GROWTH PROPERTIES OF SPHERICAL MEANS FOR MONOTONE BLD FUNCTIONS IN THE UNIT BALL

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Abstract. Let \( u \) be a monotone (in the sense of Lebesgue) function on the unit ball \( B \) of \( \mathbb{R}^n \) satisfying
\[
\int_B |\nabla u(x)|^p (1 - |x|)^\alpha \, dx < \infty,
\]
where \( \nabla \) denotes the gradient, \( 1 < p < \infty \) and \( -1 < \alpha < p - 1 \). Then \( u \) has a boundary limit \( f(\xi) \) for almost every \( \xi \in \partial B \), and \( u \) may be considered as a Dirichlet solution for the boundary function \( f \). Our aim in this paper is to deal with growth properties of the spherical means
\[
S_q(u_r - f) \equiv \left( \int_{\partial B} |u(r\xi) - f(\xi)|^q \, dS(\xi) \right)^{1/q}.
\]
In fact, we prove that
\[
\lim_{r \to 1-0} (1 - r)^{-\omega} S_q(u_r - f) = 0
\]
when \( p \leq q \leq \infty \) and \( \omega = (n - 1)/q - (n - p + \alpha)/p > 0 \).

1. Introduction

If \( f \) is a \( p \)th summable function on the boundary \( S \) of the unit ball \( B \) in \( \mathbb{R}^n \), then we can find a (weak) Dirichlet solution \( u \) which is harmonic in \( B \) and
\[
\lim_{r \to 1-0} S_p(u_r - f) = 0,
\]
where \( 1 \leq p < \infty \), \( u_r(z) = u(rz) \) for \( z \in S \) and
\[
S_p(v) = \left( \frac{1}{\sigma_n} \int_S |v(z)|^p \, dS(z) \right)^{1/p}
\]
with \( \sigma_n \) denoting the surface area of \( S \). Furthermore, for functions \( f \) in some Lipschitz spaces \( \Lambda_{\beta,p}(S) \) (see Stein [16]), we can find Dirichlet solutions \( u \) in BLD

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(Beppo Levi and Deny) spaces for which (1) is satisfied. Conversely, if \( u \) is a harmonic function on \( B \) satisfying

\[
\int_B |\nabla u(x)|^p (1 - |x|)^\alpha \, dx < \infty,
\]

where \( 1 < p < \infty \) and \(-1 < \alpha < p - 1\), then it is well known that \( u \) has nontangential limits \( f(\xi) \) at many boundary points \( \xi \in S \) and \( f \) belongs to a certain Lipschitz space (see Stein [16] and the first author [9], [10]). Of course, \( u \) is a Dirichlet solution for \( f \).

In this paper, we are concerned with weighted limits of \( S_q(u_r - f) \) for monotone BLD functions \( u \) on \( B \) with boundary values \( f \). We say that a continuous function \( u \) is monotone in an open set \( G \), in the sense of Lebesgue, if both

\[
\max_D u(x) = \max_{\partial D} u(x) \quad \text{and} \quad \min_D u(x) = \min_{\partial D} u(x)
\]

hold for every relatively compact open set \( D \) with the closure \( \overline{D} \subset G \) (see [5]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. For these facts, see Gilbarg–Trudinger [2], Heinonen–Kilpeläinen–Martio [3], Reshetnyak [13], Serrin [14] and Vuorinen [17] and [18].

Our starting point is a result of Gardiner [1, Theorem 2] which states that if \( u \) is a Green potential in the unit ball \( B \), then

\[
\liminf_{r \to 1^-} (1 - r)^{(n-1)(1-1/q)} S_q(u_r) = 0
\]

when \((n - 3)/(n - 1) < 1/q \leq (n - 2)/(n - 1)\) and \( q > 0 \).

Our first aim in this paper is to show the following result.

**Theorem 1.** Let \( u \) be a monotone function on \( B \) satisfying (2) with \( n - 1 < p \leq n + \alpha \). If \( p \leq q < \infty \) and

\[
\omega = \frac{n - 1}{q} - \frac{n - p + \alpha}{p} > 0,
\]

then

\[
\lim_{r \to 1^-} (1 - r)^{-\omega} S_q(U_r) = 0,
\]

where \( U_r(\xi) = u(r\xi) - u(\xi) \) for \( \xi \in S \).

The sharpness of the exponent will be discussed in the final section. We also find a BLD function \( u \) satisfying (2) and

\[
\limsup_{r \to 1^-} (1 - r)^{-\omega} S_q(U_r) = \infty,
\]

when \( \alpha < 0 \).
Corollary 1. Let \( u \) be a coordinate function of a quasiregular mapping on \( B \) satisfying (2). If \( n-1 < p \leq n+\alpha, p \leq q < \infty \) and \( \omega = (n-1)/q - (n-p+\alpha)/p > 0 \), then
\[
\lim_{r \to 1^{-}} (1-r)^{-\omega} S_{q}(U_{r}) = 0.
\]

For the definition and basic properties of quasiregular mappings, we refer to [3], [13], and [17]. In particular, a coordinate function \( u = f_{i} \) of a quasiregular mapping \( f = (f_{1}, \ldots, f_{n}): B \to \mathbb{R}^{n} \) is \( \mathcal{A} \)-harmonic (see [3, Theorem 14.39] and [13]) and monotone in \( B \), so that Theorem 1 gives the present corollary.

It is well known that the coordinate functions of a bounded quasiconformal mapping on \( B \) have finite \( n \)-Dirichlet integral (see Vuorinen [18]), so that Corollary 1 gives the following result.

Corollary 2. Let \( u \) be a coordinate function of a bounded quasiconformal mapping on \( B \). If \( n \leq q < \infty \), then
\[
\lim_{r \to 1^{-}} (1-r)^{-(n-1)/q} S_{q}(U_{r}) = 0.
\]

Next we are concerned with the case \( q = \infty \). In order to give a general result, we consider a nondecreasing positive function \( \varphi \) on the interval \( [0, \infty) \) such that \( \varphi \) is of log-type, that is, there exists a positive constant \( M \) satisfying
\[
\varphi(r^{2}) \leq M \varphi(r) \quad \text{for all } r \geq 0.
\]
Set \( \Phi_{p}(r) = r^{p} \varphi(r) \) for \( p > 1 \). Our next aim is to study the boundary behavior of monotone BLD functions \( u \) on \( B \), which satisfy the weighted condition
\[
\int_{B} \Phi_{p}(|\nabla u(x)|)(1-|x|)^{\alpha} \, dx < \infty.
\]
Consider the function
\[
\kappa(r) = \left[ \int_{0}^{r} \left( t^{n-p+\alpha} \varphi(t^{-1}) \right)^{-1/(p-1)} \frac{dt}{t} \right]^{1-1/p}
\]
for \( r > 0 \). We see (cf. [15, Lemma 2.4]) that if \( n-p+\alpha < 0 \), then
\[
\kappa(r) \sim [r^{n-p+\alpha} \varphi(r^{-1})]^{-1/p} \quad \text{as } r \to 0
\]
and if \( n-p+\alpha = 0 \) and \( \varphi(r) = (\log(e+r))^{\sigma} \) with \( \sigma > p-1 \), then
\[
\kappa(r) \sim [\log(1/r)]^{(p-1-\sigma)/p} \quad \text{as } r \to 0.
\]

Theorem 2. Let \( u \) be a monotone function on \( B \) satisfying (4) with \( -1 < \alpha \leq p-n \). If \( \kappa(1) < \infty \), then
\[
\lim_{r \to 1^{-}} [\kappa(1-r)]^{-1} S_{\infty}(U_{r}) = 0,
\]
where \( U_{r}(\xi) = u(r\xi) - u(\xi) \) for \( \xi \in S \).
Corollary 3. Let $u$ be a coordinate function of a quasiregular mapping on $\mathbf{B}$ satisfying (4) with $-1 < \alpha \leq p - n$. If $\kappa(1) < \infty$, then
\[
\lim_{r \to 1^{-0}} [\kappa(1 - r)]^{-1} S_{\infty}(U_r) = 0.
\]

For related results, we also refer to Herron–Koskela [4], the first author [8], [11] and the authors [12].

Finally we wish to express our deepest appreciation to the referee for his useful suggestions.

2. Proof of Theorem 1

Throughout this paper, let $M$ denote various constants independent of the variables in question.

For a proof of Theorem 1, we need the following result, which gives an essential tool in treating monotone functions.

Lemma 1 (cf. [4], [6], [10]). Let $p > n - 1$. If $u$ is a monotone BLD function on $B(x_0, 2r)$, then
\[
|u(x) - u(y)|^p \leq Mr^{p-n} \int_{B(x_0,2r)} |\nabla u(z)|^p \, dz \quad \text{whenever } x, y \in B(x_0, r).
\]

Lemma 1 is a consequence of Sobolev’s theorem, so that the restriction $p > n - 1$ is needed; for a proof of Lemma 1, see for example [4, Lemma 7.1] or [10, Theorem 5.2, Chapter 8].

Now we are ready to prove Theorem 1, along the same lines as in the proof of [12, Theorem 2].

Proof of Theorem 1. Let $u$ be a monotone function on $\mathbf{B}$ satisfying (2) with $n - 1 < p \leq n + \alpha$. If $|s - t| \leq r < \frac{1}{2}(1 - t)$, then Lemma 1 gives
\[
|S_q(u_s - u_t)| = \left( \frac{1}{\sigma_n} \int_S [u(s\xi) - u(t\xi)]^q \, dS(\xi) \right)^{1/q} 
\leq Mr^{(p-n)/p} \left( \int_S \left( \int_{B(t\xi,2r)} |\nabla u(z)|^p \, dz \right)^{q/p} \, dS(\xi) \right)^{1/q},
\]
so that Minkowski’s inequality for integrals yields
\[
|S_q(u_s - u_t)| \leq Mr^{(p-n)/p} \left( \int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \, dz \right) 
\times \left( \int_{\xi \in S: |\xi - z/t| < 2r/t} dS(\xi) \right)^{p/q} 
\leq Mr^{(p-n)/p} (2r/t)^{(n-1)/q} \left( \int_{B(0,t+2r)-B(0,t-2r)} |\nabla u(z)|^p \, dz \right)^{1/p}.
\]
Let $r_j = 2^{-j-1}$, $t_j = 1 - r_{j-1}$ and $A_j = B(0, 1 - r_j) - B(0, 1 - 3r_j)$ for $j = 1, 2, \ldots$

For simplicity, set

$$\omega = -(n-p+\alpha)/p + (n-1)/q > 0.$$ 

Then we find

$$|S_q(u_{t_j} - u_r)| \leq Mr^\omega_{j+1} \left( \int_{A_j} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j \leq r < t_j + r_{j+1}$,

$$|S_q(u_r - u_s)| \leq Mr^\omega_{j+2} \left( \int_{A_j} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j + r_{j+1} \leq r < t_j + r_{j+1} + r_{j+2}$, and

$$|S_q(u_s - u_{t_{j+1}})| \leq Mr^\omega_{j+2} \left( \int_{A_{j+1}} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j + r_{j+1} + r_{j+2} \leq s < t_{j+1}$. Collecting these results, we have

$$|S_q(u_{t_j} - u_r)| \leq Mr^\omega_j \left( \int_{A_j} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

$$+ Mr^\omega_{j+1} \left( \int_{A_{j+1}} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j \leq r < t_{j+1}$. Hence it follows that

$$|S_q(u_r - u_{t_{j+m}})| \leq M \sum_{l=j}^{j+m} r_l^\omega \left( \int_{A_l} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j \leq r < t_{j+m}$. Since $A_l \cap A_k = \emptyset$ for $l \geq k + 2$, Hölder's inequality gives

$$|S_q(u_r - u_{t_{j+m}})| \leq M \left( \sum_{l=j}^{j+m} r_l^{p'} \right)^{1/p'} \left( \sum_{l=j}^{j+m} \int_{A_l} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

$$\leq Mr^\omega_j \left( \int_{B(0,1-r_{j+m})-B(0,1-3r_j)} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j \leq r < t_{j+m}$, where $1/p + 1/p' = 1$. Now, letting $m \to \infty$, we establish

$$|S_q(U_r)| \leq M(1-r)^\omega \left( \int_{B(0,1-r)} |\nabla u(z)|^p (1 - |z|)^{\alpha} \, dz \right)^{1/p}$$

for $t_j \leq r < t_{j+1}$, which implies that

$$\lim_{r \to 1} (1-r)^{-\omega} S_q(U_r) = 0,$$

as required.
3. Proof of Theorem 2

If \( B(x, 2r) \subset \mathcal{B} \), then, applying Lemma 1 and dividing the domain of integration into two parts

\[ E_1 = \{ z \in B(x, 2r) : |\nabla u(z)| > r^{-\delta} \}, \]
\[ E_2 = B(x, 2r) - E_1, \]

we have

\[ |u(x) - u(y)|^p \leq M r^{(1-\delta)} + M r^{p-n} \left[ \varphi(r^{-1}) \right]^{-1} \int_{B(x, 2r)} \Phi_p(|\nabla u(z)|) \, dz \]

for \( y \in B(x, r) \), where \( M \) may depend on \( \delta, 0 < \delta < 1 \).

Let \( r_j = 2^{-j-1}, j = 0, 1, \ldots \). For \( x \in \mathcal{B} \), let \( x_j = (1 - 2r_j) \xi \) with \( \xi = x/|x| \). Then we find

\[ |u(x_j) - u(x)| \leq M r_j^{1-\delta} + M r_j^{p-n} \left[ \varphi(r_j^{-1}) \right]^{-1/p} \left( \int_{B(x_j, r_j)} \Phi_p(|\nabla u(z)|) \, dz \right)^{1/p} \]

when \( \frac{3}{2} r_j < \varrho(x) < \frac{5}{2} r_j \), where \( \varrho(x) = |1 - |x|| \). Hölder’s inequality gives

\[ |u(x_{j+m}) - u(x_j)| \leq M \sum_{l=j}^{j+m} r_l^{1-\delta} + \sum_{l=j}^{j+m} r_l^\omega \left[ \varphi(r_l^{-1}) \right]^{-1/p} \]

\[ \times \left( \int_{B(x_l, r_l)} \Phi_p(|\nabla u(z)|) \varrho(z)^\alpha \, dz \right)^{1/p} \]

\[ \leq M r_j^{1-\delta} + M \left( \sum_{l=j}^{j+m} r_l^{p'\omega} \left[ \varphi(r_l^{-1}) \right]^{-p'/p} \right)^{1/p'} \]

\[ \times \left( \sum_{l=j}^{j+m} \int_{B(x_l, r_l)} \Phi_p(|\nabla u(z)|) \varrho(z)^\alpha \, dz \right)^{1/p} \]

\[ \leq M r_j^{1-\delta} + M r_j^\alpha \left( \int_{B-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|) \varrho(z)^\alpha \, dz \right)^{1/p}, \]

where \( \omega = (p - n - \alpha)/p \). Since \( \lim_{r \to 0} \kappa(r) = 0 \), we see that \( \{ u(x_j) \} \) is a Cauchy sequence. Thus (6) implies that \( \lim_{r \to 1} u(r \xi) \) exists and is finite for every \( \xi \in \mathcal{S} \). Now, letting \( m \to \infty \), we have

\[ |U(x_j)| = |u(x_j) - u(\xi)| \]

\[ \leq M r_j^{1-\delta} + M r_j^\alpha \left( \int_{B-B(0,1-3r_j)} \Phi_p(|\nabla u(z)|) \varrho(z)^\alpha \, dz \right)^{1/p}. \]
Hence we establish

\[ |U(x)| \leq M \varrho(x)^{1-\delta} + M \kappa(\varrho(x)) \left( \int_{B-B(0,1-3\varrho(x))} \Phi_p(|\nabla u(z)|\varrho(z)^\alpha \, dz \right)^{1/p}. \]

Note that

\[ \kappa(r) \geq \left[ \int_{r/2}^r (t^{n-p+\alpha} \varphi(t^{-1}))^{-1/(p-1)} \frac{dt}{t} \right]^{1-1/p} \geq [r^{n-p+\alpha} \varphi(r^{-1})]^{-1/p}. \]

If we take \( \delta \) so that \((n - p + \alpha)/p + 1 > \delta > 0\), then

\[ \lim_{r \to 0} [\kappa(r)]^{-1} r^{1-\delta} = 0. \]

Now it follows that

\[ \limsup_{|x| \to 1} [\kappa(\varrho(x))]^{-1} |U(x)| = 0, \]

as required.

The above proof also shows the following (see [7] and [9]).

**Proposition 1.** Let \( u \) be a monotone function on \( B \) satisfying (4) with \(-1 < \alpha \leq p - n\). If \( \kappa(1) < \infty \), then \( u \) has a finite nontangential limit at every \( \xi \in \mathbf{S} \).

**Proposition 2.** Let \( u \) be a monotone function on \( B \) satisfying (4) with \( n - 1 < p \leq n + \alpha \). If \( \kappa(1) = \infty \), then

\[ \lim_{r \to 1} (1-r)^{\omega} \left[ \varphi((1-r)^{-1}) \right]^{1/p} S_q(U_r) = 0 \]

whenever \( p \leq q < \infty \) and \( \omega = (n-1)/q - (n - p + \alpha)/p > 0 \).

### 4. Sharpness

1. **The sharpness of the exponent \(-\omega\).** Let \(-1 < \alpha < p - 1\). For \( \delta > 0 \), consider the function \( u \) on \( B \) defined by

\[ u(x) = (1 - |x|)^{1+a}|x-e|^{-b}, \]

where \( a = \delta - (\alpha + 1)/p \), \( b = (n - 1)/p \) and \( e = (1,0,\ldots,0) \). Then the function \( u \) is monotone on \( B \). To show this, let \( D \) be a relatively compact open set with the closure \( \overline{D} \subset B \), and suppose \( u \) attains a maximum on \( \overline{D} \) at an interior point \( c \in D \), and set

\[ E = \{ x \in \overline{B} : (1 - |x|)^{1+a} = u(c)|x-e|^{-b} \}. \]
Since $e \in E$,
\[
\max_{\overline{D}} u(x) = \max_{\partial D} u(x)
\]
holds. We also see that $u$ attains a minimum on $\overline{D}$ at a boundary point. Hence $u$ is monotone on $B$. Further we have
\[
\int_B |\nabla u(x)|^p \varrho(x)^\alpha dx < \infty.
\]
If $k(x) = |x - e|^{-b}$, then
\[
S_q(u_r) = M(1 - r)^{1 + \alpha} S_q(k_r) \geq M(1 - r)^{1 + \alpha + (1/q-1/p)(n-1)} = M(1 - r)^{\omega + \delta}.
\]
This implies that the exponent $-\omega$ is sharp in Theorem 1.

2. The limits for BLD functions. Theorem 1 fails to hold for BLD functions, when $\alpha < 0$ and $(n - p + \alpha)/p < (n - 1)/q < (n - p)/p$.

For $0 < r_j < 1$ and $\gamma > 1$, set
\[
C_j = \{x : |x - r_j e| \leq (1 - r_j)^\gamma\},
\]
where $|e| = 1$. We take $\{r_j\}$ such that $\{C_j\}$ are mutually disjoint. For $\varphi \in C_0^\infty(B(0,1))$ such that $\varphi = 1$ on $B(0, \frac{1}{2})$, define
\[
u_j(x) = \varphi(|x - r_j e|/(1 - r_j)^\gamma)|x - r_j e|^{-a},
\]
where
\[
(n - p + \alpha)/p < a < (n - p + \alpha/\gamma)/p.
\]
Then
\[
\int_B |\nabla \nu_j(x)|^p(1 - |x|)^\alpha dx \leq M(1 - r_j)^{\alpha + \gamma(n-(a+1)p)}.
\]
Note further that
\[
S_q((\nu_j)_{r_j}) \geq c(1 - r_j)^{\gamma(-a+(n-1)/q)}
\]
with a positive constant $c > 0$, when $a < (n - 1)/q$. If we set $u = \sum_j \nu_j$, then
\[
\lim_{j \to \infty} (1 - r_j)^{-\omega} S_q(u_{r_j}) = \infty,
\]
when $a$ and $\gamma$ are chosen so that
\[
(n - p + \alpha/\gamma)/p + (1 - \gamma^{-1})\{(n - 1)/q - (n - p)/p\} < a < (n - 1)/q.
\]
Now we suppose that $a$ satisfies (7) and (8). Then, since $\alpha + \gamma(n - (a + 1)p) > 0$, we can take $\{r_j\}$ so that

$$\int_B |\nabla u(x)|^p(1 - |x|)^\alpha \, dx < \infty.$$  

Thus $u$ satisfies (2) and

$$\limsup_{r \to 1} (1 - r)^{-\omega} S_q(U_r) = \infty.$$  

References


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