\textbf{\Large $\delta$-STABLE FUCHSIAN GROUPS}

\textbf{Christopher J. Bishop}

SUNY at Stony Brook, Mathematics Department
Stony Brook, NY 11794-3651, U.S.A.; bishop@math.sunysb.edu

\textbf{Abstract.} We call a Fuchsian group, $G$, $\delta$-stable if $\delta(G') = \dim(\Lambda(G'))$ for every quasi-Fuchsian deformation $G'$ of $G$. It is well known that every finitely generated Fuchsian group has this property. We give examples of infinitely generated Fuchsian groups for which it holds and others for which it fails.

\section{Introduction}

Associated to a Kleinian group $G$ are two numbers, the Hausdorff dimension of its limit set, $\dim(\Lambda)$, and the critical exponent of its Poincaré series, $\delta$ and it is natural to ask when they are equal (all definitions will be given in Section 2). The question has a simple geometric interpretation. Points of the limit set $\Lambda$ naturally correspond to geodesic rays with a fixed base point $z_0 \in M = \mathbb{B}/G$. The limit set $\Lambda$ can be written as the disjoint union of two special subsets; the conical limit set, $\Lambda_c$, which corresponds to geodesics which return to some compact set infinitely often (the recurrent geodesics) and the escaping limit set, $\Lambda_e$, which corresponds to geodesics which escape to infinity. Thus we always have $\dim(\Lambda) = \max(\dim(\Lambda_c), \dim(\Lambda_e))$. It is a theorem from [16] that $\delta = \dim(\Lambda_c)$ for any non-elementary group and hence $\delta \leq \dim(\Lambda)$ for all such groups with equality if and only if $\dim(\Lambda_c) = \dim(\Lambda_e)$. Moreover, $\delta = \dim(\Lambda_b)$, where $\Lambda_b \subset \Lambda_c$ is the part of the limit set corresponding to geodesics that remain bounded in $M$ [16].

If a Kleinian group is geometrically finite then $\Lambda_e$ is at most countable (see [6], [8]) and so $\dim(\Lambda_e) = 0 \leq \dim(\Lambda_c)$ clearly holds and hence $\delta = \dim(\Lambda)$. This equality is conjectured to hold for all finitely generated Kleinian groups; the best result so far says it holds if we also assume $\Lambda$ has zero area (see [16]). For infinitely generated groups, the equality can fail, but it is still interesting to find conditions under which it holds.

We will restrict our attention to quasi-Fuchsian groups, i.e., groups $G'$ which are conjugate to a Fuchsian group $G$ of the first kind via a quasiconformal map of the plane. We shall say that a Fuchsian group $G$ is $\delta$-stable if $\delta(G') = \dim(\Lambda(G'))$ for every quasi-Fuchsian deformation $G'$ of $G$. Clearly every finitely generated Fuchsian group has this property (because the deformations are geometrically
finite), but it is not obvious whether or not any infinitely generated group does. The purpose of this note is to show both possibilities occur.

From our remarks above, we see that $G$ is $\delta$-stable if and only if

$$\dim(\Lambda_e(G')) \leq \dim(\Lambda_e(G')) = \delta(G')$$

for every quasiconformal deformation $G'$ of $G$. In other words, we want the part of the limit set corresponding to escaping geodesics to be less distorted than the part corresponding to reccurent geodesics. One way to ensure this is for the dilatation to be compactly supported (modulo $G$); in [15] we showed that if the deformation is compactly supported then $\dim(\Lambda_e) = 1$ (indeed, has sigma finite length) and hence $\dim(\Lambda) = \delta$. Thus every Fuchsian group with $\delta = 1$ is $\delta$-stable with respect to compactly supported deformations (by Lemma 2.4, $\delta$ cannot decrease under a deformation in this case). A criterion which applies to certain non-compactly supported deformations is given in Lemma 5.1 of [10].

There is another way of ensuring that $\Lambda_e$ is not distorted too much. It is well known that the thrice punctured sphere has a trivial deformation space, i.e., any quasiconformal conjugation gives another circle as the limit set. In [12] I give a quantified version of this fact, showing that if part of a Riemann surface “looks like” a thrice punctured sphere (i.e., consists of a union of $Y$-pieces, all with short boundary curves) then any deformation supported there cannot raise the dimension of the limit set very much. We will use this idea to construct $\delta$-stable groups. To be more precise, suppose $R = \mathbb{D}/G$ is a union of $Y$-pieces and for any $\varepsilon > 0$, only a finite number are not $\varepsilon$-bounded (see Section 2 for definitions). Following the terminology in [12], we shall say such a $R$ “approximates a thrice punctured sphere near infinity”.

**Theorem 1.1** Suppose $R = \mathbb{D}/G$ approximates a thrice punctured sphere at infinity. Then $G$ is $\delta$-stable.

In fact we will prove the stronger statement that for any quasi-Fuchsian deformation $G'$ of $G$, $\dim(\Lambda_e(G')) = 1$. Then $\delta$-stability follows because it is known that if $\delta(G) = 1$ then $\delta(G') \geq 1$ for any quasi-Fuchsian deformation (see Lemma 2.4). It is easy to see these examples may be taken to be either divergence type or convergence type.

In order to build an example which is not $\delta$-stable, we will take two bordered Riemann surfaces, $R_1$ with $\delta = 1$ and $R_2$ with $\delta < 1$ and join them along a boundary component. The deformation is chosen so the dilatation has support in $R_2$ which is at least distance $d$ from $R_1$ and we show that $\dim(\Lambda_e) > 1 + \varepsilon > 1$ independent of $d$. Bounded geodesics which stay in $R_2$ correspond to a piece of the limit set with dimension $< 1$, which still has dimension $< 1$ if the deformation has small $L^\infty$ norm. Geodesics which stay in $R_1$ are far from the support of the deformation and hence correspond to a part of the limit set whose dimension is hardly changed if $d$ is large enough. Combining these ideas gives:
Theorem 1.2. There is a Fuchsian group $G$ with $\delta = 1$ which is not $\delta$-stable.

As noted above, we require $\delta = 1$ to avoid trivial examples where $\delta(G) \neq \dim(A(G))$. The example in Theorem 1.2 is a convergence type group and I do not know of any divergence type examples. Is every divergence type group $\delta$-stable?

I thank the referee for carefully reading the manuscript and numerous comments and suggestions which improved it.

The rest of the paper is organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we prove that a quasiconformal map with Beltrami coefficient $\mu$ satisfies a pointwise Hölder type estimate with exponent close to 1 far from the support of $\mu$. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2. Definitions and background

If $A$ and $B$ are quantities that depend on some parameter we write $A \lesssim B$ if the ratio $B/A$ is bounded uniformly independent of the parameter. Similarly for $\gtrsim$. We write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold and say $A$ and $B$ are comparable.

The terms “dist” and “diam” will always refer to Euclidean distances in this paper, except when we explicitly state otherwise (e.g., for a set on a Riemann surface, $\text{diam}(E)$ would refer to hyperbolic diameter).

Suppose $\varphi$ is an increasing continuous function from $[0, \infty)$ to itself such that $\varphi(0) = 0$. We define the Hausdorff content of a set $E \subset \mathbb{R}^2$ as

$$\mathcal{H}_\infty^{\varphi} = \inf \left\{ \sum \varphi(r_j) : E \subset \bigcup_j D(x_j, r_j) \right\}.$$ 

If $\varphi(t) = t^\alpha$ we denote $\mathcal{H}_\infty^{\varphi}$ by $\mathcal{H}_\infty^{\alpha}$. The Hausdorff dimension of $E$ is

$$\dim(E) = \inf \{ \alpha : \mathcal{H}_\infty^{\alpha}(E) = 0 \},$$

and the Hausdorff measure of $E$ is

$$\mathcal{H}^{\varphi}(E) = \lim_{\delta \to 0} \left[ \inf \left\{ \sum \varphi(r_j) : E \subset \bigcup_j D(x_j, r_j), r_j \leq \delta \right\} \right].$$

A discrete group $G$ of isometries of the hyperbolic metric on $B^d$ is called a Kleinian group if $d = 3$ and Fuchsian if $d = 2$. A Kleinian group can also be considered as a group of linear fractional transformations on $S^2$. $G$ is called elementary if it contains a finite index Abelian subgroup. In this paper we are only concerned with non-elementary groups. For a non-elementary $G$, the accumulation of any orbit in $B^d \cup S^{d-1}$ is a closed set $\Lambda \subset S^{d-1}$ which is independent of the particular orbit. This is the limit set. The conical limit set $\Lambda_c$ is the set of points
$x \in \Lambda$ for which there is a sequence of orbit points of 0 converging to $x$ within a non-tangential cone in $B^d$. It is easy to see that $x$ is a conical limit point if and only if a geodesic ray in $B$ ending at $x$ projects to a geodesic in $M = B^d/G$ which returns to some compact set of $M$ infinitely often. The set $\Lambda_b \subset \Lambda_c$ denotes the subset corresponding to geodesic rays that remain bounded. We define $\Lambda_e = \Lambda \setminus \Lambda_c$ as the “escaping” part of the limit set. Points of $\Lambda_e$ correspond to geodesic rays which eventually leave every compact set.

A Fuchsian group $G$ is called first kind if $\Lambda = T$ and otherwise it is second kind. It is called cocompact if $R = D/G$ is compact and cofinite if $R$ has finite hyperbolic area. A Fuchsian group is called divergence type if

$$\sum_{g \in G} (1 - |g(0)|) = \infty,$$

and otherwise it is called convergence type. The latter occurs if and only if $R = D/G$ has a finite Green’s function, which is given by the series

$$G_R(z, w) = \sum_{g \in G} G_D(x, g(y)) = \sum_{g \in G} \log \left| \frac{x - g(y)}{1 - \bar{x}g(y)} \right|,$$

where $x$ and $y$ project to $z$ and $w$ respectively.

The Poincaré exponent (or critical exponent) of the group is

$$\delta(G) = \inf \left\{ s : \sum_{g \in G} \exp(-sg(0, g(0))) < \infty \right\},$$

where $\rho$ is the hyperbolic metric in $B^3$. A result from [16] says that

**Theorem 2.1.** If the Fuchsian group $G$ is a non-elementary Kleinian group then $\delta(G) = \operatorname{dim}(\Lambda_c(G))$.

For Fuchsian groups and geometrically finite Kleinian groups this was previously known, e.g., [31] and [33]. The proof given in [16] also shows

**Corollary 2.2.** If $G$ is any non-elementary, discrete Möbius group, $x \in M = D/G$ and $\varepsilon > 0$ then there is a $R = R(\varepsilon, x) < \infty$ such that the set of directions (i.e., unit tangents at $x$) which correspond to geodesic rays starting at $x$ which never leave the ball of radius $R$ around $x$ has dimension $\geq \delta(G) - \varepsilon = \operatorname{dim}(\Lambda_c(G)) - \varepsilon$. In particular $\dim(\Lambda_b) = \dim(\Lambda_c) = \delta(G)$.

For Fuchsian groups this is due to Fernández and Melián in [25].

If $E \subset \mathbb{R}^d$ is compact, let $N(E, \varepsilon)$ be the minimal number of $\varepsilon$-balls needed to cover $E$. The upper Minkowski dimension of $E$ is

$$\dim_M(E) = \limsup_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}.$$
This is clearly an upper bound for the Hausdorff dimension of $E$. If $E \subset \mathbf{T}$ is compact, let $\{I_n\}$ be an enumeration of the complementary intervals, i.e., the components of $\mathbf{T} \setminus E$. The Besicovitch–Taylor index is defined as

$$\inf \left\{ s : \sum_n \operatorname{diam}(I_n)^s < \infty \right\},$$

and is well known to equal the upper Minkowski dimension of $E$ if $E$ has zero Lebesgue measure (e.g., [9], [35]). If $G$ is a finitely generated Fuchsian group, it is known that the upper Minkowski and Hausdorff dimensions of the limit set agree and both agree with $\delta$. These remarks give

**Lemma 2.3.** If $\Lambda$ is the limit set of a finitely generated Fuchsian group $G$, $\delta$ is the critical exponent and $\{I_n\}$ is an enumeration of the components of $\mathbf{T} \setminus \Lambda$ then

$$\sum_n \operatorname{diam}(I_n)^{\delta + \varepsilon} \leq C(\varepsilon, G) < \infty,$$

for every $\varepsilon > 0$.

The critical exponent $\delta$ also has a close relationship to $\lambda_0$, the base eigenvalue of the Laplacian on the quotient manifold $M$ which is defined as

$$\lambda_0 = \sup \{ \lambda : \exists f \in C^\infty(M) \text{ such that } \Delta f = -\lambda f \text{ and } f > 0 \} = \inf_{f \in C^\infty_0(M)} \frac{\int_M |\nabla f|^2}{|f|^2}.$$

If $G$ acts on hyperbolic $n$-space, then the Elstrodt–Patterson–Sullivan formula says $\lambda_0 = \delta(n-1-\delta)$ if $\delta \geq \frac{1}{2}(n-1)$ and $\lambda_0 = \frac{1}{4}(n-1)^2$ if $\delta \leq \frac{1}{2}(n-1)$. See Theorem 2.17 of [34].

The base eigenvalue, in turn, can be bounded by the geometry of $M$ using Cheeger’s constant $h(M)$. This is defined as the infimum over all compact $n$-submanifolds $N$ of $M$ of $\frac{\operatorname{vol}_{n-1}(\partial(N))}{\operatorname{vol}_{n}(N)}$. Cheeger [23] proved that $\lambda_0(M) \geq \frac{1}{4}h(M)^2$ and Buser [21] showed that $\lambda_0 \leq Ch(N)$ for manifolds of bounded negative curvature ($C$ depends on the dimension and a lower bound for the curvature). See [22] for a different proof of Buser’s result. Combining these comments, we see that a Fuchsian group $G$ has $\lambda_0 = 0$ (and hence $\delta = 1$) if the quotient surface $R$ contains subregions $S_n$ with $l(\partial S_n)/\text{area}(S_n) \to 0$.

A homeomorphism of the plane is called $K$-quasiconformal if

$$\limsup_{r \to 0} \frac{\sup_{|y-x|=r} |f(y) - f(x)|}{\inf_{|y-x|=r} |f(y) - f(x)|} \leq K.$$
Christopher J. Bishop

Such maps are known (see [1]) to be differentiable almost everywhere and \( \mu = f_z/f_\bar{z} \) is called the Beltrami coefficient of \( f \) and is in \( L^\infty \) with norm

\[
k = (K - 1)/(K + 1).
\]

It is also true (see [1] again) that any quasiconformal map \( f \) is Hölder continuous with an exponent \( \alpha > 0 \) depending only on \( K \). In particular, we will use the fact that if \( f \) is \( K \)-quasiconformal and \( \gamma_1 \) and \( \gamma_2 \) are hyperbolic geodesics in the unit disk such that \( \gamma_1 \) separates 0 from \( \gamma_2 \) then

\[
\frac{\operatorname{diam}(f(\gamma_2))}{\operatorname{diam}(f(\gamma_1))} \leq C \left( \frac{\operatorname{diam}(\gamma_1)}{\operatorname{diam}(\gamma_2)} \right)^\alpha,
\]

for some \( \alpha \) depending only on \( K \) and \( C \) depending on \( \operatorname{diam}(f(\gamma_2))/\operatorname{diam}(f(\Omega)) \), where \( \Omega \subset \mathbb{D} \) is a region separated from 0 by \( \gamma_1 \).

If \( G \) is a Fuchsian group and \( \mu \) is a bounded measurable function on the unit disk, \( \mathbb{D} \), which satisfies \( ||\mu||_\infty < 1 \) and \( \mu(g(z)) = \mu(z)g'(z)/\overline{g'(\overline{z})} \), for every \( g \in G \), then we say \( \mu \) is a \( G \)-invariant Beltrami coefficient (or complex dilatation). There is a corresponding quasiconformal mapping \( f_\mu \) which is analytic outside the disk and which conjugates \( G \) to a quasi-Fuchsian group \( G_\mu \).

A conformal mapping \( f: \mathbb{D} \to \Omega \) is called a deformation of the Fuchsian group \( G \) if for every \( g \in G \), \( f \circ g \circ f^{-1} \) is Möbius transformation restricted to \( \Omega \). It is called a quasiconformal deformation if \( f \) has a quasiconformal extension to the whole plane.

Bowen’s theorem [18] says that if \( G \) is a cocompact Fuchsian group then for any quasi-Fuchsian deformation \( G' \) of \( G \) either \( \Lambda(G') = \emptyset \) or \( \dim(\Lambda(G')) > 1 \). This was extended to all divergence type groups in [11] and is false for all convergence type groups (see [3], [4], [5]). See [17], [19] and [32] for alternate proofs of Bowen’s theorem.

It is easy to see that if \( G' \) is a deformation of \( G \) then \( \Lambda_c(G') = f(\Lambda_c(G)) \) and \( \Lambda_e(G') = f(\Lambda_e(G)) \). A theorem of Makarov [29] says that if \( E \subset \mathbb{T} \) has \( \dim(E) = 1 \) then \( \dim(f(E)) \geq 1 \) for any conformal map of the disk. Applying this to \( E = \Lambda_c \) we see that

**Lemma 2.4.** If \( G \) is a Fuchsian group with \( \delta(G) = 1 \) then \( \delta(G') \geq 1 \) for any deformation of \( G \).

An alternate proof is described in [13]. Of course, the same result holds for the escaping limit set as well. Indeed, by a result of Fernández and Melián in [26], \( \dim(\Lambda_e) = 1 \) for any infinitely generated Fuchsian group of the first kind. Thus

**Lemma 2.5.** If \( G \) is an infinitely generated Fuchsian group of the first kind then \( \dim(\Lambda_e(G')) \geq 1 \) for any deformation of \( G \).
A theorem of Astala [2] says that if $f$ is a $K$-quasiconformal map then 
$$\dim(f(E)) \geq \frac{2\dim(E)}{(2K + (1 - K)\dim(E))}.$$ 
This is a sharper version of a result of Gehring and Väisälä in [27].

A generalized $Y$-piece in a Riemann surface $R$ is a region bounded by three simple closed geodesics (or punctures) which is homeomorphic to a 2-sphere minus three disks (or points). If all three boundary components have length $\leq L$ we say the $Y$-piece is $L$-bounded (punctures count as zero length). We say that $R$ has a $L$-bounded $Y$-piece decomposition if it can be written as a union of $L$-bounded $Y$-pieces with disjoint interiors. Let $\Gamma$ be the union of all simple closed geodesics which occur as boundary arcs in the $Y$-piece decomposition and let $\Gamma^\varepsilon \subset \Gamma$ denote the union of all those with lengths $\geq \varepsilon$. By the collar lemma (e.g., [28], [30]) there is a $C > 0$ (depending only on $L$) so that the hyperbolic $C$-neighborhoods of elements of $\Gamma$ are pairwise disjoint. The following is the result from [7] which we will use to prove Theorem 1.1.

**Theorem 2.6** Given $L, K < \infty$ and $\eta > 0$ there are $\varepsilon > 0$ and $r < \infty$ so that the following holds. Suppose $R = D/G$ is a Riemann surface which has a decomposition into $L$-bounded $Y$-pieces. Suppose $F: R \to S$ is a $K$-quasiconformal map with Beltrami coefficient $\mu$ and $\dist(\supp(\mu), \Gamma^\varepsilon) > r$ (here “dist” denotes hyperbolic distance on $R$). Then the corresponding quasi-Fuchsian deformation of $G$ has limit set of dimension $\leq 1 + \eta$.

3. A Hölder type estimate

It is well known that quasiconformal maps satisfy a Hölder condition with exponent that depends only on the quasiconformal constant (e.g. p. 47 of [1]). In this section we wish to prove that they satisfy a pointwise Hölder type estimate with exponent close to 1 if the support of the Beltrami coefficient $\mu$ is sufficiently “thin” near the point.

**Theorem 3.1.** Suppose $\Omega$ is a $K$-quasidisk and $\mu$ is Beltrami coefficient supported on $\Omega$ (hence zero outside $\Omega$) with $\|\mu\|_\infty \leq k < 1$. Let $f_\mu$ be the corresponding quasiconformal map of the plane fixing 0, 1 and $\infty$. Given $\varepsilon > 0$ there is a $r = r(K, k, \varepsilon) < \infty$ and a $C = C(K, k, \varepsilon) < \infty$ so that the following holds. Suppose $x \in \partial \Omega$, $z \in \Omega$, $s = \dist(z, \partial \Omega)$ and $\gamma \subset \Omega$ is a hyperbolic geodesic connecting $z$ to $x$. Suppose the hyperbolic distance (in $\Omega$) from $\gamma$ to the support of $\mu$ is at least $r$. Then for any $0 < t < s$,

$$\frac{1}{C} \left( \frac{t}{s} \right)^{1+\varepsilon} \leq \frac{\diam(f(B(x,t)))}{\diam(f(B(x,s)))} \leq C \left( \frac{t}{s} \right)^{1-\varepsilon}.$$

Before giving the proof, we will recall a few facts which we will need. If $0 < a < b < \infty$ we let $R(a, b) = \{ x \in \R^n : a < |x| < b \}$. Let $L_f$ denote the inner
dilatation
\[ L_f(x) = \frac{J_f(x)}{l(f'(x))}, \]
where \( l(f'(x)) = \inf_{|h|=1} |f'(x)h| \). For \( n = 2 \) (the only case we will use here) this agrees with the usual dilatation of \( f \). Let
\[ M_f(r) = \max_{|x|=r} |f(x)|, \quad m_f(r) = \min_{|x|=r} |f(x)|. \]
Also let \( \omega_{n-1} \) be the \( n-1 \) measure of the unit sphere in \( \mathbb{R}^n \). The following combines Corollaries 2.21 and 2.34 of [14].

**Lemma 3.2.** Suppose \( f: \mathbb{R}^n \to \mathbb{R}^n \) is quasiconformal, \( f(0) = 0 \) and \( n \geq 2 \). Then
\[
\log \frac{b}{a} - \log \frac{M_f(b)}{m_f(a)} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} \, dx
\]
and
\[
\log \frac{m_f(b)}{M_f(a)} - \log \frac{b}{a} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} \, dx.
\]

The following is an easy consequence of this.

**Lemma 3.3.** Given \( K < \infty \) and \( \epsilon > 0 \) there is a \( \delta > 0 \) and \( C < \infty \) so that the following holds. Suppose \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) is a \( K \)-quasiconformal map with Beltrami coefficient supported on a set \( E \). Suppose that for \( 0 < r \leq 1 \), \( E \) satisfies \( \text{area}(E \cap B(x, r)) \leq \delta r^2 \). Then
\[
\frac{1}{C} r^{1+\epsilon} \leq \frac{\text{diam}(f(B(x, r)))}{\text{diam}(f(B(x, 1)))} \leq Cr^{1-\epsilon}.
\]

**Proof.** Let \( N \) be the smallest integer such that \( 2^{-N} \leq r \). Then
\[
\int_{R(r,1)} \frac{L_f(x) - 1}{|x|^2} \, dx \leq \sum_{n=0}^{N} \int_{E \cap R(2^{-n-1}, 2^{-n})} (K - 1)2^{2n+2} \, dx
\]
\[
\leq \sum_{n=0}^{N} (\delta 2^{-2n})(K - 1)2^{2n+2}
\]
\[
\leq 4\delta(K - 1) \sum_{n=0}^{N} 1 \leq 4\delta(K - 1) \left( \log_2 \frac{1}{r} + 1 \right).
\]
Thus by Lemma 3.2,
\[
\frac{M_f(1)}{m_f(r)} \geq \left( \frac{1}{r} \right)^{1+O(\delta)}, \quad \frac{m_f(1)}{M_f(r)} \leq \left( \frac{1}{r} \right)^{1+O(\delta)}.
\]
For quasiconformal maps \( M_f(r) \simeq m_f(r) \simeq \text{diam}(f(B(0, r))) \), so this proves the lemma. \( \square \)
We also need a few easy facts about quasidisks. The proofs are included for completeness, but the results are well known.

**Lemma 3.4.** Suppose $\Omega$ is a $K$-quasidisk and $z_0 \in B(x, r) \cap \Omega$ is a point which satisfies $\text{dist}(z_0, \partial \Omega) \geq Mr$. Then there is a $C < \infty$ and an $a > 0$ so that 
\[ \{ w \in \Omega \cap B(x, r) : \varrho(w, z_0) \geq s \} \subset \{ w \in \Omega \cap B(x, r) : \text{dist}(w, \partial \Omega) \leq Cr \exp(-as) \}. \]

**Proof.** By rescaling we may assume $r = 1$. Any point $w_0$ in $B(1) \setminus \Omega$ which is distance $d$ from the boundary can be joined to $z_0$ by a path of length at most $C$ which satisfies $\text{dist}(z_0, \partial \Omega) \geq c|z - w_0|$. The quasi-hyperbolic length of such a path is at most $C \log(1/d)$ and hence the same is true of its hyperbolic length. Thus $\varrho(z_0, w_0) \leq C \log(1/d)$ or $d \leq \exp(-\varrho(z_0, w_0)/C)$, as desired. $\blacksquare$

**Lemma 3.5.** If $\Gamma$ is a $K$-quasiarc then there is an $a > 0$ and $C < 1$ (both depending only on $K$) so that the $\varepsilon$-neighborhood of $\Gamma$ has area $\leq C \varepsilon^a \text{diam}(\Gamma)^2$.

**Proof.** This follows from the fact that quasicircles are porous, i.e., there is an $N$ (depending on $K$), such that if we divide a square $Q$ into $N^2$ disjoint subsquares, then $\Gamma$ misses at least one of them (this follows from the Ahlfors 3-point condition, e.g., [1]). After dropping down $k$ times, the squares of size $\varepsilon_k = N^{-k} \gamma$ hits have area at most $(1-N^{-2})^k \varepsilon_k^a$ where $\alpha = -\log(1-N^{-2})/\log N$. Tripling each square covers an $\varepsilon$-neighborhood of $\Gamma$ and only increases the area by a factor of 9. $\blacksquare$

The proof of Theorem 3.1 is immediate from the previous lemmas. We will now deduce some consequences that we will need later. In what follows $\Omega$ will be a bounded $K$-quasicircle with a fixed base point $z_0$ which satisfies $\text{dist}(z_0, \partial \Omega) \simeq \text{diam}(\Omega)$. We will let $W \subset \Omega$ be a subdomain bounded by hyperbolic geodesics. Moreover we assume there is an $a > 0$ so that any two geodesics in $\partial W$ are at least hyperbolic distance $a$ apart. If $W$ does not contain $z_0$ then the component of $\partial W$ which separates $z_0$ from $W$ will be called the “top edge” of $W$ and the other components will be called “bottom edges”. The following are easy consequences of the results above.

**Corollary 3.6.** With notation as above, let $\gamma$ denote the top edge of $W$ and let $\{ \gamma_k \}$ be an enumeration of the bottom edges. Let $E = \partial W \cap \partial \Omega$. Suppose $\mu$ is a Beltrami coefficient supported on $\Omega$ with $\| \mu \|_{\infty} \leq k < 1$. Then given $\eta > 0$ there is an $r = r(K, k, \eta)$ so that if the support of $\mu$ is at least hyperbolic distance $r$ from $W$ then

\[ \sum_k \left( \frac{\text{diam}(f(\gamma_k))}{\text{diam}(f(\gamma))} \right)^{1+\eta} \leq C \sum_k \frac{\text{diam}(\gamma_k)}{\text{diam}(\gamma)}, \]

and

\[ \frac{\text{dim}(E)}{1+\eta} \leq \text{dim}(f(E)) \leq (1+\eta)\text{dim}(E). \]
Corollary 3.7. Suppose $\mu$ and $\nu$ are two Beltrami coefficients on $D$ with disjoint supports and $E \subset \mathbf{T}$ is such that for every $x \in E$,

$$\liminf_{r \to 1} g(rx, \text{supp}(\nu)) = \infty.$$  

Then $\dim(f_\mu(E)) = \dim(f_{\mu+\nu}(E))$.

4. Proof of Theorem 1.1

Suppose $R = D/G$ is a Riemann surface which approximates the thrice punctured sphere near infinity and suppose $f$ is a $K$-quasiconformal deformation of $G$ to a quasi-Fuchsian group $G'$. By Lemma 2.5, $\dim(E_e(G')) = \dim(f(E_e(G))) \geq 1$, so we only have to prove the opposite inequality.

Fix some $\eta > 0$ and use Theorem 2.6 to choose $\varepsilon$ and $r$. Let $\mu$ be the Beltrami coefficient of $f$ and write $\mu = \mu_1 + \mu_2$ where each $\mu_i$ is $G$ invariant, they have disjoint supports, $\mu_2$ is supported on a finite union of $Y$-pieces and $g(\text{supp}(\mu_1), \Gamma^\varepsilon) > r$. Let $f_1$ be the deformation of $G$ to a quasi-Fuchsian group $G_1$ corresponding to $\mu_1$. By Theorem 2.6, $\dim(f_1(T)) \leq 1 + \eta$.

Let $E$ denote $E_e(G)$, minus the (at most countably many) parabolic fixed points of $G$. Then each geodesic corresponding to a point of $E$ moves arbitrarily far from $\text{supp}(\mu_2)$. Thus by Corollary 3.7,

$$\dim(E_e(G)) = \dim(f(E)) = \dim(f_{\mu_1}(E)) \leq 1 + \eta.$$  

Since this holds for every $\eta > 0$, we deduce $\dim(E_e(G')) = 1$, as desired.

5. Proof of Theorem 1.2

Let $Y_0$ be a $Y$-piece that has three equal length boundary components. Construct an $X$-piece, $X_0$, by identifying two copies of $Y_0$, which we will denote $Y_1$ and $Y_2$, along one boundary component of each. We will consider several Riemann surfaces which are unions of copies of $X_0$ with various boundary identifications. The first is the compact, genus two Riemann surface $R_0$ we obtain by identifying the two remaining boundary components of $Y_1$ and the two remaining boundary components of $Y_2$ (there are many ways to do this, but any choice will be sufficient for our purposes). Let $G_0$ be a Fuchsian group such that $R_0 = D/G_0$. We will think of $R_0$ as being labeled by a multi-graph $\Gamma_0$ with one vertex and two edges. See Figure 1. Given any multigraph $\Gamma$ which covers $\Gamma_0$ we can define an associated Riemann surface $R = D/G$ which covers $R_0$ and hence $G$ is a subgroup of $G_0$.

One such covering graph is $\Gamma_2$, the infinite regular, degree four tree. It is easy to check that Cheeger’s constant for the corresponding surface $R_2 = D/G_2$ is positive and hence the corresponding critical exponent satisfies $\delta(G_2) < 1$. See
also \[20\] and \[24\]. For future reference we will denote this number as \(\delta_2 = \delta(G_2)\). Since \(R_2\) has a finite upper bound for its injectivity radius, the limit set of \(G_2\) is the whole circle and hence \(G_2\) is a Fuchsian group for which \(\delta < \dim(\Lambda)\).

The example in Theorem 1.2 is obtained by modifying \(R_2\) in order to make \(\delta = 1\). Choose a vertex \(z_0 \in \Gamma_2\) to be the root and let \(\Gamma'_2\) be the component containing \(z_0\) when two of the four edges adjacent of \(z_0\) is removed (thus \(\Gamma'_2\) is a union of two of the four “branches” which meet at \(z_0\)). Let \(\Gamma_3\) be the multigraph with vertex set \(\mathbb{N} = \{1, 2, 3, \ldots\}\) and such that vertex \(n\) is connected to \(n + 1\) by exactly two edges. Define \(\Gamma_1\) to be the union of \(\Gamma'_2\) and \(\Gamma_3\) with \(z_0\) and \(\{1\}\) joined by two edges.

Clearly \(\Gamma_1\) covers \(\Gamma_0\) and is covered by \(\Gamma_2\). Thus the associated Riemann surface \(R_1\) covers \(R_0\) and is covered by \(R_2\) and the corresponding Fuchsian groups satisfy \(G_2 \subset G_1 \subset G_0\). We claim that \(G_1\) has the properties claimed in Theorem 1.2.

To prove this, we need to show \(\delta_1 = \delta(G_1) = 1\) and to construct a \(G_1\) invariant Beltrami coefficient \(\mu\) so that the corresponding quasiconformal deformation \(f\) satisfies \(\dim(f(\Lambda_c(G_1))) < \dim(f(\Lambda_c(G_1)))\).

The first part is easy. Considering the part of \(R_2\) corresponding to \(\Gamma_3 \subset \Gamma_1\), one easily shows that the the Cheeger constant for \(R_2\) is zero. Thus \(\delta_1 = 1\) as desired by the Elstrodt–Patterson–Sullivan formula.

To prove the second part, we will actually construct a sequence of \(G_1\)-invariant coefficients \(\{\mu_n\}\) and show that the corresponding deformations \(\{f_n\}\) satisfy

\[
\dim(f_n(\Lambda_c)) \geq 1 + \varepsilon
\]

for some \(\varepsilon\) independent of \(n\) and

\[
\dim(f_n(\Lambda_c)) \to 1
\]
as $n \to \infty$. Together, these clearly imply the desired result if $n$ is large enough.

We now define $\mu_n$. Choose $\eta_1$ so that $\delta_2 < \eta_1 < 1$. By taking $k > 0$ small enough, we may assume that any map with Beltrami coefficient bounded by $k$ is Hölder of order $\eta_1$. Taking $k$ smaller, if necessary, we may also assume $k \leq 1 - \frac{1}{2}\delta_2$.

Since $R_0$ is a compact surface, Bowen’s theorem says every deformation of $G_0$ gives a quasi-Fuchsian group whose limit set is either a circle or has critical exponent $\delta > 1$. Thus we can choose a non-trivial deformation $G'_0$ of $G_0$ so that $\|\mu\|_\infty \leq k$ and an $\varepsilon > 0$ so that $\delta(G'_0) > 1 + \varepsilon$. Let $f$ denote the corresponding deformation of $G_0$. By Astala’s theorem and the fact that $k < 1 - \frac{1}{2}\delta_2$,

$$\dim(E) \leq \delta_2 \Rightarrow \dim(f(E)) < 1.$$  \hspace{1cm} (7)

The $G_0$-invariant coefficient $\mu$ is also $G_2$-invariant (since $G_2$ is a subgroup of $G_0$). Let $G'_2$ be the corresponding quasi-Fuchsian deformation of $G_2$. Note that

$$\delta(G'_2) = \dim(\Lambda_c(G'_2)) = \dim(f(\Lambda_c(G_2))) < 1$$

by (7). Thus

$$\dim(\Lambda_c(G'_2)) = \dim(\Lambda(G'_2)) = \dim(f(T)) = 1 + \varepsilon.$$

Moreover, $\Lambda_c(G_2)$ breaks into four pieces depending on which branch (i.e., component of $R_2 \setminus X_0$ the corresponding geodesic ray eventually stays in). We claim that the $f$-image of each of the four sets has dimension equal to $\dim(f(\Lambda_c))$. Given one such piece $E$, there is clearly an element $g \in G_0$ so that $\bigcup_n g^n(E)$ is all of $\Lambda_c(G_2)$ except for one point (the attracting fixed point of $g$). Since $g$ is conjugated to a Möbius transformation by $f$ (since it is a deformation of $G_0$), this says that $f(\Lambda_c)$ is the union of one point and a countable number of Möbius images of $f(E)$. Thus $\dim(f(E)) = \dim(f(\Lambda_c))$, as desired.

The coefficient $\mu$ is also $G_1$ invariant (since $G_1 \subset G_0$) and the sequence $\{\mu_n\}$ will be defined by restricting $\mu$ to certain subregions of the disk.

Label the vertices of $\Gamma'_2$ by their distance to the root $z_0$. This gives a labeling of the corresponding $X$-pieces in $R_2$ and we will let $S^+_n \subset R_2$ be the union of all $X$-pieces with labels $\geq n$. Let $\Omega^+_n \subset \mathbb{D}$ be the preimage of $S^+_n$ under the quotient map. Similarly, let $S^-_n = R_2 \setminus S^+_n$ and let $\Omega^-_n$ be the lift of $S^-_n$. Let $\Gamma_n = \partial \Omega_n \cap \mathbb{D}$. Note that $\Gamma$ is a union of infinite geodesics (each a lift a boundary geodesic of an $X$-piece) and that any two components are a uniform hyperbolic distance apart. Note for future use that the hyperbolic distance from $S^-_1$ to $S^+_n$ is $\geq cn$ for some fixed $c > 0$.

Let $\mu_n$ be the restriction of the Beltrami coefficient $\mu$ to $\Omega^+_n$ and let $f_n$ be the corresponding quasiconformal map. We claim that (5) and (6) hold for these maps.
First we prove (5). Consider a component $W$ of $\Omega^+_0$ and let $E = \Lambda_c(G_1) \cap \partial W$. Then $f_n$ can be written as a composition of $f$ with a quasiconformal map whose dilatation is supported on $f(D \setminus \Omega_{2n})$. Applying Corollary 3.7, we see that $\dim(f_n(E)) = \dim(f(E))$. Since $\dim(f(E)) = \dim(\Lambda_c(G'_2)) = 1 + \varepsilon$ and $f_n(E) \subset \Lambda_c(G'_1)$ we clearly have $\dim(\Lambda_c(G'_1)) \geq 1 + \varepsilon$, as desired.

Next we prove (6). For any $m < \infty$, let $\Lambda_b^m \subset \Lambda(G_1)$ denote the subset corresponding to geodesic rays which never enter $\Omega^+_n$. Then $\Lambda_b(G_1) \subset \bigcup_{m=1}^{\infty} \Lambda_b^m$ and so $\delta(G_1) \leq \sup_m \dim(\Lambda_b^m)$ by Theorem 2.1. Similarly, $\delta(G'_1) = \sup_m \dim(f_n(\Lambda_b^m))$. Thus it suffices to show that given $\eta > 0$ there is a $n_0$ (independent of $m$) so that $\dim(f_n(\Lambda_b^m)) \leq 1 + \eta$ for all $n \geq n_0$.

Suppose $x \in \Lambda_b^m$ is a point corresponding to a geodesic $\gamma$. Then one of the following must be hold for $\gamma$:

1. $\gamma$ eventually never leaves $S^+_1$,
2. $\gamma$ eventually never leaves $S^-_n$ or
3. $\gamma$ alternately leaves $S^+_n$ and $S^-_n$ infinitely often.

Let $E_1$, $E_2$ and $E_3$ denote the subsets of $\Lambda_b^m$ which correspond to each of these possibilities. Note that $E_1 \subset \partial \Omega_{n,m} = \partial(\Omega^+_n \cap \Omega^-_m)$. Moreover, if $\Omega$ is a component of $\Omega_{n,m}$ then $\partial \Omega \cap T$ is the limit set of a finitely generated Fuchsian group $G$ corresponding to the Riemann surface obtained by attaching funnels to boundary components of $S_{n,m} = S^+_n \cap S^-_m$. Since $\delta(S_{n,m}) \leq \delta(R_1) = \delta_2 < 1$ we see that $\dim(\partial \Omega) \leq \delta_2$. Since $E_1$ is contained in a countable union of such sets, its dimension is also $\leq \delta_2$. Hence $\dim(f_n(\Gamma_1)) \leq 1$ by (7), independent of $n$ and $m$.

Next consider the set $E_2 = \partial \Omega^-_n \cap T$. Since the hyperbolic distance from $\Omega^-_n$ to the support of $\mu_n$ is at least $cn$ (since $\mu_n$ is supported in $\Omega_{2n}$), (4) of Corollary 3.6 says that $\dim(f(E_2)) < 1 + \frac{1}{2}\varepsilon$ if $n$ is large enough.

Finally, consider the set $E_3$. It is separated from $0$ by infinitely many alternating curves from $\Gamma_0$ and $\Gamma_n$. Given a component $\gamma \subset \Gamma_n$ the maximal components of $\Gamma_0$ separated from $0$ by $\gamma$ will be denoted $\{\gamma_k\}$. Since $\eta_1 > \delta_2$, by Lemma 2.3, they satisfy

$$\sum_k \left( \frac{\diam(\gamma_k)}{\diam(\gamma)} \right)^{\eta_1} \leq C.$$ 

By our choice of $\mu$, $f_n$ is $\eta_1$-Hölder, so we get for any $\eta > 0$

$$\sum_k \left( \frac{\diam(f_n(\gamma_k))}{\diam(f_n(\gamma))} \right)^{1+\eta} \leq \sum_k \left( \frac{\diam(\gamma_k)}{\diam(\gamma)} \right)^{\eta_1(1+\eta)} \leq C e^{-\eta_1 \eta cn} \sum_k \left( \frac{\diam(\gamma_k)}{\diam(\gamma)} \right)^{\eta_1} \leq C e^{-\eta_1 \eta cn}.$$
This is $\leq 1$ if $n$ is large enough.

On the other hand, if $\gamma \subset \Gamma_0$ and $\{\gamma_j\}$ are the maximal components of $\Gamma_n$ separated from 0 by $\gamma$ then trivially

$$\sum_j \frac{\text{diam}(\gamma_j)}{\text{diam}(\gamma)} \leq 1.$$ 

If we apply Theorem 3.1, with $\varepsilon$ so small that $(1 - \varepsilon)(1 + \eta) \geq 1 + \frac{1}{2}\eta$, then we see that if $n$ is large enough

$$\sum_j \left[ \frac{\text{diam}(f_n(\gamma_j))}{\text{diam}(f_n(\gamma))} \right]^{1+\eta} \leq C \left( \sum_j \frac{\text{diam}(\gamma_j)}{\text{diam}(\gamma)} \right)^{1+\eta/2}$$

$$\leq C e^{-\eta cn/2} \left( \sum_j \frac{\text{diam}(\gamma_j)}{\text{diam}(\gamma)} \right)$$

$$\leq C e^{-\eta cn},$$

which is $\leq 1$ if $n$ is large enough. Thus if $n$ is large (depending on $\eta$), there is a cover of $f_n(E_3)$ with $(1+\eta)$-Hausdorff sum bounded by 1. Taking $\eta$ small enough completes the proof that $\dim(f_n(A^m_b)) \leq 1 + \frac{1}{2}\varepsilon$ if $n$ is large enough (independent of $m$) and hence finishes the proof of Theorem 1.2.

Bibliography


δ-stable Fuchsian groups


Received 21 May 2002