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POROSITY OF JULIA SETS OF NON-RECURRENT AND PARABOLIC COLLET–ECKMANN RATIONAL FUNCTIONS

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Abstract. It is proved that the Julia set of a rational function on the Riemann sphere whose critical points contained in the Julia set are non-recurrent (but parabolic periodic points are allowed) is porous. Next, new classes of rational functions: parabolic Collet–Eckmann and topological parabolic Collet–Eckmann are introduced and mean porosity of Julia sets for functions in these classes is proved. This implies that the upper box-counting dimension of the Julia set is less than 2.

1. Introduction

A bounded subset X of a Euclidean space (or Riemann sphere) is said to be *porous* if there exists a positive constant $c > 0$ such that each open ball B centered at a point of X and of an arbitrary radius $0 < r \le 1$ contains an open ball of radius cr disjoint from X.

If only balls B centered at a fixed point $x \in X$ are discussed above, X is called *porous at* x.

X as above is said to be *mean porous* if there exist $P, c > 0$ such that for every $x \in X$ there exists an increasing sequence of integers n_j and a sequence of points x_j such that $n_j \le Pj$, $\operatorname{dist}(x, x_j) \le 2^{-n_j}$ and $B(x_j, c2^{-n_j}) \cap X = \emptyset$.

In this paper we deal with $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, a rational function of the Riemann sphere of degree ≥ 2. In Section 3 we consider functions whose all critical points contained in the Julia set are non-recurrent. Recall that a point is non-recurrent if it is not a member of its ω-limit set. We call all the maps defined above NCP maps (abbreviation for *non-recurrent critical points*). We prove the following.

Theorem 1.1. *The Julia set of each* NCP *map, if different from* $\overline{\mathbf{C}}$, *is porous.*

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In Section 4 we introduce two classes of rational functions: parabolic Collet–Eckmann maps (abbreviated by PCE) and topological parabolic Collet–Eckmann maps (abbreviated by TPCE). Recall from [P1] that f is called Collet–Eckmann (abbreviated by CE) if there exist $\lambda > 1$, $C > 0$ such that for every f-critical point $c \in J(f)$ whose forward trajectory does not contain any other critical point and every positive integer n

$$|(f^n)'\big(f(c)\big)| \geq C\lambda^n. \tag{1.1}$$

This notion was introduced for the first time for unimodal maps of an interval in [CE1] and [CE2].

In presence of parabolic points a weaker definition seems appropriate. Instead of n at the right-hand side of (1.1) we put smaller integers, which we call *rescaled times*. Namely when the forward trajectory of c passes close to parabolic points, instead of iterating by f we iterate by $f^{a_1}, f^{a_2}, \ldots$ so that the derivatives $|(f^{a_i})'|$ are about 2. Analogously with the use of the rescaled time we generalize from [P3] and [PR2] the notion of topological Collet–Eckmann maps to TPCE maps. This class is larger than NCP. We prove the following

Theorem 1.2. *The Julia set of each* PCE *or* TPCE *map, if different from* $\overline{\mathbf{C}}$, *is mean porous.*

As an immediate consequence of this result we get, due to [KR], the following.

Corollary 1.3. *The upper box-counting dimension of the Julia set of each* NCP, PCE *or* TPCE *map is less than* 2 ($\overline{BD}\big(J(f)\big) < 2$).

The notions of porosity and mean porosity have appeared in several contexts and for a short survey and some bibliographical references the reader may see the paper [KR]. Koskela's and Rohde's theorem implying Corollary 1.3 says that if X is mean porous then $\overline{BD}(X) < 2$. In fact, instead of referring to [KR], we could prove the so-called *box mean porosity* as in [PR1] and then refer to the easy theorem saying that the box mean porosity of X implies that $\overline{BD}(X) < 2$, whose simple proof (by Michal Rams) was provided in [PR1].

For rational functions expanding on a Julia set the proof of porosity is easy (it was folklore since a long time). Just pull-back large scale holes to all small scales by iteration of inverse branches of f. For NCP maps without parabolic periodic points the proof is similar. One pulls back large disks, meeting critical points only finite number of times (bounded by the number of critical points in $J(f)$), hence resulting small disks are boundedly distorted. The same goes through for CE maps, for every $z \in J(f)$. Namely for a positive lower density set of positive *good* integers n one can pull a large disk B with the origin at $f^n(z)$ back to a neighbourhood of z, meeting critical points only uniformly bounded number of times (the bound depending only on f). This is called the topological Collet–Eckmann property (abbreviated by TCE). Therefore, for every $z \in J(f)$, in most scales around z, one finds boundedly distorted holes that yield mean porosity [PR1].

In presence of parabolic periodic points but in absence of critical points in $J(f)$ porosity was proved by Lucas Geyer [G]. The idea was to use additionally porosity at points close to parabolic ω in scales comparable to the distance from ω, true since the Julia set close to ω is confined in cusp-like channels.

Here, in Section 3 we prove Theorem 1.1 combining this idea with finite criticality while pulling back, as for non-parabolic NCP.

In Section 4 we introduce PCE and TPCE properties and prove that TPCE is topologically invariant.

In Section 5 we apply the ideas of Section 3 to prove Theorem 1.2. The hardest point is to prove that PCE implies TPCE; the latter is a version of TCE in presence of parabolic points, with *good* integers considered with respect to the rescaled time.

Some technical difficulties appear. We need to improve the estimate of an average distance of any trajectory from critical points from [DPU], applied in [PR1]. This is done in Appendix A. We need also to prove that diameters of components of preimages under iterates of f of any small disk are uniformly small (*backward Lyapunov stability*) to know that the rescaled times along blocks of a trajectory and shadowing critical trajectory coincide. This was sketched in [P1] in the non-parabolic case under so called summability condition, weaker than CE. Here we provide a precise proof, in Appendix B.

Rational PCE functions and TPCE functions are introduced here for the first time. Similarly to TCE the TPCE property is topologically invariant. In Section 6 we continue sketching a theory analogous to the theory of CE and TCE in [PR1], [PR2] and [P3].

Historical remarks on dimension. The class of NCP maps forms a joint extension of parabolic and semihyperbolic maps (the former without critical points in $J(f)$, the latter without parabolic points). For parabolic maps Corollary 1.3 follows from the results obtained in [ADU] and [DU]. It has been mentioned in [U1] that the Hausdorff dimension of the Julia set of each NCP map is less than 2 and in [U2] some number of sufficient conditions was provided for the Hausdorff dimension and the upper box-counting dimension of the Julia set of an NCP map to coincide. This therefore gave a partial contribution towards the inequality $\overline{BD}(J(f)) < 2$ (denote it by $(*)$) for NCP maps. $(*)$ was proved for CE maps with only one critical point in the Julia set in [P1] and [P2]. This was the first class of rational maps containing reccurrent critical points, for which this property was verified. The proofs used ergodic theory. Later, as we already mentioned, $(*)$ was proved in [PR1] for all CE maps, without using ergodic theory.

Independently a different class was provided by C. McMullen [McM]. A large class of maps satisfying $(*)$, including CE, was provided recently by J. Graczyk and S. Smirnov [GS2].

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2. Preliminaries on distortion, parabolic points and non-recurrent dynamics

If $h\colon D \to \overline{\mathbf{C}}$ is an analytic map, $z \in \overline{\mathbf{C}}$, and $r > 0$, then by $\operatorname{Comp}(z,h,r)$ we denote the connected component of $h^{-1}\big(B(h(z),r)\big)$ that contains z. Fix now $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, a rational function. Denote by Ω, or $\Omega(f)$, the set of all periodic parabolic points for f, where $\omega \in J(f)$ is called parabolic if there exists $q \geq 1$ such that $f^q(\omega) = \omega$ and $(f^q)'(\omega) = 1$. Passing to a sufficiently high iterate does not change the Julia set, so we may assume that for every $\omega \in \Omega$, $f(\omega) = \omega$ and $f'(\omega) = 1$. Assume that the spherical metric on $\overline{\mathbf{C}}$ is scaled so that $\operatorname{diam}(\overline{\mathbf{C}}) = 1$. All diameters and absolute values of derivatives are considered with respect to this spherical metric. However, we assume that $\Omega \subset \mathbf{C}$ and close to Ω we use the euclidean distance $|x-y|$.

Fix for the rest of the paper a number $\delta > 0$ so small that for each $\omega \in \Omega$, $B(\omega, 2\delta) \cap \operatorname{Crit}(f) = \emptyset$ (where by $\operatorname{Crit}(f)$ we denote the set of all f-critical points, i.e., points where $f' = 0$), $f^\tau\big(B(\omega_1,\delta)\big) \cap B(\omega_2,\delta) = \emptyset$ for $\omega_1 \neq \omega_2$ and $\tau = 0, 1$, $f|_{B(\omega,\delta)}$ is injective and $|f'|\,|_{B(\omega,\delta)} < 2$. We also require $\delta > 0$ to be so small that there exists a unique holomorphic inverse branch $f_\omega^{-1}\colon B(\omega, 2\delta) \to \mathbf{C}$ of f mapping ω to ω. This inverse branch is contracting when restricted to $J(f)$. We may even require for an arbitrary $0 < \sigma < 1$ that $\delta > 0$ is so small that the following is true (use Fatou coordinates to see this, [DH, Exposé IX, I2]).

Lemma 2.1. *For every $x \in B(\omega,\delta) \cap J(f)$ and every $n > 0$ there exists the inverse branch $f_\omega^{-n}\colon B = B(x, \sigma|x-\omega|) \to B(\omega, 2\delta)$. Moreover, $(f_\omega)^{-n}(B) \subset B\big(f_\omega^{-n}(x), \sigma|f_\omega^{-n}(x) - \omega|\big)$.*

Other restrictions on δ will appear in the course of the paper. By the Fatou flower theorem and the classification theorem of connected components of the Fatou set, we may find also $\operatorname{Const}\delta < \theta < \frac{1}{2}\delta$ such that for all $x \in J(f) \setminus B(\omega,\delta)$, the ball $B(x,\theta)$ is disjoint from the forward orbit of all critical points contained in the Fatou set.

Finally, we assume $\theta = \theta(f)$ to be small enough to satisfy the following.

Lemma 2.2. *For every* NCP *function f there exist $\theta > 0$ and $M \geq 0$ such that for every $x \in J(f) \setminus B(\Omega,\delta)$ every integer $n \geq 0$, every component V of $f^{-n}\big(B(x,\theta)\big)$ is simply connected and the restriction $f^n|_V$ has at most M critical points* (*counted with multiplicities*).

Proof. This lemma follows from [Ma, Theorem II]. See also [CJY, Theorem 2.1] or [U1, Lemma 2.12 and Lemma 5.1]. A crucial step in the proof of Lemma 2.2 is that given an arbitrary $\varepsilon > 0$ there exists θ so that the diameters of all the above components are less than ε. The latter property is called ([Le]) *backward Lyapunov stability.* □

Remark 2.3. If θ is small enough the assertion of Lemma 2.2 holds also for $x \in B(\omega, \delta)$ if V is any component of $f^{-(n-1)}(W)$ for W any component of $f^{-1}\big(B(x,\theta)\big)$ different from $f_\omega^{-1}\big(B(x,\theta)\big)$.

In the sequel we shall apply this to B as in Lemma 2.1 so we assume that σ is small enough to satisfy $2\sigma\delta \le \theta$. Assume moreover (needed later at one place): $\sigma\delta\|f'\| \le \theta$, where $\|f'\| := \sup_{z\in\overline{\mathbf{C}}} |f'(z)|$.

A step in proving Lemma 2.2 is the following lemma by Ricardo Mañé [Ma] (see also [P1, Lemma 1.1]) true for any rational function f.

Lemma 2.4. *For every integer $M \ge 0$ and $0 < r < 1$ the following holds*:

1. *For every $\varepsilon > 0$ there exists $\theta > 0$ such that for every $x \in J(f) \setminus B(\Omega, \delta)$, every integer $n \ge 0$ and every component V of $f^{-n}\big(B(x,\theta)\big)$ such that the restriction $f^n|_V$ has at most M critical points* (*counted with multiplicities*), *for every component V' of $f^{-n}\big(B(x,r\theta)\big)$ one has* $\operatorname{diam}(V') \le \varepsilon$.
2. $\operatorname{diam}(V') \to 0$ *for $n \to \infty$ uniformly* (*i.e., independently of x and V'*).

The following result is a part of the "bounded distortion" lemma that has been proved in [P1, Lemma 1.4] and [PR1, Lemma 2.1].

Lemma 2.5. *For each $\varepsilon > 0$ and $D < \infty$ there are constants C_1 and C_2 such that the following holds for all rational maps $F\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, all $x \in \overline{\mathbf{C}}$, all $\frac{1}{2} \le r < 1$ and all $0 < \gamma \le \frac{1}{2}$*:

Assume that V (*or V'*) *is a simply connected component of $F^{-1}\big(B(x,\gamma)\big)$* (*or $F^{-1}\big(B(x,r\gamma)\big)$*) *with $V \supset V'$. Assume further that $\overline{\mathbf{C}} \setminus V$ has diameter at least ε and F has at most D critical points* (*counted with multiplicities*) *in V. Then*

$$\text{(a)} \qquad |F'(y)| \operatorname{diam}(V') \le C_1(1-r)^{-C_2}\gamma$$

for all $y \in V'$. Furthermore, if $r = \frac{1}{2}$ and $0 < \tau < \frac{1}{2}$, let $B'' = B(z, \tau\gamma)$ be any disk contained in $B(x, \frac{1}{2}\gamma)$ and let V'' be a component of $F^{-1}(B'')$ contained in V'. Then

$$\text{(b)} \qquad \operatorname{diam}(V'') \le C_3 \operatorname{diam}(V')$$

with $C_3 = C_3(\tau, \varepsilon, D)$ and $\lim_{\tau\to 0} C_3(\tau, \varepsilon, D) = 0$, and

$$\text{(c)} \qquad V'' \text{ contains a disk of radius } \ge C_4 \operatorname{diam}(V')$$

around every preimage of $F^{-1}(z)$ contained in V''. Here $C_4 = C_4(\tau, \varepsilon, D) > 0$.

In Section 3, to consider NCP maps, we will only need part (c) of this lemma, (a) and (b) will be needed in Section 4. In applications we will skip the dependence of constants on ε.

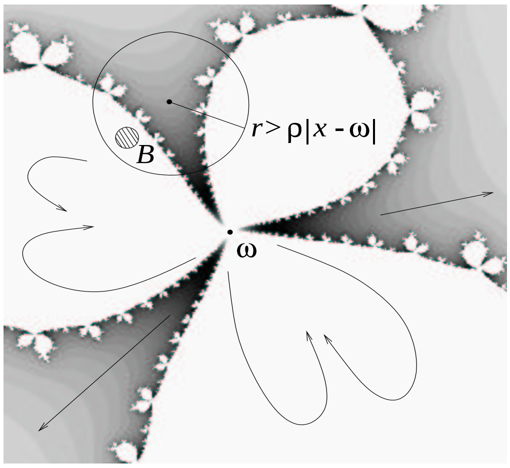

Figure 1. A "flower" for "rabbit", $f(z) = z^2 - 0.12 + 0.66i$.

The last fact stated in this section follows for points x close to Ω from the Fatou flower theorem, $J(f)$ confined in cusp-like sectors, see Figure 1.

Lemma 2.6. *If $\Omega \neq \emptyset$ then for every $\varrho > 0$ there exists $c = c(\varrho) > 0$ such that for each $x \in J(f)$ and each r, $\varrho \operatorname{dist}(x, \Omega) \leq r \leq 1$, there exists an open ball $B \subset B(x, r) \setminus J(f)$ with radius cr.*

3. The proof of Theorem 1.1

Fix $z \in J(f)$ and define

$$T(z) = \{n \geq 0 : f^n(z) \notin B(\Omega, \delta)\}$$

and

$$S(z) = \{n \geq 0 : n - 1 \in T(z) \text{ if } n > 0 \text{ and } f^n(z) \in B(\Omega, \delta)\}.$$

Given $m \in S(z)$ we find a unique $\omega \in \Omega$ such that $f^m(z) \in B(\omega, \delta)$ and then we define

$$R_m(z) = \{k \geq 0 : 2^{-k}\theta > \sigma |f^m(z) - \omega|\}$$

and

$$S_m(z) = \big\{n \geq m : n < \min\{T(z) \setminus [0, m-1]\}\big\}.$$

We first equip all the sets $T(z)$, $R_m(z)$ and $S_m(z)$, $m \in S(z)$, with the natural order inherited from the set of non-negative integers and then we further order the disjoint union

$$W(z) = T(z) \bigoplus \bigoplus_{m \in S(z)} \Big(R_m \bigoplus S_m\Big)$$

by declaring that for each $m \in S(z)$ the element $m - 1$ of $T(z)$ precedes all the elements of R_m, the last element of R_m (if it exists, i.e., if $f^m(z) \notin \Omega$) precedes the first element of S_m and the last element of S_m precedes the first element of $T(z)\setminus[0,m-1]$. In this way we have equipped $W(z)$ with a linear order isomorphic to the natural order of positive integers. For each $t\in W(z)$ we write $n(t):=n$ if $t\in\widetilde{R}_n(z)$, or $\widetilde{S}_m(z)$ or $\widetilde{T}(z)$ if $t=t(n)$, $n\in S_m(z)$ or $T(z)$, and $k(t):=k$ if $t\in\widetilde{R}_n(z)$. Here the tilde means the appropriate set is considered as embedded by t in $W(z)$. We set $k(t)=0$ for $t\notin\widetilde{R}_n$. Note that if for $m\in S(z)$, $f^m(z)\in\Omega$, then $R_m(z)$ is infinite and there are no elements of $W(z)$ after $\widetilde{R}_m$. We do not treat t's as integers here, we need only the order in $W(z)$. See Figure 2.

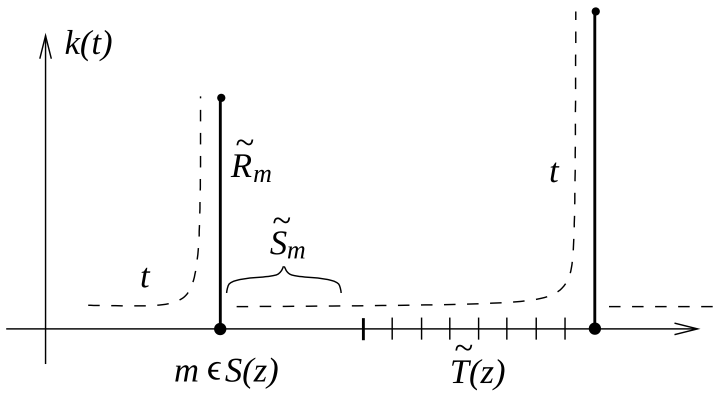

Figure 2. Order t and coordinates k,n in $W(z)$.

Now, to each $t\in W(z)$ we ascribe a number $r(z,t)>0$ as follows.

(a) $r(z,t)=\operatorname{diam}\big(\mathrm{Comp}(z,f^{n(t)},\tfrac12\theta)\big)$ if $t\in\widetilde{T}(z)$.

(b) $r(z,t)=\operatorname{diam}\big(\mathrm{Comp}(z,f^{n(t)},2^{-(k(t)+1)}\theta)\big)$ if $t\in\widetilde{R}_{n(t)}(z)$.

(c) $r(z,t)=\operatorname{diam}\big(\mathrm{Comp}(z,f^{n(t)},\tfrac12\sigma|f^{n(t)}(z)-\omega|\big)$ if $t\in\widetilde{S}_m(z)$ for some $m\in S(z)$.

Each of the connected components appearing in the definition of $r(z,t)$, with $\tfrac12\theta, 2^{-(k+1)}\theta$, $\tfrac12\sigma$ in (a), (b), (c) respectively replaced by numbers twice larger, will be denoted by $V_t(z)$.

Our next goal is to prove the following.

Lemma 3.2. *There exists a constant $C>0$ such that for all $z\in J(f)$ and all $t\in W(z)$*

$$\frac{r(z,\bar t)}{r(z,t)}\geq C,$$

where $\bar t$ is the successor of t in the order introduced in $W(z)$.

Proof. Suppose first that $t\in\widetilde{T}(z)$. Then for $n=n(t)$

$$\mathrm{Comp}\big(f^n(z),f,\tfrac12\theta\big)\supset B\bigg(f^n(z),\frac{\theta}{2\|f'\|}\bigg)$$

and therefore it follows from Lemma 2.2 and Lemma 2.5(c) applied with $\gamma=\theta$, $\tau=1/(2\|f'\|)$ and $D=M$ that

$$\begin{aligned} r(z,\bar t) &\ge \operatorname{diam}\left(\operatorname{Comp}\left(z,f^n,\frac{\theta}{2\|f'\|}\right)\right) \\ &\ge C_4\big((2\|f'\|)^{-1},M\big)\operatorname{diam}\big(\operatorname{Comp}\big(z,f^n,\tfrac12\theta\big)\big) \\ &= C_4\big((2\|f'\|)^{-1},M\big)r(z,t). \end{aligned}$$

Suppose in turn that $t\in\widetilde R_n(z)$ for some $n\in S(z)$. Then using Remark 2.3 and Lemma 2.5(c) applied with $\gamma=2^{-k}\theta$ and $\tau=\frac12$, if also $\bar t\in\widetilde R_n(z)$, we obtain

$$\begin{aligned} r(z,\bar t) &= \operatorname{diam}\big(\operatorname{Comp}\big(z,f^n,2^{-(k+2)}\theta\big)\big) \\ &\ge C_4(\tfrac12,M)\operatorname{diam}\big(\operatorname{Comp}(z,f^n,2^{-(k+1)}\theta)\big) \\ &= C_4(\tfrac12,M)r(z,t). \end{aligned}$$

If $\bar t\in\widetilde S_n(z)$, then the first equality is replaced by the inequality $\ge$.

Finally suppose that $t\in\widetilde S_m(z)$ for some $m\in S(z)$. Then for $n=n(t)$

$$\begin{aligned} \operatorname{Comp}\left(f^n(z),f,\frac{\sigma}{2}|f^{n+1}(z)-\omega|\right) &\supset B\left(f^n(z),\frac{\sigma}{2\|f'\|}|f^{n+1}(z)-\omega|\right) \\ &\supset B\left(f^n(z),\frac{\sigma}{2\|f'\|}|f^{n}(z)-\omega|\right). \end{aligned}$$

Hence, as $\sigma\delta\le\theta$, using again Lemma 2.2 and Remark 2.3, it follows from Lemma 2.5(c) applied with $\gamma=\sigma|f^n(z)-\omega|$ and $\tau=1/(2\|f'\|)$ that

$$r(z,\bar t)\ge\operatorname{diam}\left(\operatorname{Comp}\left(z,f^n,\frac{\sigma}{2\|f'\|}|f^n(z)-\omega|\right)\right)\ge C_4\big((2\|f'\|)^{-1},M\big)r(z,t).$$

provided $\bar t\in\widetilde S_m(z)$. If $\bar t\in\widetilde T(z)$, we obtain the same inequality since $\theta\ge\sigma\|f'\|\delta\ge\sigma|f^{n+1}(z)-\omega|$. So, the proof is complete by setting $C=C_4\big((2\|f'\|)^{-1},M\big)$. ▫

Since $z\in J(f)$ and $J(f)$ contains only non-recurrent critical points, it follows from Lemma 2.2, Lemma 2.4 and from the local behaviour around parabolic points that we obtain the following lemma.

Lemma 3.3. *For every $z\in J(f)$*

$$\lim_{t\to\infty}r(z,t)=0.$$

Proof. If $t\to\infty$ implies $n(t)\to\infty$ then for $t\in\widetilde T(z)$, i.e., $f^{n(t)}(z)\notin B(\Omega,\delta)$, we can use Lemma 2.2 and Lemma 2.4 with $r=\frac12$. To cope with $t\in\widetilde S_m$ we use

Lemma 2.1 which allows to consider only s preceding $\bar{t} \in \widetilde{R}_m$, $k(\bar{t}) = 0$, hence refer to the previous case, since $f^s(z) \notin B(\Omega, \delta)$.

If $n(t) \not\to \infty$, then $f^m(z) = \omega \in \Omega$ for some $m = m(t)$, so that $\bar{t} \in \widetilde{R}_m$ for all $\bar{t} \geq t$. Then for $\bar{t} \to \infty$, $k(\bar{t}) \to \infty$. Hence

$$r(z, \bar{t}) = \operatorname{diam} \operatorname{Comp}(z, f^m, 2^{-(k(\bar{t})+1)}\theta) \to 0. \ \square$$

We now want to do the last step in the proof of Theorem 1.1. So, fix $z \in J(f)$ and consider an arbitrary radius $0 < r \leq \frac{1}{2}\theta$. Note that $r(z, 0) = \frac{1}{2}\theta$, where 0 is the least element of $W(z)$. It follows from Lemma 3.3 that there exists a maximal element $t \in W(z)$ such that $r \leq r(z, t)$. Using Lemma 3.2 we then conclude that for $\bar{t}$, the successor of t,

$$r(z, \bar{t}) < r \leq r(z, t) \leq C^{-1} r(z, \bar{t}).$$

Combining now Lemma 2.5(c), Lemma 2.2, Remark 2.3 and the definition of the numbers $r(z, t)$ we conclude that there exists a constant $\eta > 0$ independent of z and t and a ball $B \subset B\bigl(z, r(z, \bar{t})\bigr) \setminus J(f) \subset B(z, r) \setminus J(f)$ with radius $\geq \eta r(z, \bar{t}) \geq \eta C r$. This ball is in the pullback of a ball existing by Lemma 2.6.

In the case $\Omega = \emptyset$, due to the assumption $J(f) \neq \overline{\mathbf{C}}$ implying that $J(f)$ is closed nowhere dense, there exists $c > 0$ such that for each $x \in J(f)$ there exists a ball $B \subset B(x, \frac{1}{2}\theta) \setminus J(f)$ with radius c. This remark plays the role of Lemma 2.6 for x far from Ω in the former case. $\square$

4. Parabolic Collet–Eckmann maps

The Collet–Eckmann property for rational maps was introduced in [P1] as follows. There exist $\lambda > 1$, $C > 0$ such that for every f-critical point $c \in J(f)$ whose forward trajectory does not contain any other critical point (we call later on such a critical point *exposed*) and every positive integer n

$$|(f^n)'\bigl(f(c)\bigr)| \geq C\lambda^n.$$

In presence of parabolic periodic points we shall consider an adequate weaker property: *parabolic Collet–Eckmann.* Let us first introduce an adequate *rescaled time.* Consider an arbitrary $z \in J(f)$ as in the previous sections. Suppose that $0 \leq i < j$ are such integers that for all $i \leq \tau \leq j$ we have $f^\tau(z) \in B(\omega, \delta)$. Suppose

$$f(x) = x + a_\omega (x - \omega)^{p+1} + \cdots \tag{4.0}$$

for $a_\omega \neq 0$ and an integer $p = p_\omega \geq 1$, in a neighbourhood of ω. Then define

$$n(i, j) = E\bigl((p+1)\log_2(|f^j(z) - \omega| / |f^i(z) - \omega|)\bigr) + 1, \tag{4.1}$$

where E stands for *Entier*, i.e., $E(x)$ is the least integer not exceeding x. We have $n(i,j) > 0$ since ω is "weakly" repelling in the cusp-like sectors containing $J(f) \cap B(\omega, \delta)$, see Figure 1. It is rigorously visible in the Fatou coordinates [DH, Exposé IX, I.2]. By the inequality $|f'| \le \frac{3}{2}$ in $B(\Omega, \delta)$ true for every $\delta > 0$ small enough, we have on the other hand $n(i,j) \le j - i$. This is so since

$$(4.2)\qquad \begin{aligned} 2^{n(i,j)} &\le 2 \cdot \frac{|f^j(z) - \omega|^{p+1}}{|f^i(z) - \omega|^{p+1}} \sim 2 \frac{|f^{j+1}(z) - f^j(z)|}{|f^{i+1}(z) - f^i(z)|} \\ &\sim 2 |(f^{j-i})'\big(f^i(z)\big)| \le 2 \cdot \Big(\frac{3}{2}\Big)^{j-i}. \end{aligned}$$

The similarity symbol $\sim$ means the equality up to a factor close to 1. The first similarity follows directly from (4.0). The second similarity follows for example from Koebe's distortion lemma estimate for $f_\omega^{-(j-i)}$ on $B\big(f^j(z), \sigma|f^j(z) - \omega|\big)$, see Lemma 2.1. Since $|f^{j+1}(z) - f^j(z)|$ is much smaller than $\sigma|f^j(z) - \omega|$ for δ small, the map $f^{-(j-i)}$ is almost conformal affine on $B\big(f^j(z), |f^j(z) - f^{j+1}(z)|\big)$.

Given $n \ge 0$ let $0 \le i_s < n$ be consecutive integers for $s = 0, 1, \ldots$, such that $f^{i_s}(z) \in B(\Omega, \delta)$ and $i_s = 0$ or $f^{i_s - 1}(z) \notin B(\Omega, \delta)$ in the case $i_s > 0$. In the terminology of Section 3, i_s are consecutive integers in $S(z)$. For each s denote the point $\omega \in \Omega$ such that $f^{i_s}(z) \in B(\omega, \delta)$, by ω_s. Let $j_s : i_s \le j_s \le n$ be the largest integer such that $f^\tau(z) \in B(\omega_s, \delta)$ for all $i_s \le \tau \le j_s$. We define the *rescaled time* $\phi(n) = \phi(n, z)$ by

$$(4.3)\qquad \phi(n) := \sum_{s : i_s \le n} n(i_s, j_s) + n - \sum_{s : i_s \le n} (j_s - i_s).$$

Sometimes we denote $\phi(n)$ by $\hat{n}$ or $\hat{n}(z)$, or use the notation $\hat{n}$ for integers in the range of ϕ (i.e., interpreted as the *rescaled time*).

Definition 4.1. We call a rational map f *parabolic Collet–Eckmann* (abbreviated by PCE) if there exist $\lambda > 1$, $C > 0$ such that for every exposed f-critical point $c \in J(f)$ for $z = f(c)$ and for every positive integer n

$$|(f^n)'(z)| \ge C \lambda^{\hat{n}(z)}.$$

Note that this property does not depend on the base of logarithm in the definition of $\phi(i,j)$ (and $\phi(n) = \hat{n}$). Indeed such a change of the base would multiply $\hat{n}$ by a bounded factor, so it would change only λ in Definition 4.1.

Analogously to [PR1, Lemma 2.2] (*uniform density of good times* property), [P3] and [PR2] (*topological Collet–Eckmann*) we shall define *topological parabolic Collet–Eckmann* property.

First we introduce more notation. Similarly to ϕ we shall define the *rescaled parameter* Φ on $W(z)$, see Section 3 for the definition of $W(z)$. We shall consider this linearly ordered set as the set of non-negative integers (equipped with the arithmetic operations). Then define

$$\Phi(t) = \sum_{s: i_s \le n} n(i_s, j_s) + t - \sum_{s: i_s \le n} (j_s - i_s).$$

Given $\delta > 0$ and $\theta > 0$ denote for $t \in W(z)$ similarly as in Section 3 (with $\theta = \sigma$) the sets

$$\mathrm{Comp}(z, f^{n(t)}, \theta) \quad \text{for } t \in \widetilde{T}(z), \tag{4.4a}$$

$$\mathrm{Comp}(z, f^{n(t)}, 2^{-k(t)}\theta) \quad \text{for } t \in \widetilde{R}_{n(t)}, \tag{4.4b}$$

$$\mathrm{Comp}\big(z, f^{n(t)}, \theta|f^{n(t)}(z) - \omega|\big) \quad \text{for } t \in \widetilde{S}_m \tag{4.4c}$$

by $V_t(z)$.

Again we denote sometimes $\Phi(t)$ by $\hat{t}$ or use the notation $\hat{t}$ for integers in the range of Φ. We denote a right inverse of Φ by Ψ. Let us choose for example as $\Psi(\hat{t}\,)$ the least t such that $\Phi(t) = \hat{t}$.

Write finally $\widehat{V}_{\hat{t}}(z) := V_{\Psi(\hat{t}\,)}(z)$. Given $\hat{t}$ we sometimes write t for $\Psi(\hat{t}\,)$, $V_t(z)$ for $\widehat{V}_{\hat{t}}(z)$ etc.

Given $z \in J(f)$, $\delta > 0$, $\theta > 0$ and $M < \infty$ we call $\hat{t}$ a *good hat-integer* and denote the set of good hat-integers by $G(z)$, if $f^{n(\Psi(\hat{t}\,))}$ has at most M critical points (counted with multiplicity) in $\widehat{V}_{\hat{t}}(z)$.

Definition 4.2. We call f *topological parabolic Collet–Eckmann* (abbreviated by TPCE) if there exist $\delta, \theta > 0$, $0 < \kappa \le 1$ and $M < \infty$ such that for every $z \in J(f)$ the lower density of $G(z)$ in $\mathbf{N}$ is at least κ,

$$\inf_{\hat{t}} \frac{\#(G(z) \cap [1, \hat{t}\,])}{\hat{t}} \ge \kappa. \tag{4.5}$$

Remark 4.3. (a) One can call t a *good integer* if $f^{n(t)}$ has at most M critical points (counted with multiplicity) in $V_t(z)$. Then TPCE means that if we divide $[0, t_0]$ into blocks of Φ-preimages of points $\hat{t} \le \Phi(t_0)$ then at least κ proportion of blocks contains good integers. Note that if an integer $t \in \Phi^{-1}(\hat{t}\,)$ is good then all $s > t$, $s \in \Phi^{-1}(\hat{t}\,)$ are good, since for δ small, $f_\omega^{-1}\big(B\big(f^{n(s)}(z), \theta|f^{n(s)}(z) - \omega|\big)\big) \subset B(f^{n(s-1)}(z), \theta|f^{n(s-1)}(z) - \omega|)$ by Lemma 2.1.

In fact, by the same argument, all $s : t \le s \le s(t)$ for $s(t)$ the last element of $\widetilde{S}_m(z)$ where m is defined by $t \in \widetilde{T}_m \cup \widetilde{S}_m$, are *good.*

Note that the inclusions in Lemma 2.1 with $\sigma = \theta$ hold for adequate δ and θ. Indeed, first shrink the original δ in the definition of TPCE to some δ' so that these inclusions hold. Unfortunately t good may become not good if $f^n(t) \notin B(\omega, \delta')$

since $\theta > \theta|f^{n(t)}(z) - \omega|$. We avoid this trouble by setting any new $\theta' \le \theta\delta'$. Then t is good.

Finally find for this θ' a new δ'' so that the inclusions hold. In case $f^n(t) \in B(\omega, \delta') \setminus B(\omega, \delta'')$, by the inclusions for θ in Lemma 2.1, $s(t)$ found for δ' is good, hence $s(t)+1$, for $\theta' = \frac{1}{2}\theta\delta'$, is good. Though t can be not good for δ'', θ' it is accompanied by good $s(t)+1$. Thus the property PTCE is preserved, with maybe different κ resulted from not accounting to *good* the elements of blocks of length $\log_2(\delta'/\delta'')$ preceding good elements.

(b) For each z and good t we can assume that all $f^j(V_t(z))$, $j = 0, \dots, n(t)$, have small diameters if θ is small enough. This follows from the bounded criticality for θ replaced by, say, 2θ; see Lemma 2.4. For $f^{n(t)} \in B(\Omega, \delta)$ use Lemma 2.1, compare the proof of Lemma 3.3. In particular, all $f^j\big(V_t(z)\big)$ are topological discs.

(c) κ in Definition 4.2 can be arbitrary at the cost of M, see [PR2, Section 2]. The idea of the proof is that each gap between two consecutive good $\hat{t}$ and $\hat{t}'$ can be in κ proportion filled with good hat-integers for $G\big(f^{n(\Psi(\hat{t}))}(z)\big)$. This gives in Definition 4.2 the proportion of non-good hat-integers $1-\kappa$ decreased to $(1-\kappa)^2$. M is replaced by $2M$. We can continue this procedure. The proof uses the observation made in (b).

The name *topological* preceding Collet–Eckmann is explained by the following.

Proposition 4.4. *Topological Collet–Eckmann is a topological property, namely if there exists h: $U(f) \to U(g)$ a homeomorphism between neighbourhoods of $J(f)$ and $J(g)$ that conjugates f to g on $U(f)$, i.e. $hf = gh$ and g is* TPCE *then f is also* TPCE.

(This proposition is placed here to explain the definition; it is not needed in the further course of Section 4.)

Proof. Suppose there exists h: $U(f) \to U(g)$ a conjugating homeomorphism as above. First notice that h is bilipschtz at $\omega, h(\omega)$ in Julia set. Namely

$$\log_2|x-\omega| - C \le \log_2|h(x) - h(\omega)| \le \log_2|x-\omega| + C \tag{4.6}$$

for a constant C and every $x \in J(f)$. Moreover (4.6) holds for $x \in Q$ where $Q := \{x \in B(\omega, \delta) :$ there exists $j > 0,\ f^j(x) \notin B(\omega,\delta)\}$. To prove this, note first that p, the number of petals at ω, is preserved by h. Use next Fatou's coordinates $w = w(z) = 1/(z-\omega)^p$, [DH, Exposé IX, I.2]. In these coordinates f takes the form $F(w) = w - pa_\omega + o(1)$ for $|w| \to \infty$ and g takes the form $G(w) = w - pa_{h(\omega)} + o(1)$, with a_ω, $a_{h(\omega)}$ defined in (4.0). Let n be the least positive integer so that $f^n(x) \notin B(\omega, \delta)$. Then write $h(x) = g^{-n}_{h(\omega)} hf^n(x)$. If δ is small enough, then $|hf^n(x) - h(\omega)| < \delta'$ for an arbitrarily small δ', hence indeed $hf^n(x)$ is in the domain of the branch $g^{-n}_{h(\omega)}$. We obtain in the Fatou coordinates $|w - F^n(w)| = pa_\omega n + o(n)$ and $|v - G^n(v)| = pa_{h(\omega)} n + o(n)$ for $w = w(x)$ and $v = w\big(h(x)\big)$. If δ, δ' are small enough we can assume $o(n) < p\min\{a_\omega, a_{h(\omega)}\}$.

This yields in the original coordinates $|x-\omega|/|h(x)-h(\omega)| \leq (4a_\omega/a_{h(\omega)})^{1/p}$. Analogously we estimate from above $|h(x)-h(\omega)|/|x-\omega|$.

Let g be TPCE with constants θ_g, δ_g, M and κ.

By the continuity of h there exists $\theta>0$ such that for every $x\in J(f)$ one has

$$h\big(B(x,\theta)\big)\subset B\big(h(x),\theta_g\big). \tag{4.7}$$

Moreover for $x\in J(f)\cap B(\omega,\delta)$, for $\omega\in\Omega$ and all $0\leq k$ such that $2^{k-1}|x-\omega|\leq 1$

$$h\big(B(x,2^k\theta|x-\omega|)\cap Q\big)\subset B\big(h(x),2^{k-1}\theta_g|h(x)-h(\omega)|\big) \tag{4.8}$$

and for $x\in J(f)\setminus B\big(\Omega(f),\delta\big)$, $h(x)\in B\big(\omega_g,\delta_g\big)$ for $\omega_g\in\Omega(g)$

$$h\big(B(x,\theta)\big)\subset B\big(h(x),\theta_g|h(x)-\omega_g|\big). \tag{4.9}$$

The proof of (4.8) is similar to the proof of (4.6) with the use of Fatou coordinates. The proof of (4.9) makes use of the continuity of h^{-1}. Note that k here are not the same as in (4.4b).

Let t be good for $h(z)$ and g. Then, in the case when $x=f^{n(t)}(z)\notin B(\Omega,\delta)$, $f^{n(t)}$ is, by (4.7) or (4.9), at most M-critical on $\operatorname{Comp}\big(z,f^{n(t)},B(x,\theta)\big)$ since $g^{n(t)}$ is at most M-critical on $\operatorname{Comp}\big(h(z),g^{n(t)},B\big(h(x),r\big)\big)$, with $r=\theta_g$ or $\theta_g|h(x)-\omega_g|$.

In the case when $x=f^{n(t)}(z)\in B(\omega,\delta)$ consider $B=B(x,2^k\theta|x-\omega|)$, with k nonzero if $t\in\widetilde{R}_n$. Then by (4.8), $f^{n(t)}$ is at most M critical on $\operatorname{Comp}(z,f^{n(t)},B\cap Q)$. The latter set is well defined if $B\cap Q$ is connected. This is the case for all except maybe a bounded by a constant (related to p) number of k's, where B (and k) is so large that it intersects more than one cusp-like sector of Q, but so small that it does not contain ω. (Omitting the related finite blocks of t's do not have influence on the TPCE property.)

In the case of connected $B\cap Q$ far from ω the components of $B\setminus Q$ are disjoint from forward trajectories of critical points in the Fatou set, see Lemma 2.1. Hence all branches of $f^{-n(t)}$ involved in $\operatorname{Comp}\big(z,f^{n(t)},B\cap Q\big)$ extend to these components, so $f^{n(t)}$ is at most M-critical on $\operatorname{Comp}\big(z,f^{n(t)},B\big)$.

If $t\in\widetilde{R}_{n(t)}(z)$ and k large the forward trajectory of a critical point in the Fatou set enters B but it is irrelevant since, by the first composant f^{-1} of $f^{-n(t)}$, we jump out of $B(\omega,\delta)$. So, we do not capture the critical point. Thus the proof is the same as before.

Note that if t is the first element of $\widetilde{S}_{n(t)}(h(z))$, then for $\omega\in\Omega(f)$, $2^{-k(t-1)}>|g^{n(t)}\big(h(z)\big)-h(\omega)|$ but it can happen that $2^{-k(t-1)}\leq|f^{n(t)}(z)-\omega|$. In such a case $t-1\in W\big(h(z)\big)$, in $\widetilde{R}_n(t)$, but it is skipped in $W(z)$. Analogously, having reverse inequalities, it can happen that after $\widetilde{R}_{n(t)}\big(h(z)\big)$ we need to add some elements to build $\widetilde{R}_{n(t)}(z)$. Also involvement of δ in the definition of W causes the disappearance of some blocks $\widetilde{R}_n\big(h(z)\big)$ in $W(z)$, if we choose $\delta=\delta_f$ sufficiently small compared to δ_g, or appearance of new $\widetilde{R}(z)$'s if δ_f is large compared to δ_f.

Let us pass now to good hat-integers. In order to simplify notation suppose that the above complications do not happen, namely that $W(z)$ and $W\big(h(z)\big)$ have the same divisions into $\widetilde{T}$, $\widetilde{S}_m$ and $\widetilde{R}_n$. (The above complications have influence to the value of κ_f, which is fortunately positive if κ is close enough to 1, compare Remark 4.3(c).)

Note that each block $\Phi_g^{-1}(\hat{t}\,)$ (see Remark 4.3(b)), intersects at most $2C+1$ blocks $\Phi_f^{-1}(\hat{t}\,)$, by (4.6), so $\hat{t}$ good for g implies at least one of the $2C+1$ blocks (hat-elements) good for f. Hence the density in Definition 4.2 for $G(z)$ for f is bounded below by $\kappa(2C+1)^{-1}$. □

5. PCE implies TPCE and mean porosity

Definition 5.1 ([Ma], [Le]). We say a rational $f\colon \overline{\mathbf{C}}\to\overline{\mathbf{C}}$ is *backward Lyapunov stable* if for all $\varepsilon,\delta>0$ there exists $\theta>0$ such that for every $x\in J(f)\setminus B(\Omega,\delta)$ every $n\ge 0$ and every component W of $f^{-n}\big(B(x,\theta)\big)$ we have $\operatorname{diam}(W)\le\varepsilon$.

Recall that NCP functions satisfy this property, by Lemmas 2.2 and 2.4.

In Appendix B we prove Theorem B.1 saying in particular that PCE implies backward Lyapunov stability. This fact, crucial in the proof of Theorem 5.2 below to deal with *rescaled time*, was stated (in absence of parabolic points) in [P1, Remark 3.2].

So we shall prove the following.

Theorem 5.2. *The parabolic Collet–Eckmann property implies the topological parabolic Collet–Eckmann property* (PCE *implies* TPCE).

Theorem 5.3. *The Julia set of each* TPCE *rational map is mean porous.*

As we mentioned in the introduction, mean porosity implies by [KR] that the upper box-counting dimension of the Julia set is less than 2.

The strategy to prove Theorems 5.2 and 5.3 will be similar to [PR1]. To prove Theorem 5.2 we shall use the following lemma.

Lemma 5.4. *There exist $C=C_f>0$ and $P>0$ such that for every $z\in J(f)$ and every integer $n\ge 0$, for $K(n):=\max\{0,-\log_2\big(P\operatorname{dist}\big(f^n(z),\operatorname{Crit}(f)\cap J(f)\big)\big)\}$ we have*

$$\sum_{j=0}^{n}{}'K(j)\le \hat{n}C_f, \tag{5.1}$$

$\hat{n}=\phi(n)$, see (4.3) and where $\sum{}'$ denotes the summation taken over all but at most $\sharp\big(\mathrm{Crit}(f)\cap J(f)\big)$ indices j.

Note that the larger P the smaller $K(j)$'s.

This is a stronger version of an important inequality [DPU, (3.3)] where there was n rather than $\hat{n}$ on the right-hand side. The version with n has been used in [PR1]. The proof of Lemma 5.4 is a slight modification of the proof from [DPU]. We provide it in Appendix A.

Proof of Theorem 5.2. *Step* 1. *Shadows.* Fix $z\in J(f)$. One can consider $K(n)$ in Lemma 5.4 as a function on $\hat{n}$, the *rescaled time*, since each block of integers $\phi^{-1}(\hat{n})$ longer than 1 corresponds to a piece of the trajectory of z in $B(\Omega,\delta)$, where $K=0$, if P is large enough.

To each vertical interval $I=\{x\}\times[0,y]\subset\mathbf{R}^2$, $x,y\geq 0$, we associate for $\lambda>1$ (as in Definition 4.1) its *shadow*, namely the closed triangle $\Delta(I)$ with the following vertices: the top and bottom of I and the point $\big(x+4y/\log_2(\lambda),0\big)$ in $\mathbf{R}^+_x$, the non-negative part of the first coordinate axis $\mathbf{R}_x$ in $\mathbf{R}^2$.

Let $J_{\hat{n}}=\{\hat{n}\}\times\big[0,\max\{k(t):t\in\widetilde{R}_n\}\big]\subset\mathbf{R}^2$ for $n\in S(z)$. (If $f^{n(t)}(z)\in\Omega$ then $J_{\hat{n}}$ is infinitely high.) Note that $\phi^{-1}(\hat{n})$ is a singleton, so we may write n for it. Finally define

$$X=\mathbf{R}^+_x\cup\bigcup_{n\in S(z)}J_{\hat{n}},$$

$$X'=X\setminus\{(x,0):\text{there exists }\hat{n},\ \hat{n}<x,\ n\in S(z),\ (x,0)\in\Delta(J_{\hat{n}})\}.$$

If P in Lemma 5.4 is large enough and $K(\hat{n})\geq 1$, then a critical point in $J(f)$, closer to $f^n(z)$ than other critical points, is distinguished. Denote it by $c(\hat{n})$. Denote then by $\nu(\hat{n})$ the multiplicity of f at $c(\hat{n})$. To each $\hat{n}$ with positive $K(\hat{n})$ we associate the interval $I_{\hat{n}}=\{\hat{n}+1\}\times[0,\nu(\hat{n})K(\hat{n})]$. Denote by $\Delta(\hat{n})$ its shadow. In the case $K(\hat{n})=\infty$ we set $\Delta(\hat{n})$ the upper right quarter of $\mathbf{R}^2$ with the corner at $(\hat{n}+1,0)$.

We shall study how much the union $\bigcup I_{\hat{n}}$ shadows X'.

First, as in [PR1], we consider shadows on $\mathbf{R}^+_x$. For each $\hat{n}$ we obtain from Lemma 5.4

$$\sum_{j=0}^{\hat{n}-1}|\Delta(j)\cap\big([0,\hat{n}]\times\{0\}\big)|\leq\hat{n}\Big(\sup_j\nu(j)\Big)\frac{C_f4}{\log_2\lambda}+\hat{n}\sharp\big(\mathrm{Crit}(f)\cap J(f)\big):=\hat{n}N_f,$$

where $|\cdot|$ stands for the length of intervals. Hence for

$$A=\big\{x:(x,0)\text{ belongs to at most }2N_f\text{ shadows }\Delta(j)\big\},$$

we conclude that for every $x > 0$

$$\frac{|A \cap [0, x]|}{x} \geq \frac{1}{2},$$

where by $|\cdot|$ we denote the sum of the lengths of the intervals composing $A \cap [0, x]$, or $A'_{(x,y)}$ below.

Let $\mathscr{P}$ be the projection of $\mathbf{R}^2$ to $\mathbf{R}_x$ defined by $\mathscr{P}(x, y) = x + 4y/\log_2 \lambda$. For an arbitrary $(x_0, y_0) \in X$ define

$$A'_{(x_0,y_0)} := X' \cap \mathscr{P}^{-1}\big(A \cap [0, \mathscr{P}(x_0, y_0)]\big) \cap \{(x, y) \in \mathbf{R}^2 : x \leq x_0\}.$$

(We intersect with $\{x \leq x_0\}$ to cut off $J_{\hat{n}}$, $\hat{n} > x_0$. Note that $\{(x, y) : x > x_0,\ y = 0\}$ has been already cut off by the definition of X'.)

By the definitions each point of $A'_{(x_0,y_0)}$ belongs to at most $2N_f$ shadows $\Delta(j)$. Note also that

$$|A'_{(x_0,y_0)}| \geq \min\{\tfrac{1}{4}\log_2 \lambda, 1\}\big|A \cap \big[0, \mathscr{P}(x_0, y_0)\big]\big|.$$

The number $\frac{1}{4}\log_2 \lambda$ appears when we project by $\mathscr{P}^{-1}$ to vertical intervals, the number 1 for subintervals of $A \cap [0, \mathscr{P}(x_0, y_0)]$ not shadowed by any $J_{\hat{n}}$. Hence, as we could assume that $\log_2 \lambda \leq 4$,

$$|A'_{(x_0,y_0)}| \geq \big(\tfrac{1}{8}\log_2 \lambda\big) \cdot \mathscr{P}(x_0, y_0).$$

The components of A' are open intervals in $\mathbf{R}_x^+$ with integer right-hand side ends (at the bottoms of $I_{\hat{n}}$'s), or intervals in $\mathbf{R}_x^+$ with the right-hand side ends at the bottoms of $J_{\hat{n}}$'s followed by vertical intervals in the respective $J_{\hat{n}}$'s. Using the notation

$$H(\hat{t}\,) := \big(\hat{n}\big(\Psi(\hat{t}\,)\big), k\big(\Psi(\hat{t}\,)\big)\big)$$

and for an arbitrary t_0,

$$A_{\hat{t}_0}^{\text{integer}} := \{\hat{t} : H(\hat{t}\,) \in \operatorname{cl} A'_{H(\hat{t}_0)}\},$$

we obtain the inequality $\sharp A_{\hat{t}_0}^{\text{integer}} \geq \frac{1}{2}|A'_{H(\hat{t}_0)}|$. Hence

$$\sharp A_{\hat{t}_0}^{\text{integer}} \geq \frac{1}{16}(\log_2 \lambda) \cdot \mathscr{P}\big(H(\hat{t}_0)\big). \tag{5.2}$$

Note that each point in $A_{\hat{t}_0}^{\text{integer}}$ belongs to at most $2N_f + 1$ shadows $\Delta(\hat{n})$ (+1 may happen at the bottom of some $I_{\hat{n}}$. Remember that this point may belong to $\operatorname{cl} A'_{H(\hat{t}_0)} \setminus A'_{H(\hat{t}_0)}$).

Note now that by our definition of rescaled time, if $\hat{n}_j$, $j=1,2,\ldots$, are all consecutive integers with non-empty $\widetilde{R}_{n_j}$ then, if δ is small enough, we have for the corresponding $\hat{t}_j$, with $H(\hat{t}_j)=(\hat{n}_j,0)$,

$$\hat{t}_{j+1}-\hat{t}_j\le 2(\hat{n}_{j+1}-\hat{n}_j);$$

compare with the inequality in the opposite direction than (5.6) (true up to a constant summand). Note also that $\hat{t}_1\le\hat{n}_1$.

Assume from now on that $t_0\notin\widetilde{S}_m$ for any m. So, we can substitute in (5.2) $\mathscr{P}\big(H(\hat{t}_0)\big)=\hat{n}_1+\big(\sum_j\hat{n}_{j+1}-\hat{n}_j\big)+4k(t_0)/\log_2\lambda$ and obtain (using (5.7))

$$\sharp(A_{\hat{t}_0}^{\text{integer}})\ge\Big(\frac{1}{32}\log_2\lambda\Big)\big(\hat{t}_0-k(t_0)\big)+\frac{1}{4}k(t_0)\ge\Big(\frac{1}{32}\log_2\lambda\Big)\hat{t}_0. \tag{5.3}$$

Note that the case $\{\hat{n}_j\}=\emptyset$ is the case $f^n(z)\notin B(\Omega,\delta)$ for all $0\le n\le n(t_0)$, where (5.3) immediately follows from (5.2).

Step 2. Good hat-integers. Now we show that all $\hat{t}\in A^{\text{integer}}:=\bigcup_{\hat{t}_0}A_{\hat{t}_0}^{\text{integer}}$ are good hat-integers. This can be done as in [PR1] with the use of 'shrinking neighbourhoods' procedure [P1] (the name comes from [GS]).

For each exposed critical point $c\in J(f)$ let $s_j(c)$, $j=1,2,\ldots$, be the increasing sequence of all positive integers such that $f^{s_j(c)}(c)\notin B(\Omega,\frac{1}{2}\delta)$ or $f^{s_j(c)-1}(c)\notin B(\Omega,\frac{1}{2}\delta)$.

By the definitions of ϕ and $s_j(c)$ we have $j-1\le a\phi\big(s_j(c)-1,f(c)\big)$. (We need a constant $a>0$ here since for $f^{s_j(c)}(c)\in B(\Omega,\delta)\setminus B(\Omega,\frac{1}{2}\delta)$ for $j=j_0$, $j_0+1,\ldots,j_0+T$, we have s_j all consecutive integers, so the rescaled time can be shorter than T. We can set $a=\sup T+1$.)

Define for C_2 from Lemma 2.5(a),

$$b_{s_j(c)}=|(f^{s_j(c)-1})'(f(c))|^{-1/(2C_2)}. \tag{5.4}$$

Hence, using Definition 4.1, we obtain

$$b_{s_j(c)}\le C^{-1}\lambda^{-((j-1)/a)^{1/(2C_2)}}.$$

In particular the series $\sum_j b_{s_j(c)}$ is convergent.

Next organize $\bigcup_c\{s_j(c)\}$, the union over all exposed critical points in J, into an increasing sequence s_j and let

$$b_{s_j}=C\max_{c:\,\text{there is }j',\,s_j=s_{j'}(c)}b_{s_{j'}(c)},$$

where C in the latter formula is a normalizing factor such that, say,

$$\prod_{j=1}^{\infty}(1-b_j)=\frac{1}{2}. \tag{5.5}$$

Finally for all positive integers s not belonging to $\{s_j\}$ set $b_s := 0$. Fix $x \in J(f)$ (it plays the role of z from Step 1) and $\hat{t} \in A^{\text{integer}}$. Assume also that for $n = n\big(\Psi(\hat{t}\,)\big)$ we have

Case 1. $f^n(x) \notin B(\Omega,\delta)$

or

Case 2. $f^{n-1}(x) \notin B(\Omega,\delta)$.

In Case 2, $f^n(x)$ can be very close to $\omega \in \Omega$, i.e., $k = k\big(\Psi(\hat{t}\,)\big)$ can be nonzero. These are the only possibilities. Indeed, if $f^n(x) \in B(\Omega,\delta)$ and $f^{n-1}(x) \in B(\Omega,\delta)$, then there is m such that $t \in \widetilde{S}_m(x)$. This $m < n$ is the smallest integer such that for every $s : m \le s \le n$ one has $f^s(x) \in B(\omega,\delta)$. Then

$$|J_{\widehat{m}}| \ge -\log_2|f^m(x)-\omega|-1$$

and for δ small enough

$$\phi\big(n-m, f^m(x)\big) \le -(p+1)\log_2|f^m(x)-\omega|-1.$$

Hence

$$|J_{\widehat{m}}| \ge \frac{1}{p+1}\phi\big(n-m, f^m(x)\big). \tag{5.6}$$

So, if

$$\log_2\lambda \le 4/(p+1), \tag{5.7}$$

then $4|J_{\widehat{m}}|/\log_2\lambda \ge \phi(n-m, f^m(x))$, i.e., $H(\hat{t}\,) \in \Delta(J_{\widehat{m}})$, so $\hat{t} \notin X'$, a contradiction. (It is paradoxical that we assume λ to be small in (5.7). This is caused by our rough definition of shadows $\Delta(\hat{n})$. They could be smaller. Along periods of rescaled time when the trajectory stays close to Ω, the slope of the edge line of the shadow could be $\log_2 2 = 1$ rather than $\frac{1}{4}\log_2\lambda$, so the upper-right edge of the shadow could be piecewise affine, below our affine edge. This is related with the inclusion in Lemma 2.1.)

Consider the sequence

$$B_s = B\bigg(f^n(x), 2\cdot 2^{-k}\theta\prod_{i=1}^{s}(1-b_i)\bigg)$$

of neighborhoods of $B = B\big(f^n(x), 2^{-k}\theta\big)$, where $k = 0$ in Case 1, together with connected components

$$W_s = \mathrm{Comp}\big(f^{n-s}(x), f^s, B_s\big) \quad \text{and} \quad W'_{s-1} = f(W_s).$$

Recall the main idea of shrinking neighborhoods approach from [P1], in our setting with the subsequence s_j. If along backwards iteration from $f^n(x)$ a critical point c is captured by W_s then $f^s(c) \in B\big(f^n(x), 2\theta\big)$, so $f^s(c) \notin B(\Omega, \frac{1}{2}\delta)$ if $2\theta < \frac{1}{2}\delta$, in Case 1. (We can assume the latter inequality, since we do not need the inclusions in Lemma 2.1.) Similarly in Case 2 we have $f^{s-1}(c) \notin B(\Omega, \frac{1}{2}\delta)$. Hence $s = s_j(c)$ for some j, hence $b_s \neq 0$. So,

$$f^{s-1}\big(W_{s-1} \setminus W'_{s-1}\big) = B_{s-1} \setminus B_s$$

is a non-trivial Koebe's space for the appropriate branch of $f^{-(s-1)}$ allowing us to use $|(f^{s-1})'\big(f(c)\big)|$ to control the diameter as in Lemma 2.5(a).

Let us be more precise now. We want to show that if W_s contains an exposed critical point, then $H(\hat{t})$ belongs to the shadow $\Delta(\phi(n-s))$. Assume this is not the case. Then there is a smallest such s, with an exposed critical point $c \in W_s$. Hence there are at most $2N_f$ integers $0 < s' < s$ such that $W_{s'}$ contains an exposed critical point (as $\hat{t} \in A^{\mathrm{integer}}$ and s is the smallest). Note again that this s is of the form $s_j(c)$. This is a tricky place in the proof.

The number of all $s' < s$ of captures by $W_{s'}$ of critical points (not only exposed ones) is bounded above by $(2N_f + 1)N$ where N is the maximal positive integer for which there exist critical points $c \neq c'$ in $J(f)$ with $f^N(c) = c'$. W_{s-1} is simply connected since all the sets $\mathrm{Comp}\big(f^{s'}(x), f^{n-s'}, B\big(f^n(x), 2\theta 2^{-k}\big)\big)$ for $0 \le s' \le n$ have small diameters if θ is small, by backward Lyapunov stability (see Definition 5.1 and Appendix B). (We can see the simple-connectedness also directly, as in [PR1], by induction, proving only that all W_{s_j} containing critical points have diameters so small that each could capture at most one critical point.) This implies in particular that there exists a constant $D = D(N_f)$ such that we can apply Lemma 2.5(a) to $F = f^{s-1}$ for our $s = s_j(c)$ and to $V = W_{s-1}$. We obtain

$$|(f^{(s-1)})'\big(f(c)\big)|\,\mathrm{diam}(W'_{s-1}) \le C_1 b_s^{-C_2} 2\theta 2^{-k}.$$

Hence, by Definition 4.1 and by (5.4), we can write with a constant C depending on C_1 and C_2

$$\mathrm{diam}(W'_{s-1}) \le C_1 2\theta 2^{-k} b_s^{-C_2} |f^{(s-1)'}\big(f(c)\big)|^{-1} \le C 2\theta 2^{-k} \lambda^{-\phi(s-1,f(c))/2}.$$

Since both points, $f^{n-s}(x)$ and critical c, are in W_s, the above expressions give also upper bounds for the distance between these points. So, for $\nu = \nu\big(\phi(n-s, x)\big)$, after applying $-\log_2$, we obtain for θ small enough

$$\begin{aligned}|I_s| &= \nu K\big(\phi(n-s,x)\big) = -\nu\log_2\big(P\,\mathrm{dist}\big(c, f^{n-s}(x)\big)\big)\\ &\ge -\nu\log_2\big(2P(\mathrm{diam}(W'_{s-1}))^{1/\nu}\big)\\ &\ge -\nu\log_2(2P) - \log_2 C + k - \log(2\theta) + \tfrac{1}{2}\phi(s-1, f(c))\log\lambda.\end{aligned}$$

Hence, again for θ small enough to compensate other constants here,

$$|I_s| \ge k + \tfrac{1}{2}\phi\big(s-1, f(c)\big)\log_2\lambda. \tag{5.8}$$

Note now that $\phi\big(s-1, f^{n-s+1}(x)\big) \le 2\phi\big(s-1, f(c)\big)$. This estimate says that the rescaled times for the trajectories $f(c), f^2(c), \ldots, f^s(c)$ and $f^{n-s+1}(x), f^{n-s+2}(x), \ldots, f^n(x)$ are similar. This is so since for θ small, all diameters $\mathrm{diam}\, f^{s'}(W_s)$, $s' = 1, 2, \ldots, n-s$, are small. Here is the place where we substantially use the backward Lyapunov stability. Thus by (5.8)

$$|I_s| - k \ge \tfrac{1}{4}\phi\big(s-1, f^{n-s+1}(x)\big)\log_2\lambda,$$

so $H(\hat{t}\,) \in \Delta\big(\phi(n-s)\big)$, a contradiction.

Let us summarize. We have proved in this way that there are at most $2N_f$ integers $s : 0 \le s < n$ such that $\mathrm{Comp}\big(f^s(x), f^{n-s}, B\big)$ captures an exposed critical point. So the number of all times of captures of critical points (not only exposed ones) is bounded above by $(2N_f + 1)N$. Hence f^n has at most $D(N_f)$ critical points in $W_n \supset V_t(x)$. The last inclusion follows from (5.5). This proves that $\hat{t}$ is good. So, (5.3) yields (4.5) with $\kappa = \frac{1}{32}\log_2\lambda$. Remember that at the end of Step 1 we assumed that $t_0 \notin \widetilde{S}_m(x)$. For $t_0 \in \widetilde{S}_m(x)$, if $m = 0$ (i.e., if $x, \ldots, f^{n(t_0)}(x) \in B(\omega, \delta)$), Theorem 5.2 is trivial, i.e., all hat-integers $\le \hat{t}_0$ are good. If $m > 0$, then by (5.6), for t' the largest in $\widetilde{R}_m(x)$, we have

$$\hat{t}' \ge \frac{1}{p+1}(\hat{t}_0 - \hat{t}')$$

i.e.,

$$\hat{t}' \ge \frac{1}{p+2}\hat{t}_0.$$

Hence (4.5) for $\hat{t}'$ yields

$$\frac{\sharp(G(x) \cap [0, \hat{t}_0]}{\hat{t}_0} \ge \frac{\kappa}{p+2}. \ \square$$

Proof of Theorem 5.3. This repeats roughly an adequate part of the proof of the mean porosity in [PR1, Theorem 1.1] and the proof of Theorem 1.1.

We need to pass from good times (more precisely from good hat-integers) for $x \in J$ to good scales, in which J^c contains some definite disk. We use the notation $V_t(x)$ for $\hat{t} \in G(x)$ as in (4.4a)–(4.4c) and $V_t'(x)$ if θ is replaced by $\frac{1}{2}\theta$.

We claim that there is an integer $\overline{N}$ such that the following holds: For all $x \in J$ and for all $\hat{t}, \hat{t}' \in G(x)$ with $\hat{t} - \hat{t}' \geq \overline{N}$

$$\operatorname{diam}\bigl(V_t'(x)\bigr) \leq \tfrac{1}{2} \operatorname{diam}\bigl(V_{t'}'(x)\bigr). \tag{5.9}$$

For $\hat{t}' = 0$ and for $0, t \in \widetilde{R}_0(x)$ the inequality (5.9) is trivial. For $0, t \in \widetilde{S}_0(x)$, it follows from the definition of the rescaled time close to a parabolic point. Indeed, in the latter case, for $n = n(t)$

$$\frac{\operatorname{diam}\bigl(V_t'(x)\bigr)}{\operatorname{diam}\bigl(V_0'\bigl(f^n(x)\bigr)\bigr)} \leq 2|(f^n)'(x)|^{-1} \leq 2^{-\hat{n}+2}$$

and

$$\frac{\operatorname{diam}\bigl(V_0'(x)\bigr)}{\operatorname{diam}\bigl(V_0'\bigl(f^n(x)\bigr)\bigr)} = \frac{|x-\omega|}{|f^n(x)-\omega|} \geq (2^{-\hat{n}})^{1/(p+1)}.$$

we obtain $\operatorname{diam}\bigl(V_t'(x)\bigr) / \operatorname{diam}\bigl(V_0'(x)\bigr) \leq 2^{-p(\hat{n}+2)/(p+1)}$.

Finally for $0, t \in \widetilde{T}(x)$ this follows from Lemma 2.4. Other cases are combinations of the above cases.

In fact, for each $0 < \tau < 1$ there is $\overline{N} = \overline{N}(\tau, f, M, \theta)$ such that, for $t \geq \overline{N}$, $\operatorname{diam}\bigl(V_t'(x)\bigr) \leq \tau \operatorname{diam} V_0'(x)$.

For $\hat{t}' > 0$ use backward iteration. Write n, n' for $n(t)$ and $n(t')$ respectively. As f^n is M-critical on $V_t'(x)$, the iterate $f^{n-n'}$ is M-critical on $V_{t-t'}'\bigl(f^{n'}(x)\bigr)$. So

$$f^{n'}(V_t') = V_{t-t'}'\bigl(f^{n'}(x)\bigr) \subset B\bigl(f^{n'}(x), \tau \operatorname{diam} V_0'\bigl(f^{n'}(x)\bigr)\bigr)$$

by the first case. Applying $f^{-n'}$ we obtain (5.9) provided τ is small enough, by Lemma 2.5(b).

Observe also that for every t we have

$$\operatorname{diam} V_t'(x) \geq 2^{-S} \max\{L, 2\}^{-\hat{n}(t)} 2^{-k(t)} \theta \quad \text{for } \mathscr{L} = \sup |f'|, \tag{5.10}$$

where S is the number of n's $0 \leq n \leq n(t)$ such that $f^{n-1}(x) \notin B(\Omega, \delta)$ (provided $n > 0$), but $f^n(x) \in B(\Omega, \delta)$.

The proof of (5.10) uses the definition of the rescaled time in which, close to Ω, the rate of shrinking for backward iterates is $\frac{1}{2}$; see (4.2) the first inequality in the opposite direction and no factor 2. The factor 2^{-S} comes from a bound on distortion that can be 2 on each block (i_s, j_s); see (4.3).

Consider the increasing sequence $\hat{t}_j$ of all good hat-integers of x, $\{\hat{t}_j\} = G(x)$, and set $\hat{k}_j = \hat{t}_{\overline{N}j}$, i.e., consider only every $\overline{N}$-th good hat-integer. By the definition of TPCE we have $\hat{k}_j \le \kappa^{-1}\overline{N}j$, and as $\hat{k}_{j+1} - \hat{k}_j \ge \overline{N}$, inequality (5.9) implies that

$$\operatorname{diam}\bigl(V'_{k_{j+1}}(x)\bigr) < \tfrac{1}{2}\operatorname{diam}\bigl(V'_{k_j}(x)\bigr),$$

where $k_i = \Psi(\hat{k}_i)$. Hence, taking into account also (5.10), we obtain a strictly increasing sequence $u_j \le Pj$ (with $P \le 4\kappa^{-1}\overline{N}\log_2 \max\{2, L\}$) such that

$$\operatorname{diam}\bigl(V'_{k_j}(x)\bigr) \sim 2^{-u_j}.$$

($\sim$ means: up to a factor between $\frac{1}{2}$ and 2.)

Finally, as in Section 3, we find inside each V'_{k_j} a disk of radius $\eta \operatorname{diam} V'_{k_j}$ disjoint from the Julia set. This proves the mean porosity of $J(f)$. □

6. Final remarks on PCE maps

In [P3], [PR2] and [NP] some other properties equivalent to TCE are listed. The same can be introduced for TPCE.

Definition 6.1. (a) (*Exponential shrinking of components.*) There exist $\lambda_1 > 1$ and $\theta, \delta > 0$ such that for every $z \in J(f) \setminus \Omega$, every $t \in W(z)$, see Section 3, and $V_t(z)$ as in (4.4), one has $\operatorname{diam} V_t(z) \le \lambda_1^{-\hat{n}(t)}$.

(b) (*Exponential shrinking of components at critical points.*) The same as above, but only for $V_t(z)$ containing a critical point.

(c) (*Uniform hyperbolicity on periodic orbits*, abbreviated: PUHPer.) There exists $\lambda_2 > 1$ such that for every periodic $x \in J(f) \setminus \Omega$ one has $|(f^n)'(x)| \ge \lambda_2^{\hat{n}}$, where n is a period of x.

Theorem 6.2. (a), (b) *and* TPCE *are equivalent. They imply* (c). [1]

Proof. TPCE implies (a) by (5.9). The proof of the implication (b) $\Rightarrow$ TPCE is similar to the proof of Theorem 5.2 (one need not even use 'shrinking neighbourhoods'). The proof of (a) $\Rightarrow$ (c) is easy. It is the same as in [P3] and [NP]. □

An analysis of the proof of TCE $\Rightarrow$ CE in [P3] might give a positive answer to the following question:

Question 6.3. Does TPCE imply PCE, provided there is only one critical point in $J(f)$?

[1] Recently it was proved in [PRS] that UHPer (PUHPer in absence of parabolic points) is equivalent to TCE. In view of this, the equivalence of PUHPer and TPCE seems probable.

Remark 6.4. Analogously to TCE $\Leftrightarrow$ (A_∞ is Hölder) for f polynomial and A_∞ the basin of attraction to ∞, [GS], [P3], there would exist a natural property of A_∞ equivalent to TPCE.

Remark 6.5. A theory analogous to our PCE is possible for iteration of maps of interval, compare [NP].

Appendix A

Average distance from critical points. Proof of Lemma 5.4. For every $c \in \mathrm{Crit}(f) \cap J(f)$ and arbitrary $a > 0$ define the function k_c: $\overline{\mathbf{C}} \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ by setting

$$k_c(x) = \min\{n \geq 0 : x \notin B(c, a2^{-(n+1)})\} \quad \text{if } x \neq c$$

and $k_c(x) = \infty$ if $x = c$.

The following lemma can be easily deduced from the fact that up to a biholomorphic change of coordinates every holomorphic function is of the form $z \mapsto z^q$ in some neighborhood of a critical point of order $q \geq 2$.

Lemma A.1. *There exists $\vartheta > 0$ (depending on a and f) such that*

$$|f'(x)| \leq 2^{\vartheta}\, 2^{-k_c(x)} \tag{A.1}$$

for every $x \in \overline{\mathbf{C}}$.

Lemma A.2 (the main lemma). *Suppose $a < 2\delta$. Then there exists a constant $Q > 0$ such that for every integer $n \geq 1$, if $x \in J$ satisfies*

$$k_c\big(f^j(x)\big) \leq k_c\big(f^n(x)\big) \quad \text{for every } j = 1, 2, \ldots, n-1, \tag{A.2}$$

then

$$\min\big\{k_c(x), k_c\big(f^n(x)\big)\big\} + \sum_{j=1}^{n-1} k_c\big(f^j(x)\big) \leq Q\hat{n}, \tag{A.3}$$

where the rescaled time $\hat{n} = \phi(n)$ has been defined in Section 4.

Proof. Notice that since $\frac{1}{2}a < \delta$ and $B(\Omega, 2\delta) \cap \mathrm{Crit}(f) = \emptyset$, if $f^j(x) \in B(\Omega, \delta)$ then $k_c\big(f^j(x)\big) = 0$. The proof of Lemma A.2 is carried through by induction with respect to n. For $n = 1$ the statement is obvious since $f(c) \neq c$. The procedure for the inductive step will be the following: Given $x, f(x), \ldots, f^n(x)$ satisfying (A.2) we shall decompose this string into two blocks: (a) $x, f(x), \ldots, f^m(x)$, $m \leq n$, for which we shall prove (A.3); (b) $f^m(x), \ldots, f^n(x)$ for which we can apply the inductive hypothesis. Summing these two estimates we prove (A.3) for $x, f(x), \ldots, f^n(x)$.

Let $k' = \min\{k_c(x), k_c(f^n(x))\}$ and $B = B(c, a2^{-(k'-1)})$.

If $k' \geq 1$, let $1 \leq m' \leq n$ be the first positive integer such that

$$\text{(i)} \qquad \operatorname{diam}\big(f^{m'}(B)\big) \geq a2^{-k'} (= \tfrac{1}{4} \operatorname{diam} B)$$

or

$$\text{(ii)} \qquad k_c\big(f^{m'}(x)\big) \geq k'.$$

If $k' = 0$ set

$$\text{(iii)} \qquad m' = 1.$$

Now, if $f^{m'}(x) \notin B(\Omega, \delta)$ we set $m := m'$. If $f^{m'}(x) \in B(\omega, \delta)$ we set m the smallest $m \geq m'$ such that $f^m(x) \notin B(\omega, \delta)$ or $m := n$ if $f^j(x) \in B(\omega, \delta)$ for all $m' \leq j \leq n$.

In all these cases the sequence $y = f^m(x), f(y), \ldots, f^{n-m}(y)$ satisfies the assumption (A.2) automatically, with x, n replaced by y, $n - m$, and moreover $k_c(y) = \min\{k_c(y), k_c(f^{n-m}(y))\}$. Hence by the inductive hypothesis

$$\text{(A.4)} \qquad \sum_{j=m}^{n-1} k_c\big(f^j(x)\big) \leq Q\phi(n-m).$$

with the latter $\phi(n-m) = \phi(n-m, y)$; see the notation preceding (4.3).

Assume $k' \geq 1$ (the cases (i) or (ii)). By the definition of m we have for every $0 < j < m'$,

$$\text{(A.5)} \qquad \operatorname{diam}\big(f^j(B)\big) < a2^{-k'}$$

so $f^j(B)$ cannot intersect at the same time $\partial B(c, a2^{-k''})$ and $\partial B(c, a2^{-(k''-1)})$ for $k'' \leq k'$. Hence for every $w \in B$,

$$k_c\big(f^j(w)\big) \geq k_c\big(f^j(x)\big) - 1.$$

Hence by Lemma A.1 it follows that

$$\text{(A.6)} \qquad |f'\big(f^j(w)\big)| \leq 2^{\vartheta} 2^{-(k_c(f^j(x))-1)}.$$

For $j = 0$ we replace here $k_c(x)$ by k'.

For $i_s \leq j < j_s < m'$; see (4.1), we have by (A.5), provided a has been chosen small enough, a better estimate:

$$\text{(A.7)} \qquad |(f^{j_s-i_s})'\big(f^{i_s}(w)\big)| \leq 2|(f^{j_s-i_s})'\big(f^{i_s}(x)\big)| \leq 4 \cdot 2^{n(i_s, j_s)}.$$

The first inequality follows from $\operatorname{diam} f^{j_s}(B) < \sigma|f^{j_s}(x) - \omega|$, Lemma 2.1, and distortion of $f_\omega^{-(j_s-i_s)}$ bounded on $f^{j_s}(B)$ by 2 for a small enough.

The second inequality follows from the definition of $n(i_s, j_s)$ and from

$$|(f^{j_s-i_s})'(f^{i_s}(x))| \Big/ \left(\frac{|f^{j_s}(x) - \omega|}{|f^{i_s}(x) - \omega|}\right)^{p+1} \leq 2$$

for δ small enough; see (4.2) and the arguments following it.

For $i_s < m' \leq j_s$ we cannot repeat the above considerations as $\operatorname{diam}\big(f^{i_s}(B)\big)$ can be large in comparison to $|f^{i_s}(x) - \omega|$. In particular we can have $f^{i_s}(B) \ni \omega$, and we get troubles with bounded distortion. Instead, provided $|f^{i_s}(w) - \omega| \geq |f^{i_s}(x) - \omega|$, we estimate as follows

$$\begin{aligned} |(f^{m'-i_s})'\big(f^{i_s}(w)\big)| &\leq 2\left(\frac{2\delta}{|f^{i_s}(w) - \omega|}\right)^{p+1} \leq 2^{p+2}\left(\frac{|f^m(x) - \omega|}{|f^{i_s}(x) - \omega|}\right)^{p+1} \\ &\leq 2^{p+3}2^{n(i_s,m-1)+1}. \end{aligned} \tag{A.8}$$

If $|f^{i_s}(w) - \omega| \leq |f^{i_s}(x) - \omega|$, we estimate as follows

$$|(f^{m'-i_s})'\big(f^{i_s}(w)\big)| \leq 2|(f^{m'-i_s})'\big(f^{i_s}(x)\big)| \leq 4 \cdot 2^{n(i_s,m')} \leq 4 \cdot 2^{n(i_s,m-1)+1}. \tag{A.9}$$

Gathering (A.6)–(A.9) we obtain for $C = \vartheta + p + 3$

$$\frac{\operatorname{diam}\big(f^{m'}(B)\big)}{\operatorname{diam}(B)} \leq 2^{C\phi(m) - \left(k' + \sum_{j=1}^{m'-1} k_c(f^j(x))\right)}. \tag{A.10}$$

In case (i) but not (ii) we have

$$\frac{\operatorname{diam} f^{m'}(B)}{\operatorname{diam} B} \geq \frac{1}{4}. \tag{A.11}$$

This together with (A.10) gives

$$\frac{1}{4} \leq 2^{C\phi(m) - \left(k' + \sum_{j=1}^{m'-1} k_c(f^j(x))\right)}.$$

Hence

$$k' + \sum_{j=1}^{m'-1} k_c\big(f^j(x)\big) \leq C\phi(m) + 2 \leq (C+2)\phi(m). \tag{A.12}$$

In the case (ii) we also obtain (A.11). Since otherwise $f^{m'}(B) \subset B$ and consequently the family of functions $(f^{tm}|_B)_{t=1,2,\ldots}$ is normal, which contradicts the fact that $c \in J(f)$. Therefore (A.12) holds in this case too. In the case (iii), we have (A.12) trivially by $k' = 0$, $m' = 1$. Finally we can change in (A.12) m' to m since $k_c\big(f^j(x)\big) = 0$ for all $j = m', \ldots, m-1$. This together with (A.4) proves the lemma. □

The proof of Lemma 5.4 is now the same as in [DPU]. The idea is to find $0 \le n_1 \le n$, where $k_c(f^{n_1}(z))$ attains its maximum, next $n_2 = r$ where $k_c\big(f^r(z)\big)$ on $n_1 < r \le n$ attains its maximum, etc., and to use Lemma A.2 for blocks $[0, n_1], [n_1, n_2], \ldots, [n_k, n]$. The index omitted in the sum is n_1. Next do the same with every other critical point c and sum up the resulting inequalities over $c \in J(f)$.

Appendix B

Backward asymptotic stability. Recall that for a rational map $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ a critical point in $J(f)$ is called *exposed* if its forward trajectory does not contain any other critical point. Let Ω denote the set of all periodic parabolic points. Given $x \in J(f)$ and $\delta > 0$ denote by $\mathscr{N}(x,\delta)$ the set of all integers $n \ge 0$ such that $f^n(x) \notin B(\Omega, \delta)$.

Theorem B.1. *Suppose that for every* exposed *f-critical point $c \in J(f)$ and every $\delta > 0$*

$$\sum_{n \in \mathscr{N}(f(c),\delta)} |(f^n)'\big(f(c)\big)|^{-1} < \infty. \tag{B.1}$$

Then there exists $\theta > 0$ and $\alpha_n \to 0$ such that for every non-parabolic $x \in J(f)$, every $n \ge 0$, and every component W of $f^{-n}\big(B\big(x, \theta \operatorname{dist}(x,\Omega)\big)\big)$, we have $\operatorname{diam}(W) \le \alpha_n$.

Proof. In absence of parabolic points, with $\theta \operatorname{dist}(x,\Omega)$ replaced by θ, this theorem coincides with [P1, Remark 3.2]. The proof was not explicitly written there, only ingredients were provided. We will first prove Theorem B.1 in this case and only at the end we will indicate the modifications that should be done to prove the parabolic case.

Let N be the maximal nonnegative integer such that there exist $c, c' \in \operatorname{Crit}(f) \cap J(f)$ for which $f^N(c) = c'$. Denote the set of all exposed critical points in $J(f)$ by $\operatorname{Crit}^e(f)$. It is easy to find a sequence $b_n > 0$, $n = 1, 2, \ldots$, such that

$$\lambda_n := b_n \min\left\{ \left(\sum_{j=0}^{N+1} |(f^{n-1+j})'(f(c))|^{-1} \right)^{-1} : c \in \operatorname{Crit}^e(f) \right\} \to \infty, \tag{B.2}$$

$\sum b_n < \infty$ and in addition

$$\prod_{n=1}^{\infty} (1 - b_n) > \frac{3}{4}. \tag{B.3}$$

Set also $\lambda_0 = 1$ and then set

$$C_5 = \inf \lambda_n \quad \text{and} \quad C_6 = \sup \frac{\operatorname{diam}(W)}{\operatorname{diam}(B)} |(f^j)'(y)|,$$

where the supremum is taken over all disks B and components W of $f^{-j}(B)$ for all $j = 0, 1, \dots, N+1$, and $y \in W$. It is easy to see that $C_6 < \infty$, [P1, Lemma 1.3].

Let $N_0 \geq N$ be such that $\lambda_n \geq 8C_1C_6$ for all $n \geq N_0$. Here C_1 is the constant from Lemma 2.5(a) for $D = 0$. Then $C_2 = 1$, see for example [P1, (1.1)].

We consider θ_0 so small that if $B(x, 2\theta_0)$ contains a critical point in $J(f)$ but no other critical point has this critical former point in its forward trajectory, and if $f^{-n}\big(B(x, 2\theta_0)\big) \cap \mathrm{Crit}(f) \neq \emptyset$, then $n > N_0$.

For every $\theta > 0$ let $\theta' := \sup \mathrm{diam}(W)$, where the supremum is taken over all components W of $f^{-j}(B)$ for all $j = 0, 1, \dots, N+1$, and B, disks of radius θ. If $\theta \to 0$, then of course $\theta' \to 0$. We consider only θ small enough so that $\theta' \leq \theta_0$. Now given a disk $B = B(x, 2\theta)$ and given a backward trajectory $x = x_0, x_1, \dots$, $f(x_j) = x_{j-1}$ we consider disks

$$B_s = B\Big(x, 2\theta \prod_{j=1}^{s} (1 - b_j)\Big)$$

together with connected components W_s of $f^{-s}(B_s)$ such that $W_s \ni x_s$ (this is a so-called "shrinking neighbourhoods" procedure: ([P1], [GS]) until W_s, for $s = s_0$, contains a critical point for the first time. Denote this critical point by c. Of course it can happen that we never capture a critical point; then we set $s_0 = \infty$ and say that we have an infinite block. By Lemma 2.5(a) we have for $s = s_0$

$$\begin{aligned} \text{(B.4)} \qquad & \mathrm{diam}\big(\mathrm{Comp}\big(x_{s-1}, f^{s-1}, B(x,\theta)\big)\big) \leq \mathrm{diam}\big(f(W_s)\big) \\ & \leq C_1 |(f^{s-1})'(f(c)|^{-1} b_s^{-1} 2\theta \leq C_1 \lambda_s^{-1} 2\theta \leq 2C_1C_5^{-1}\theta. \end{aligned}$$

Suppose now that $s_0 = 1$, i.e., $c \in W_1$. Then let $0 \leq j \leq N$ be the largest integer such that $\mathrm{Comp}(x_{j+1}, f^j, W_1) \cap f^{-j}(\{c\})$ consists of a critical point. (This is a singleton if θ is small enough.) We call $(x_0, \dots, x_{j+1})$ a *critical block*, with respect to θ. Then by definition of θ'

$$r := \mathrm{diam}\big(\mathrm{Comp}\big(x_{j+1}, f^{j+1}, B(x,\theta)\big)\big) \leq \theta' \leq \theta_0.$$

The critical block $(x_0, \dots, x_{j+1})$ is followed by a block $(x_{j+1}, \dots, x_{j+1+s_0})$ defined as above for the sequence $x_{j+1}, x_{j+2}, \dots$ and the disk $B = B(x_{j+1}, 2r)$. By the definition of θ_0 this block is *long*, namely $s_0 > N_0$. Denote the critical point captured by W_{s_0} by c'. Then, as in (B.4), for $s = s_0$

$$\begin{aligned} & \mathrm{diam}\big(\mathrm{Comp}(x_{(j+1)+(s-1)}, f^{j+s}, B\big(x,\theta)\big)\big) \leq \mathrm{diam}\big(f(W_s)\big) \\ & \quad \leq C_1 |(f^{s-1})'\big(f(c')\big)|^{-1} b_s^{-1} 2r \\ & \quad \leq C_1 |(f^{(s-1)+(j+1)})'\big(f(c')\big)|^{-1} |(f^{j+1})'\big(f^s(c')\big)| b_s^{-1} 2r. \end{aligned}$$

We have

$$|(f^{(s+j)})'(f(c'))|^{-1} b_s^{-1} \le \lambda_s^{-1} \tag{B.5}$$

and $|(f^{j+1})'(f^s(c'))|r \le C_6 2\theta$. Hence

$$\operatorname{diam}\big(\operatorname{Comp}(x_{j+s}, f^{j+s}, B(x,\theta))\big) \le 4C_1 C_6 \lambda_s^{-1}\theta \le \tfrac{1}{2}\theta. \tag{B.6}$$

Fix now $x \in J(f)$ and its backward trajectory (x_s) and divide it into blocks as follows. The first block is $x_0, \dots, x_{s_0-1}$ defined as above for the disk $B(x, 2\theta^-)$ for $\theta^- := \frac{1}{2}C_1^{-1}C_5\theta$. If $s_0 = 1$, this block does not appear. By (B.4)

$$\operatorname{Comp}\big(x_{s_0-1}, f^{s_0-1}, B(x,\theta^-)\big) \subset B(x_{s_0-1}, \theta). \tag{B.7}$$

The next block is composed of a critical block followed by a long block, for the sequence $x_{s_0-1}, x_{s_0}, \dots$ and the disk $B = B(x_{s_0-1}, 2\theta)$. By (B.6), for $s+j$ appearing there, denoted below by s_1, we obtain

$$\operatorname{Comp}\big(x_{s_1}, f^{s_1}, B(x_{s_0-1},\theta)\big) \subset B(x_{s_1}, \tfrac{1}{2}\theta). \tag{B.8}$$

We continue with blocks composed of critical blocks followed by long ones and we obtain for $\Sigma_n = \sum_{j=0}^n s_j - 1$

$$\operatorname{Comp}\big(x_{\Sigma_n}, f^{\Sigma_n}, B(x,\theta^-)\big) \subset B(x_{\Sigma_n}, 2^{-n}\theta).$$

Note that if $\theta \to 0$, $r \to 0$ (see the paragraph where we introduced the *long* block notion), then

$$\sup_{0\le s'<s} \big\{\operatorname{diam}\big(\operatorname{Comp}(x_{j+1+s'}, f^{s'}, B(x_{j+1}, r))\big)\big\} \to 0.$$

This is so by Lemma 2.4 applied with $M = 0$, since the adequate branches of $f^{-s'}$ extend to $B(x_{j+1}, \frac{3}{2}r)$; see (B.3). Therefore $\operatorname{diam}\big(\operatorname{Comp}(x_j, f^j, B(x,\theta^-)\big) \to 0$ and the convergence is uniform. Otherwise, if there existed $L > 0$ and sequences $x(k)$ and $n(k)$ such that

$$\operatorname{diam}\big(\operatorname{Comp}(x(k)_{\Sigma_{n(k)}}, f^{\Sigma_{n(k)}}, B(x(k),\theta^-)\big) \ge L$$

and $\Sigma_{n(k)} \to \infty$, then $n(k)$ would be bounded. So, there would exist k with an arbitrarily long block. Hence, in (B.7), if this is the first block, or in (B.8) for one of the blocks which follow, the diameter of the right-hand ball could be θ multiplied by an arbitrarily small factor. So, the estimate by L resulting from the composition of blocks would be false. This is a contradiction.

Finally if an infinite block occurs after a sequence of finite blocks, the convergence is also uniform since the convergence along infinite blocks is uniform (see Lemma 2.4).

Consider now the case when $\Omega \neq \emptyset$. Then, similarly as in the proof of Theorem 5.2, we set $b_n = 0$ for all n such that n and $n-1 \notin \mathscr{N}(\delta)$, where $\mathscr{N}(\delta) := \bigcup_{c \in \mathrm{Crit}^e} \mathscr{N}\bigl(f(c), \delta\bigr)$. For other n, i.e., such that there exists $c \in \mathrm{Crit}^e$ for which $f^{n-1}\bigl(f(c)\bigr) \notin B(\Omega, \delta)$ or $f^n\bigl(f(c)\bigr) \notin B(\Omega, \delta)$, we find b_n such that $\sum b_n < \infty$, moreover (B.3) holds, and (B.2) holds with $\sum_{j=0}^{N+1}$ replaced by the sum only over $j \in [0, N+1] \cap \mathscr{N}(\delta)$.

Let n_j be all consecutive integers in $\mathscr{N}(\delta)$. Then we modify the definition of N_0 so that $\lambda_n \geq 8C_1C_6$ for all $n = n_j \geq N_0$.

We consider x or $x_1 \notin B(\Omega, \delta)$ and proceed as in the case when $\Omega = \emptyset$ case, getting $b_n \neq 0$ whenever W_n captures a critical point. This gives the needed Koebe's space (see the proof of Theorem 5.2, Step 2). Finally for a critical block $x, x_1, \dots, x_{j+1}$ followed by $x_{j+1}, \dots, x_{j+1+s}$ we can prove (B.6) due to (B.5). The latter holds since $s+j \in \mathscr{N}\bigl(f(c'), \delta\bigr)$ which is true since $f^{s+j-1}\bigl(f(c')\bigr) \in B(c, \theta_0)$. So $f^{s+j-1}\bigl(f(c')\bigr) \notin B(\Omega, \delta)$, for θ, δ small enough. □

Added in proof. Theorem 1.1 was proved independently by Yin Yongcheng: Geometry and dimension of Julia sets. - In: The Mandebrot Set, Theme and Variations, edited by Tan Lei. London Mathematical Society Lecture Notes Series 274, Cambridge Univ. Press, 2000, 281–287.

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