h(a), h(b), h(c), h(d)) = \sigma(a, b, c, d)
\]

which proves (2.18) and hence the first part of the theorem.

Notice that equality in (2.18) implies that \( |a| = |b| = s \) and \( |c| = |d| = t \).
To prove the second part of the theorem, we let \( f \in \text{M"ob}(\mathbb{R}^n) \) be such that

\[
f(a) = -e_1, \quad f(b) = e_1, \quad f(c) = -w \quad \text{and} \quad f(d) = w
\]

for some \( w \in \mathbb{R}^n \) with \(|w| \geq 1\). Since \( p > 1 \), we have \(|w| = p + \sqrt{p^2 - 1} > 1\) by Lemma 2.8. Then the equality in (2.17) holds if we take \( u = f^{-1}(0) \) and \( v = f^{-1}(\infty) \). This establishes the existence of the pair \( \{u, v\} \).

To prove the uniqueness, we assume that \( \{u_1, v_1\} \) is another pair for which the equality in (2.17) holds and show that \( u_1 = u \) and \( v_1 = v \). Let \( g \in \text{M"ob}(\mathbb{R}^n) \) be such that

\[
g(u_1) = 0, \quad g(v_1) = \infty \quad \text{and} \quad g(a) = -e_1.
\]

By our assumption we have

\[
\min\{0, -e_1, g(c), \infty, 0, -e_1, g(d), \infty, 0, g(b), g(c), \infty, 0, g(b), g(d), \infty \} = \min\{\|g(c)\|, \|g(d)\|\} = \max\{\|g(a)\|, \|g(b)\|\} = p + \sqrt{p^2 - 1}.
\]

Hence

\[
|g(a)| = |g(b)| \leq |g(c)| = |g(d)|
\]

as we have noted above. Then using (2.12) we have

\[
g(a) + g(b) = g(c) + g(d) = 0.
\]

Hence

\[
g(a) = e_1, \quad g(b) = e_1, \quad g(c) = -z \quad \text{and} \quad g(d) = z
\]

for some \( z \in \mathbb{R}^n \) and by Lemma 2.8 we have \(|z| = p + \sqrt{p^2 - 1} = |w|\). Since \(|f(a), f(b), f(c), f(d)| = |a, b, c, d| = |g(a), g(b), g(c), g(d)|\) and

\[
|f(b), f(a), f(c), f(d)| = |b, a, c, d| = |g(b), g(a), g(c), g(d)|,
\]

we have

\[
|w - e_1| = |z - e_1| \quad \text{and} \quad |w + e_1| = |z + e_1|.
\]

Hence the angle \( \angle(w0e_1) \) is equal to the angle \( \angle(z0e_1) \). By means of a preliminary rotation about the \( e_1 \)-axis if necessary, we can assume that \( w = z \). Hence \( f^{-1} \) and \( g^{-1} \) agree on a set \( \{-e_1, e_1, -w, w\} \) and consequently they agree on a 2-dimensional (1-dimensional, if \( w \) lies on the \( e_1 \)-axis) subspace of \( \mathbb{R}^n \) containing these points. In particular,

\[
u = f^{-1}(0) = g^{-1}(0) = u_1 \quad \text{and} \quad v = f^{-1}(\infty) = g^{-1}(\infty) = v_1
\]
as required. \( \square \)

**Remark 2.21.** The hypothesis \( p > 1 \) in the uniqueness part of Theorem 2.16 cannot be removed. We now show that if \( p = 1 \) the uniqueness part fails.

Indeed, if \( a = -e_1, b = te_1, c = -te_1, d = e_1 \) for some \( 0 < t < 1 \), then \( \sigma(a, b, c, d) = 1 \). But equality in (2.17) holds for all points \( \{u, v\} \) with

\[
|u - a| = |v - a|, \quad |u - b| = |v - b| \quad \text{and} \quad |u - c| = |v - c|, \quad |u - d| = |v - d|.
\]
3. Möbius modulus of ring domains

In this section we define the Möbius modulus of rings. Our main result is Theorem 3.8 which gives a solution to Teichmüller’s extremal problem for the Möbius modulus. As a corollary to Theorem 3.8 we settle the conjecture of Vuorinen. We will then establish a relationship between the usual and the Möbius moduli of rings and compute the Möbius modulus of Grötzsch and Teichmüller rings.

Definition 3.1. Let $R = R(C_1, C_2)$ be a ring in $\mathbb{R}^n$. The quantity

$$\text{mod}_M R = \max_{u, v \in \mathbb{R}^n} \min_{x \in C_1, y \in C_2} \left| \log \frac{|u-y|}{|x-y|} - \log \frac{|x-v|}{|y-v|} \right|$$

is called the Möbius modulus of $R$.

Observe that if $R$ is any ring and $A$ is a Möbius annulus separating the complementary components of $R$, then

$$(3.3) \quad \text{mod}_M R \geq \text{mod} A > 0.$$ 

Indeed, we can assume that

$$A = B^n(0, t) \setminus B^n(0, 1)$$

for some $t > 1$. Then

$$\text{mod}_M R \geq \min_{x \in C_1, y \in C_2} \left| \log \frac{|0-y|}{|0-x|} \right| = \min_{x \in C_1, y \in C_2} \left| \log \frac{|y|}{|x|} \right| \geq \text{mod} A.$$ 

On the other hand, if $\text{mod}_M R > 0$, then there exists a Möbius annulus $A$ separating the complementary components of $R$ such that

$$(3.4) \quad \text{mod} A = \text{mod}_M R.$$ 

Indeed, let $u, v \in \mathbb{R}^n$ be a pair with

$$\text{mod}_M R = \min_{x \in C_1, y \in C_2} \left| \log \frac{|u-y|}{|x-y|} - \log \frac{|x-v|}{|y-v|} \right| > 0.$$ 

We can assume that $u = 0$ and $v = \infty$. Then

$$\text{mod}_M R = \min_{x \in C_1, y \in C_2} \left| \log \frac{|y|}{|x|} \right| = \log \frac{s}{t},$$

where

$$s = \min \{|x| : x \in C_2\} \quad \text{and} \quad t = \max \{|x| : x \in C_1\}.$$ 

Then the ring

$$A = \{x \in \mathbb{R}^n : t < |x| < s\}$$

is the required Möbius annulus.

Thus we have the following remark to Definition 3.1.
Remark 3.5. Let $R$ be any ring with mod$_{M}R > 0$. Then
\begin{equation}
\text{mod}_M R = \max_A \text{mod } A,
\end{equation}
where the maximum is taken over all Möbius annuli $A$ which separate the complementary components of $R$. In particular, if $R$ is a Möbius annulus, then
\begin{equation}
\text{mod}_M R = \text{mod } R.
\end{equation}

Ring domains with separating euclidean or Möbius annuli were studied in [7] ($n = 2$) and [14] ($n \geq 2$), respectively.

Theorem 3.8. Let $a, b, c, d \in \mathbb{R}^n$ be distinct points. Then
\begin{equation}
\max_R \text{mod}_M R = \arccosh (\sigma(a, b, c, d)),
\end{equation}
where the maximum is taken over all rings $R$ which separate the sets \{a, b\} and \{c, d\}.

Proof. Let $R = R(C_1, C_2)$ be a ring with $a, b \in C_1$ and $c, d \in C_2$ and assume that mod$_M R > 0$. Let $u, v \in \mathbb{R}^n$ and $x_0 \in C_1$, $y_0 \in C_2$ be such that
\begin{equation}
\text{mod}_M R = |\log |u, x_0, y_0, v||.
\end{equation}

By performing a preliminary Möbius transformation we can assume that $u = 0$, $v = \infty$ and $|x_0| < |y_0|$. Then the Möbius annulus
\begin{equation}
A = B^n(0, |y_0|) \setminus B^n(0, |x_0|)
\end{equation}
separates $C_1$ and $C_2$. In particular, we have
\begin{equation}
\max\{|a|, |b|\} \leq |x_0| < |y_0| \leq \min\{|c|, |d|\}.
\end{equation}

Hence by Theorem 2.16 we have
\begin{equation}
\text{mod}_M R = \log \frac{|y_0|}{|x_0|} \leq \log \left( \min\left\{ \frac{|c|}{|a|}, \frac{|d|}{|b|}, \frac{|c|}{|d|} \right\} \right) \leq \arccosh (\sigma(a, b, c, d)).
\end{equation}

The equality holds for the ring $R_0 = R(C_1, C_2)$ where
\begin{equation}
C_1 = f^{-1}([-w, \infty] \cup [w, \infty]) \quad \text{and} \quad C_2 = f^{-1}([-e_1, e_1])
\end{equation}
and $f$ is a Möbius transformation such that
\begin{equation}
f(a) = w, \quad f(b) = -w, \quad f(c) = e_1 \quad \text{and} \quad f(d) = -e_1
\end{equation}
for some $w \in \mathbb{R}^n$ with $|w| \geq 1$. Indeed, for $u = f^{-1}(\infty)$ and $v = f^{-1}(0)$ we have
\begin{equation}
\text{mod}_M R_0 = \text{mod}_M f(R_0) \geq \min\{ |\log |\infty, x, y, 0|| : x \in C_1, y \in C_2\}
= \log |w| = \arccosh (\sigma(a, b, c, d))
\end{equation}
using Lemma 2.8. Hence
\begin{equation}
\text{mod}_M R_0 = \arccosh (\sigma(a, b, c, d)).
\end{equation}

completing the proof. $\diamond$
Next we have the following two corollaries of Theorem 3.8. The first one settles the conjecture of Vuorinen.

**Corollary 3.10.** For \( x \in \mathbb{R}^n \setminus [0, e_1] \),
\[
\max_A \text{mod } A = \text{arccosh } (|x| + |x - e_1|),
\]
where the maximum is taken over all Möbius annuli \( A \) which separate the sets \( \{0, e_1\} \) and \( \{x, \infty\} \).

**Proof.** Since
\[
|x| + |x - e_1| = \sigma(0, e_1, x, \infty),
\]
Theorem 3.8 along with (3.6) imply that
\[
\max_A \text{mod } A = \text{mod}_M R_0 = \text{arccosh } (\sigma(0, e_1, x, \infty)) = \text{arccosh } (|x| + |x - e_1|),
\]
where \( R_0 = R([0, e_1], [x, \infty]) \). \( \Box \)

The next corollary gives a characterization of Möbius transformations of \( \mathbb{R}^n \) in terms of the Möbius modulus of rings. A similar type of characterization in terms of the modulus of rings is given in [5].

**Corollary 3.12.** A homeomorphism \( f: \mathbb{R}^n \to \mathbb{R}^n \) belongs to \( \text{Möb}(\mathbb{R}^n) \) if and only if
\[
\text{mod}_M f(R) = \text{mod}_M R
\]
for all rings \( R \) in \( \mathbb{R}^n \).

**Proof.** The necessity part follows from (2.3) and Definition 3.1. For the sufficiency part it is enough to show that
\[
\sigma(f(a), f(b), f(c), f(d)) = \sigma(a, b, c, d)
\]
for all quadruples \( a, b, c, d \in \mathbb{R}^n \). Given \( a, b, c, d \), we let \( R = R(C_1, C_2) \) be a maximal ring, i.e., a ring with
\[
\text{mod}_M R = \text{arccosh } (\sigma(a, b, c, d)).
\]
Then using Theorem 3.8 and the fact that \( f(a), f(b) \in f(C_1) \) and \( f(c), f(d) \in f(C_2) \) we get
\[
\text{arccosh } (\sigma(a, b, c, d)) = \text{mod}_M R(f(C_1), f(C_2)) \\
\leq \text{arccosh } (\sigma(f(a), f(b), f(c), f(d)))
\]
which implies
\[
\sigma(a, b, c, d) \leq \sigma(f(a), f(b), f(c), f(d)).
\]
By applying the same argument to \( f^{-1} \) we get
\[
\sigma(f(a), f(b), f(c), f(d)) \leq \sigma(a, b, c, d)
\]
as required. \( \Box \)
We have the following relation between the modulus and the Möbius modulus of a ring \( R \).

**Lemma 3.13.** For any ring \( R = R(C_1, C_2) \) in \( \mathbb{R}^n \) we have

\[
\text{mod}_M R \leq \text{mod} R \leq \text{mod}_M R + c(n),
\]

where \( c(n) \) is a constant depending only on \( n \).

**Proof.** It follows from (1.2) and (1.3) that

\[
\text{mod} R_T(t) \leq \log(\lambda^2_n(t + 1)) \quad \text{for all } t > 0.
\]

The first inequality in (3.14) follows from the monotonicity of the modulus along with Corollary 3.5. To show the second inequality in (3.14), let \( \log |x, u, v, y| = \text{mod}_M R \) and

\[
\log |x', u', v', y'| = \max_{u \in C_1, v \in C_2} \min_{x \in C_1, y \in C_2} \left| \log \frac{|u - y|}{|u - x|} \right| \left| \log \frac{|x - v|}{|y - v|} \right|.
\]

Then \( |x', u', v', y'| \leq |x, u, v, y| \) and by Theorem 8.46 \([1]\) we have

\[
\text{mod} R \leq \text{mod} R_T(|x', u', v', y'|) \leq \text{mod} R_T(|x, u, v, y|) \\
\leq \log(2\lambda^2_n|x, u, v, y|) = \text{mod}_M R + \log(2\lambda^2_n).
\]

Hence the lemma holds with \( c(n) = \log(2\lambda^2_n) \). \( \square \)

Finally, we compute the Möbius modulus of Grötzsch and Teichmüller rings.

**Example 3.15.** For \( s > 1 \)

\[
\text{mod}_M R_G(s) = \arccosh(s).
\]

**Proof.** Since \(-e_1, e_1 \in B^n(0, 1)\) and \( se_1, \infty \in [se_1, \infty] \), Theorem 3.8 implies that

\[
\text{mod}_M R_G(s) \leq \arccosh(s).
\]

Equality holds for

\[
u_0 = (s - \sqrt{s^2 - 1})e_1 \quad \text{and} \quad v_0 = (s + \sqrt{s^2 - 1})e_1.
\]

See (3.2). \( \square \)
Example 3.16. For $t > 0$

$$\text{mod}_M R_T(t) = \arccosh (2t + 1).$$

**Proof.** Since $-e_1, 0 \in [-e_1, 0]$ and $te_1, \infty \in [te_1, \infty]$, Theorem 3.8 implies that

$$\text{mod}_M R_T(t) \leq \arccosh (2t + 1).$$

Equality holds for

$$u_0 = (-\sqrt{t(t+1)} + t) e_1 \quad \text{and} \quad v_0 = (\sqrt{t(t+1)} + t) e_1.$$ 

See (3.2). □

**Remark 3.17.** We have the same relation between the Möbius modulus of Grötzsch and Teichmüller rings as in (1.1), namely

$$\text{(3.18)} \quad \text{mod}_M R_T(t) = 2\text{mod}_M R_G(\sqrt{t+1}).$$

**Acknowledgements.** I would like to thank my advisor Frederick Gehring for discussions and suggestions on this paper. I also would like to acknowledge many valuable comments of the referee.

**References**


Received 10 June 2002