ON THE $L^1$ APPROXIMATION OF $|f(z)|$
BY $\text{Re } f(z)$ FOR ANALYTIC FUNCTIONS

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Abstract. For angular regions of the plane, the integral of $|f(z)| - \text{Re } f(z)$, where $f(z)$ is analytic and $L^1$, can be estimated from below in terms of the $L^1$ norm of $|f(z)|$. We obtain necessary and sufficient conditions on the shape of a type of region of infinite area for which a natural generalization of the foregoing exists, and examine the consequences.

1. Introduction

Let $\Omega$ be a region of the complex plane. We denote by $L^1_a(\Omega)$ the class of functions $f(z)$ analytic in $\Omega$ and belonging to $L^1(\Omega)$ and set

$$
\delta_\Omega[f] = \iint_\Omega [ |f(z)| - \text{Re } f(z) ] \, dx \, dy, \quad f \in L^1_a(\Omega).
$$

Of course, $\delta_\Omega[f] \geq 0$. Also, obviously, if $\Omega$ has finite area, and $f(z)$ is identically 1 (or any non-negative constant) in $\Omega$, then $\delta_\Omega[f] = 0$. When $\Omega$ has infinite area, non-zero constants no longer belong to $L^1_a(\Omega)$. In that case we consider the sequence,

$$
\delta_\Omega[f_n] = \iint_\Omega [ |f_n(z)| - \text{Re } f_n(z) ] \, dx \, dy, \quad f_n \in L^1_a(\Omega), \ n = 1, 2, \ldots,
$$

where

$$
\lim_{n \to \infty} f_n(z) = 1 \text{ uniformly on every compact subset of } \Omega.
$$

We will see that, depending on the size of the “opening” of $\Omega$ at infinity, as measured by comparison to a class of standard openings when a comparison in the sense to be described below is possible, there are only two alternatives: Either there exists a sequence $f_n \in L^1_a(\Omega)$ satisfying (1.3) for which $\lim \delta_\Omega[f_n] = 0$, or $\lim \delta_\Omega[f_n] = +\infty$ for every sequence $f_n \in L^1_a(\Omega)$ satisfying (1.3).

The standard openings are defined in terms of the family of parabolic-shaped regions, $\Omega_\beta$, $0 \leq \beta \leq 1$, starting with the limiting case of a half-strip $\Omega_0$, and ending with the limiting case of an angular region, $\Omega_1$,

$$
\Omega_0 = \{ z = x + iy : |x| < 1, \ y > 0 \},
$$

$$
\Omega_\beta = \{ z = x + iy : |x| < y^\beta, \ y > 0 \}, \quad (0 < \beta \leq 1).
$$

The result is as follows:

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Theorem 1.1. With the conditions and notations (1.2), (1.3), (1.4),

(i) if $\beta < 1$, then $\lim_{n \to \infty} \delta_{\Omega}[f_n] = +\infty$ for every choice of $\{f_n\}$;

(ii) if $0 \leq \beta \leq \frac{1}{2}$, there exist $\{f_n\}$, such that $\lim_{n \to \infty} \delta_{\Omega}[f_n] = 0$.

Since $\Omega_{\beta}$ contains a translated copy of $\Omega_{\beta'}$ when $0 \cdot \beta' \cdot 1$, it is sufficient to restrict consideration to values of $\beta$ to any interval of the form, $\frac{1}{3} < \beta < \frac{1}{2}$, in proving (i). However, since the situation when $\beta < 1$ is much subtler than when $\beta = 1$, we obtain a better appreciation of it if we first consider the case separately when $\Omega$ is an angular region. This will be done in Section 2. The case, $\frac{1}{3} < \beta < 1$, follows in Section 3. The method used in Section 3 differs radically from that used in Section 2. It would be difficult to anticipate the key result, (3.1)–(3.2), of Section 3 from the key result (2.2) of Section 2, but once they are compared, they are seen to fit together well.

Except for the special value, $\beta = \frac{1}{3}$, the proof of (ii) is very easy: A simple computation shows that it is enough to set $f_n(z) = e^{iz/n}$. Namely,

$$
\delta_{\Omega}[e^{iz/n}] = \int_{\Omega} e^{-y/n} \left( 1 - \cos \frac{x}{n} \right) dx \, dy \leq \frac{2}{n^2} \int_{\Omega} x^2 e^{-y/n} \, dx \, dy
$$

$$
= \frac{4}{3n^2} \int_{0}^{\infty} y^{3\beta} e^{-y/n} \, dy = \frac{4}{3} n^{3\beta-1} \Gamma(3\beta + 1).
$$

So, if $0 \leq \beta < \frac{1}{3}$, then $\lim_{n \to \infty} \delta_{\Omega}[e^{iz/n}] = 0$. For $\beta = \frac{1}{3}$, $\{\delta_{\Omega}[e^{iz/n}]\}$ is bounded, but does not go to zero as $n \to \infty$. The problem is handled in a roundabout manner in Section 4.

As may be gathered from the list of references, there is a strong relationship of our problem with certain results of the theory of extremal planar quasiconformal mappings. The material in Section 2 and a part of the construction of Section 3 actually duplicate work in some of these references but, as far as I am aware, Part (i) of Theorem 1.1 is stronger than anything that has up to now been established in quasiconformal mapping theory. On the other hand, Part (ii) of Theorem 1.1 is equivalent to known facts in [5], [4], and [1]. Except for Section 4, however, it has been possible to keep the reasoning here completely elementary and the exposition self-contained.

It should be noted that for the case of arbitrary regions $\Omega$, even if we considered only simply-connected regions with infinite area, a comparison of $\Omega$ with a region of type $\Omega_{\beta}$ in the sense of Theorem 1.1 is of course, in general, not possible. Whether other alternatives for $\lim \delta_{\Omega}[f_n]$ can occur in the general case is an open problem.

2. Angular regions and the case $\beta = 1$

Let $\mathcal{R}_\alpha$ be the angular region

$$
\mathcal{R}_\alpha = \{z : 0 < \arg z < \alpha\}, \quad (0 < \alpha < 2\pi).
$$
Theorem 2.1. For every \( f \in L^1_\alpha(\mathscr{R}_\alpha) \) we have

\[
\delta_{\mathscr{R}_\alpha}[f] \geq \left(1 - \frac{\sin \alpha}{\alpha}\right) \int_{\mathscr{R}_\alpha} |f(z)| \, dx \, dy.
\]  

Proof. ([2, p. 124]) Let \( \Sigma = \{w = u + iv : 0 < v < \alpha\} \). Then

\[
A = \iint_{\mathscr{R}_\alpha} f(z) \, dx \, dy = \iint_{\Sigma} e^{-2iv} g(w) \, du \, dv, \quad g(w) = e^{2w} f(e^w),
\]

and

\[
B = \iint_{\mathscr{R}_\alpha} |f(z)| \, dx \, dy = \iint_{\Sigma} |g(w)| \, du \, dv = \int_0^\alpha dv \int_{-\infty}^{\infty} |g(u + iv)| \, du.
\]

Since \( B < \infty \), \( \int_{-\infty}^{\infty} g(u + iv) \, du \) exists for almost all \( v \), and since \( g(w) \) is analytic in \( \Sigma \),

\[
\int_{-\infty}^{\infty} g(u + iv) \, du = \kappa = \text{const} \quad \text{for a.a. } v, \ (0 < v < \alpha).
\]

Hence,

\[
A = \kappa \int_0^\alpha e^{-2iv} \, dv = \kappa e^{-i\alpha} \sin \alpha, \quad B \geq |\kappa| \alpha.
\]

It follows that

\[
\frac{|A|}{B} \leq \frac{|\sin \alpha|}{\alpha}.
\]

This implies (2.1). Therefore, since \( \mathscr{R}_{\pi/2} \) is congruent to \( \Omega_1 \),

\[
\delta_{\Omega_1}[f] \geq \left(1 - \frac{2}{\pi}\right) \iint_{\Omega_1} |f(z)| \, dx \, dy, \quad f \in L^1_\alpha(\Omega_1).
\]

This proves Part (i) of Theorem 1.1 for the case \( \beta = 1 \). When \( 0 \leq \beta < 1 \), no inequality like (2.2) holds; that is, it is not possible to replace the coefficient \((1 - 2/\pi)\) on the right side by any positive constant, even if that constant were allowed to depend on \( \beta \). This can be seen by trying, \( f_n(z) = e^{iz/n}, \ n \to \infty \). But we will see in Section 3, following, that the matter can be handled by means of a different type of relationship.  

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1 It is not difficult to show that the constant \((1 - 2/\pi)\) in (2.2) is best possible.
3. The case $\frac{1}{3} < \beta < 1$

We will see that (2.2) can be generalized as follows:

**Theorem 3.1.** When $\frac{1}{3} < \beta < 1$, there exists a non-negative function $P_\beta(z)$, measurable as a function of $z$, $z \in \Omega_\beta$, such that

\[
(\delta_{\Omega_\beta}|f| \geq \iiint_{\Omega_\beta} P_\beta(z)|f(z)| \, dx \, dy, \quad f \in L^1_a(\Omega_\beta).
\]

The function $P_\beta(z)$ has infinite $L^1$ norm,

\[
\iiint_{\Omega_\beta} P_\beta(z) \, dx \, dy = +\infty.
\]

When $0 \leq \beta \leq \frac{1}{3}$, no such function $P_\beta$ exists.

The proof will consist in an explicit construction of $P_\beta(z)$, and will take up the remainder of this section. It is clear that Part (i) of Theorem 1.1 will then follow immediately.

We start with some preliminaries.

**Pavlović’s inequality.**\(^2\) If $\zeta_1$, $\zeta_2$, $w$ are complex numbers with

\[
|\zeta_1|^2 + |\zeta_2|^2 \leq 2,
\]

then

\[
|\zeta_1 - \zeta_2|^2 |w| \leq 4 \left[|w| - \text{Re}(\zeta_1 w)\right] + 4 \left[|w| - \text{Re}(\zeta_2 w)\right].
\]

Suppose $\tau \in L^\infty(\Omega)$, $\|	au\|_\infty \leq 1$, $f \in L^1(\Omega)$, $\iiint_{\Omega} f(z) = \iiint_{\Omega} \tau(z) f(z)$. Then

\[
(\delta_{\Omega}|f| \geq \frac{1}{8} \iiint_{\Omega} |1 - \tau(z)|^2 |f(z)| \, dx \, dy.
\]

**Proof.** The right-hand side of (3.3) minus the left-hand side of (3.3) equals $2Q_1 + 4Q_2 + Q_3$, where

\[
\begin{align*}
Q_1 &= (2 - |\zeta_1|^2 - |\zeta_2|^2)|w| \geq 0, \\
Q_2 &= |(\zeta_1 + \zeta_2)w| - \text{Re}[(\zeta_1 + \zeta_2)w] \geq 0, \\
Q_3 &= (|\zeta_1 + \zeta_2| - 2)|w| \geq 0.
\end{align*}
\]

Relation (3.4) follows from (3.3) on setting $\zeta_1 = 1$, $\zeta_2 = \tau$, $w = f$, and integrating over $\Omega$.

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\(^2\) Miroslav Pavlović, personal communication.
In outline, the procedure to be used to take advantage of (3.4) is the following. We look for a complex-valued function $H$ on $\Omega \cup \partial \Omega$, vanishing on $\partial \Omega$ and sufficiently regular so that $H(z)$ is bounded in $\Omega$ and so that Green’s formula,

$$
0 = \frac{1}{2i} \int_{\partial \Omega} H(z)f(z) \, dz = \iint_{\Omega} H \bar{z}f(z) \, dx \, dy,
$$

holds whenever, say, $f$ belongs to $L^1_a(\Omega)$ and is continuous on $\Omega \cup \partial \Omega$. It is crucial to be able to determine $H$ in such a manner that, in addition,

(3.5) \quad \text{ess sup}\{\text{Re } H \bar{z} : z \in \Omega\} < 0.

By (3.5), and the boundedness of $H \bar{z}$, there will then exist a constant $c > 0$, such that $|1 + cH \bar{z}| \leq 1$. Set $\tau(z) = 1 + cH \bar{z}$. Then, by (3.4),

(3.6) \quad \delta_{\Omega}[f] \geq \frac{c^2}{8} \iint_{\Omega} |H \bar{z}|^2 |f(z)| \, dx \, dy

for all $f \in L^1_a(\Omega)$ that are continuous on $\Omega \cup \partial \Omega$. By means of an approximation argument one establishes that (3.6) holds for all $f \in L^1_a(\Omega)$.

It so happens that a function $H$ that allows us to carry out the above in slightly modified form for $\Omega = \Omega_\beta$, $\frac{1}{3} < \beta < 1$, has already been determined [3] in connection with a question of Hahn–Banach extensions. One starts with the disjoint decomposition,

$$
\Omega_\beta = E \cup \Omega_{11} \cup \Omega_{12} \cup \Omega_{22} \cup \Omega_{21},
$$

where

$$
\Omega_{mn} = \{z \in \Omega_\beta : (-1)^m \text{Re } z < 0, (-1)^n \text{Im } z > (-1)^n\},
$$

$$
E = \{z \in \Omega_\beta : \text{Re } z = 0 \text{ or } \text{Im } z = 1\},
$$

and defines $H$ by

(3.7) \quad H(x + iy) = \begin{cases} 
(y^\beta - x)(-\frac{1}{2}(1 - \beta)xy + iy^{1-\beta}), & z \in \Omega_{11}, \\
(y^\beta - x)(-\frac{1}{2}(1 - \beta)xy^{1-2} + iy^{2-1}), & z \in \Omega_{12}, \end{cases}

and

(3.8) \quad H(z) = -\overline{H(-z)}, \quad z \in \Omega_{21} \cup \Omega_{22}.

This $H$ is continuous in $\Omega_\beta$, vanishes on $\partial \Omega_\beta$, and is in $C^\infty$ in the separate regions $\Omega_{mn}$. The resulting function $H_z$ is well defined and bounded in all $\Omega_{mn}$, that is, in $\Omega_\beta$ outside a set of two-dimensional measure zero. One applies Green’s
formula to the portion of $\Omega_\beta$ between parallel horizontals. In the limit, as $\Omega_\beta$ is exhausted, the conclusion is that

$$\int \int_{\Omega_\beta} H_{\bar{z}} f(z) \, dx \, dy = 0, \quad f \in L^1_0(\Omega_\beta).$$

By (3.7), (3.8),

$$(3.9) \quad \Re H_{\bar{z}} = \begin{cases} -\frac{1}{2} \beta + \frac{1}{4} (1 - \beta)(2xy + 2xy^{-\beta} - y^{\beta+1}), & z \in \Omega_{11}, \\ \frac{1}{4} (1 - 3\beta) y^{2\beta - 2}, & z \in \Omega_{12}, \end{cases}$$

$$
(3.10) \quad \Im H_{\bar{z}} = \begin{cases} -\frac{1}{2} y^{1-\beta} + \frac{1}{4} (1 - \beta)[x - (1 + \beta)y^{\beta}] x, & z \in \Omega_{11}, \\ -\frac{1}{2} y^{\beta-1} + \frac{1}{4} (1 - \beta)[2(1 - \beta)y^{\beta} - 2(1 - \beta)x] xy^{\beta-3}, & z \in \Omega_{12}, \end{cases}
$$

$$
(3.11) \quad H_{\bar{z}}(-x + iy) = H_{\bar{z}}(x + iy), \quad z \in \Omega_{21} \cup \Omega_{22}.
$$

Evidently, $\Re H_{\bar{z}} < 0$ in $\Omega_{12} \cup \Omega_{22}$. For $z \in \Omega_{11}$, we note that $\Re H_{\bar{z}}$ is a linear function of $x$ for every fixed $y$, $(0 < x < y^\beta)$, whose maximum occurs at $x = y^\beta$. Thus, $\Re H_{\bar{z}} \leq \frac{1}{4} (1 - 3\beta) < 0$ in $\Omega_{11} \cup \Omega_{21}$. We therefore conclude that (3.1) holds with $P_{\beta}(z) = \frac{1}{8} c^2 |H_{\bar{z}}|^2$. As $P_{\beta}(x + iy)$ is bounded and an even function of $x$, it suffices to consider its behavior in $\Omega_{12}$, as $y \to +\infty$, in order to prove (3.2). By (3.9), (3.10),

$$
\int_0^{y^\beta} [\Re H_{\bar{z}}(x + iy)]^2 \, dx = \left( \frac{1 - 3\beta}{4} \right)^2 y^{5\beta-4},
$$

$$
\int_0^{y^\beta} [\Im H_{\bar{z}}(x + iy)]^2 \, dx = \frac{1}{4} y^{3\beta-2} + \frac{(2\beta - 1)(1 - \beta)}{12} y^{5\beta-4}
+ \frac{(1 - \beta)^2(2 - 7\beta + 8\beta^2)}{240} y^{7\beta-6},
$$

for $y > 1$. Thus, for $y > 1$,

$$
(3.12) \quad \int_{-y^\beta}^{y^\beta} P_{\beta}(x + iy) \, dx = Cy^{3\beta-2} + O(y^{5\beta-4}), \quad \text{as } y \to +\infty, \quad \left( \frac{1}{3} < \beta < 1 \right),
$$

where $C$ is a positive constant depending on $\beta$. Assertion (3.2) follows.

**Remark.** In the case $\frac{1}{3} < \beta < 1$, the exponent of the term $y^{3\beta-2}$ in (3.12) is best possible as $y \to +\infty$.

**Proof.** By (3.1), (3.12), $\delta_{\Omega_\beta}[e^{i\gamma/n}]$ has as lower bound $\frac{1}{2} C \int_1^{\infty} y^{3\beta-2} e^{-y/n} \, dy$, when $n$ is sufficiently large. But this term has the same order of magnitude, as $n \to \infty$, as the upper bound (1.5), namely $n^{3\beta-1}$.
4. The case $\beta = \frac{1}{3}$

Let $\mathcal{G} = \{z = x + iy : y > |x|^3\} = \Omega_{1/3}$. Our object is to prove Part (i) of Theorem 1.1, that is, to show that there exists a sequence $\varphi_n \in L^1_{a}(\mathcal{G})$, such that $\lim_{n \to \infty} \delta_\mathcal{G}[\varphi_n] = 0$. Although $\{\varphi_n\}$ will not be explicitly determined, we will point out that the conclusion follows from results of [5] and [1] about quasiconformal mappings of $\mathcal{G}$. Let $F$ denote the horizontal stretch of $\mathcal{G}$ onto $\mathcal{G}' = F(\mathcal{G})$ by the factor $K > 1$; i.e.,

$$F(x + iy) = Kx + iy, \quad z = x + iy \in \mathcal{G}.$$  

In line with standard terminology, $F$ is called extremal if the maximal dilatation $M$ of any arbitrary quasiconformal mapping $\zeta$ of $\mathcal{G}$ onto $\mathcal{G}'$ which agrees with $F$ on $\partial \mathcal{G}$ satisfies $M \geq K$. The mapping $\zeta: \mathcal{G} \to \mathcal{G}'$ is called uniquely extremal if $M = K$ implies $\zeta(z) \equiv F(z)$. We refer to the following result, first proved in [5] ³:

**Theorem 4.1** ([5, Sections 1–3]). The horizontal stretch mapping $F: \mathcal{G} \to \mathcal{G}'$ is uniquely extremal.

Since the mapping $F$ has complex dilatation

$$\frac{F_z}{\overline{F_z}} = \frac{K - 1}{K + 1} = k > 0,$$

the desired conclusion now follows from the following special case of a fundamental theorem of Božin, Lakic, Marković, and Mateljević:

**Theorem 4.2** ([1, p. 312]). Let $\varphi$ be a non-zero function analytic in the simply connected region $\mathcal{R}$ and let $F$ be a quasiconformal mapping of $\mathcal{R}$ with complex dilatation $\mu(z) = k \varphi(z)/|\varphi(z)|$, $0 < k < 1$. Then $F$ is uniquely extremal if and only if there exists a sequence $\varphi_n \in L^1_a(\mathcal{R})$ such that

(i) $\lim \varphi_n(z) = \varphi(z)$, uniformly on every compact subset of $\mathcal{R}$, and

(ii) $k \int_{\mathcal{R}} |\varphi_n(z)| \, dx dy - \text{Re} \int_{\mathcal{R}} \mu(z) \varphi_n(z) \, dx dy \to 0$.

In our case, $\varphi(z) \equiv 1$.

References


³ An alternative proof can be found in [4].


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