SEQUENTIAL CONVERGENCES AND DUNFORD–PETTIS PROPERTIES

Jesús Angel Jaramillo(1), Angeles Prieto(1), and Ignacio Zalduendo(2)

(1) Universidad Complutense de Madrid, Departamento de Análisis Matemático,
Facultad de Matemáticas, E-28040 Madrid, Spain
(2) Universidad de San Andrés, Departamento de Economía y Matemática
CC 35-1544 Victoria, Prov. de Buenos Aires, Argentina

Abstract. Several forms of the Dunford–Pettis property are studied, each related to a different mode of sequential convergence, and a different class of weakly compact functions. The relationship between these Dunford–Pettis properties is investigated, and the appearance of previously studied Dunford–Pettis properties is pointed out, giving a unifying approach to the subject.

Introduction

In recent years several forms of the Dunford–Pettis property have been studied (see [6], [2], [12], [16]) which conform more or less to the following scheme: if $\tau$ is some type of sequential convergence in a Banach space $X$, and $\mathcal{A}$ is a class of functions (linear, polynomial, holomorphic) defined on $X$, one can define a Dunford–Pettis property on $X$ by requiring

“For all $Y$, all weakly compact $F \in \mathcal{A}(X,Y)$, and all $\tau$-null sequences $(x_n)$ in $X$, $F(x_n)$ converges in norm to $F(0)$.”

Clearly, the stronger the convergence $\tau$ is, the weaker the corresponding property will be; and larger classes of functions $\mathcal{A}$ will result in stronger properties. Our aim in this paper is to clarify the relationships between some of these properties and give a unified approach to the matter.

We find that—for any fixed type of convergence $\tau$—taking $\mathcal{A}$ to be the class of linear operators, $k$-homogeneous polynomials, or holomorphic functions of bounded type produces the same Dunford–Pettis property. On the other hand, if one takes $\mathcal{A}$ to be the class of all holomorphic functions, a strictly stronger property may be obtained. We give conditions under which this happens. Finally, we concentrate on $\tau = H$ (holomorphic sequential convergence) and find that in this case even the strongest form of Dunford–Pettis property—when $\mathcal{A}$ is the class of all holomorphic functions—holds for any Banach space.

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Throughout, $X$ will be a Banach space. If $Y$ is another Banach space, $L(X,Y)$ will denote the space of continuous linear operators from $X$ to $Y$. A continuous $k$-homogeneous polynomial $P: X \to Y$ is given by $P(x) = A(x, \ldots, x)$, where $A$ is a continuous $k$-linear function. $P$ determines a unique $A$ if we require it to be symmetric. The space of all such polynomials will be denoted by $P^k(X,Y)$ (or simply $P^k(X)$ if $Y$ is the scalar field) and is a Banach space when endowed with the norm
\[\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}\]
A function $f: X \to Y$ is holomorphic if it can be locally expressed as a uniformly convergent power series. We will denote the space of all such functions by $H(X,Y)$, and by $H_b(X,Y)$ the subspace of those functions which are bounded on the bounded subsets of $X$ (i.e., the holomorphic functions of bounded type). These spaces will be denoted by $H(X)$ and $H_b(X)$ in the scalar-valued case, respectively. $H_b(X)$ is a Fréchet space when endowed with the topology $\tau_b$ of uniform convergence over bounded subsets of $X$. Note that $P^k(X)$ with the norm defined above is a closed subspace of $H_b(X)$. We consider $H(X)$ with the Nachbin topology $\tau_\omega$ (see [10, 3.14]). We will say that a mapping is weakly compact if it sends some neighborhood of the origin onto a relatively weakly compact set.

For more on polynomials and holomorphic functions see [10], and for an extensive survey on the Dunford–Pettis property see [8].

We will use several different forms of sequential convergence in $X$. Dudley [11] defines a sequential convergence $\tau$ in $X$ as a relation “$\to_\tau$” between sequences and elements of $X$ such that:

(i) If $x_n = x$ for all $n$, then $x_n \to_\tau x$.
(ii) If $x_n \to_\tau x$ and $(x_{n_k})$ is a subsequence of $(x_n)$, then $x_{n_k} \to_\tau x$.
For our purposes, we add the following two conditions:

(iii) If $x_n \to_\tau 0$ then $(x_n)$ is weakly null.
(iv) $x_n \to_\tau x$ if and only if $x_n - x \to_\tau 0$.

Many topologies and sequential convergences have appeared in the literature in accordance with our definition. Among them:

1. Weak convergence $\tau = \omega$.
2. The convergence $\tau_s$ introduced by Pełczyński [14]. A sequence $(x_n)$ is said to be $\tau_s$-null for $s \in [0,1)$ if there exists a constant $c > 0$ such that for any $k \in \mathbb{N}$ and any collection of distinct natural numbers $n_1, \ldots, n_k$, $\|\theta_1 x_{n_1} + \cdots + \theta_k x_{n_k}\| \leq c k^s$ whenever $|\theta_i| = 1$.
3. Weak $p$-summability $\tau = \omega_p$. The sequence $(x_n)$ is said to be $\omega_p$-null if $(x'(x_n)) \in l^p$ for each $x' \in X'$.
4. The convergence $\tau = P(\leq k)X$ [12]. In this case $(x_n)$ is considered $\tau$-convergent to $x$ if $P(x_n)$ converges to $P(x)$ for any continuous scalar-valued polynomial $P$ of degree $\leq k$.
5. Polynomial convergence. The sequence $(x_n)$ is polynomially convergent to $x$ if $P(x_n)$ converges to $P(x)$ for any continuous scalar-valued polynomial $P$. 


Note that this is equivalent to \( f(x_n) \) converging to \( f(x) \) for any holomorphic scalar-valued function of bounded type \( f \).

(6) Holomorphic convergence [15]. Here \( (x_n) \) is \( H \)-convergent to \( x \) when \( f(x_n) \) converges to \( f(x) \) for any holomorphic scalar-valued function \( f \).

\[ \tau \text{-Dunford–Pettis properties} \]

We begin by proving that for any fixed \( \tau \), the classes \( \mathcal{A} \) of linear, polynomi- nal, or bounded type holomorphic functions give rise to the same Dunford–Pettis property. We will use the following lemma.

**Lemma 1.** If for every \( \tau \)-null sequence \( (x_n) \) in \( X \) and for every \( k \) and every weakly null sequence \( (P_n) \) in \( P^k(X) \), \( P_n(x_n) \to 0 \), then all scalar-valued polynomials are \( \tau \)-sequentially continuous. Consequently, the same holds for scalar-valued holomorphic functions of bounded type.

**Proof.** We need only check the conclusion for homogeneous polynomials. This we do by induction on the degree \( k \) of a homogeneous polynomial \( Q \). For \( k = 1 \) this is clear because \( \tau \)-convergence implies weak convergence. Suppose then that the result holds for \( (k - 1) \)-homogeneous polynomials, and consider a \( \tau \)-null sequence \( (x_n) \) and a \( k \)-homogeneous scalar-valued polynomial \( Q \). Define \( T_Q : X \to P^{k-1}(X) \) by \( T_Q(x) = A(x, \ldots, \cdot) \) where \( A \) is the symmetric \( k \)-linear form associated to \( Q \). \( T_Q \) is a continuous linear operator, so for all \( \gamma \in P^{k-1}(X) \), \( \gamma(T_Q(x_n)) \to 0 \), thus \( (T_Q(x_n)) \) is a weakly null sequence of \( (k - 1) \)-homogeneous polynomials, and by our inductive hypothesis \( Q(x_n) = T_Q(x_n)(x_n) \to 0 \).

If \( f \in H_b(X) \), its Taylor series converges uniformly over bounded sets such as \( (x_n) \). Thus, \( f \) is also \( \tau \)-sequentially continuous. \( \square \)

**Theorem 2.** For any Banach space \( X \), the following are equivalent.

(\( a \)) For all \( Y \), all weakly compact \( T \in L(X,Y) \), and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( T(x_n) \to 0 \) in norm.

(\( a' \)) For all weakly null sequences \( \gamma_n \subset X' \) and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( \gamma_n(x_n) \to 0 \).

(\( b \)) For all \( Y \), all weakly compact \( P \in P^k(X,Y) \) \((k \geq 1)\), and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( P(x_n) \to 0 \) in norm.

(\( b' \)) For all weakly null sequences \( (P_n) \subset P^k(X) \) \((k \geq 1)\), and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( P_n(x_n) \to 0 \).

(\( c \)) For all \( Y \), all weakly compact \( F \in H_b(X,Y) \), and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( F(x_n) \to F(0) \) in norm.

(\( c' \)) For all weakly null sequences \( (f_n) \subset H_b(X) \) and all \( \tau \)-null sequences \( (x_n) \) in \( X \), \( f_n(x_n) \to 0 \).

**Proof.** We show first that (c) and (c') are equivalent. The equivalences (a) \( \iff \) (a') and (b) \( \iff \) (b') are analogous—and perhaps less technical—so we will omit them.
(c) implies (c'): Let \((f_n)\) and \((x_n)\) be as in (c'), and define \(F: X \rightarrow c_0\) by \(F(x) = (f_n(x))\). Since \((f_n)\) is weakly null and point evaluations are continuous linear forms over \(H_b(X)\), \(F\) is well defined. Since \((f_n)\) is weakly bounded, it is \(\tau_b\)-bounded (both topologies produce the same dual space), and thus bounded for the compact-open topology. Hence \((f_n)\) are uniformly bounded on compact subsets of \(X\), and so \(F\) is locally bounded. Thus, to see that \(F\) is holomorphic, we only need to prove weak holomorphicity. Take then \(a \in l^1\), and consider \((a \circ F)(x) = \sum_n a_n f_n(x)\). Since \(a \circ F\) is a uniform limit (over bounded sets) of holomorphic functions of bounded type, \(a \circ F \in H_b(X)\) for each \(a\), and thus \(F \in H_b(X, c_0)\).

Now consider the transpose \(F^t: l^1 \rightarrow H_b(X)\) given by \(F^t(a) = a \circ F\). By [17, 3.4] if we see that \(F^t\) is weakly compact, then \(F\) is also weakly compact. So take \(a\) in the unit ball of \(l^1\) and note that \(F^t(a) = a \circ F\) is in the closed absolutely convex hull of \(\{f_n : n \geq 1\} \cup \{0\}\). Since this set is weakly compact in the metric space \(H_b(X)\), so is its hull [18, IV.11.4]. Thus \(F^t\), and \(F\), are weakly compact. Since \((x_n)\) is \(\tau\)-null, \(\lim_n F(x_n) = F(0)\) exists, and

\[
|f_n(x_n)| \leq |f_n(x_n) - f_n(0)| + |f_n(0)| \leq \|F(x_n) - F(0)\| + |f_n(0)|
\]

so \(\lim_n f_n(x_n) = 0\), because \(\lim_n f_n(0) = 0\).

(c') implies (c): Let \(F\) and \((x_n)\) be as in (c). We may suppose \(F(0) = 0\), and prove that \(\lim_n F(x_n) = 0\). If this were not the case, take \(\varepsilon > 0\) and a subsequence \((x_{n_k})\) with \(\|F(x_{n_k})\| > \varepsilon\) for all \(k\). Consider \((y'_{n_k}) \subset Y'\) of norm one and such that \(y'_{n_k}(F(x_{n_k})) = \|F(x_{n_k})\|\). Now consider the transpose of \(F\), \(F^t: Y' \rightarrow H_b(X)\). This is a continuous linear operator, and is weakly compact because \(F\) is. Take then a subsequence \((y'_{n_k})\) with \(F^t(y'_{n_k})\) converging weakly to \(g \in H_b(X)\). Since \((x_n)\) is \(\tau\)-null and \(F^t(y'_{n_k}) - g\) is weakly null, then \((F^t(y'_{n_k}) - g)(x_{n_k}) \rightarrow 0\). But by Lemma 1 (note that the \(\tau_b\) topology when restricted to \(P^k(X)\) coincides with the norm topology) \(g(x_{n_k}) \rightarrow 0\), a contradiction. Thus \(\lim_n F(x_n) = 0\).

Clearly (c) implies (b) and (b) implies (a). Thus we will be done if we show that (a) \(\Rightarrow\) (b') and (b) \(\Rightarrow\) (c).

(a) implies (b']): We prove this by induction. The case of degree \(k = 1\) is simply (a) \(\Rightarrow\) (a'), so suppose the result true for \((k - 1)\)-homogeneous polynomials, and consider \((P_n)\) and \((x_n)\) as in (b') above. Define a sequence of polynomials \((Q_n)\) by setting \(Q_n = A_n(x, \ldots, \cdot)\) where \(A_n\) is the symmetric \(k\)-linear form associated to \(P_n\). We need to show that this sequence is weakly null. Define \(P: X \rightarrow c_0\) by \(P(x) = (P_n(x))\). \(P\) is weakly compact, for its transpose \(P^t: l^1 \rightarrow P^k(X)\) is weakly compact \((P^t(e_n) = P_n \rightarrow 0\) weakly). Thus \(P^t: P^k(X)' \rightarrow l^\infty\) has image contained in \(c_0\). For each \(L \in P^{k-1}(X)'\) define \(\overline{L}: X \rightarrow P^k(X)'\) by \(\overline{L}(x)(R) = L(B(x, \cdot, \ldots, \cdot))\), where \(B\) is the symmetric \(k\)-linear form associated to \(R\). Now consider \(P^t \circ \overline{L}: X \rightarrow c_0\). This is a weakly compact linear operator, and since \((x_n)\) is \(\tau\)-null, we have, by (a), that \(P^t \circ \overline{L}(x_n)\)
tends to zero in norm, that is \( \sup_j |(\mathcal{L}(x_n) \circ P^j)_j| \to 0 \) as \( n \) grows. In particular

\[
|L(Q_n)| = |L(A_n(x_n, \cdot, \cdots, \cdot))| = |\mathcal{L}(x_n)(P_n)| \to 0.
\]

Thus \( (Q_n) \) is weakly null, and by our inductive hypothesis \( P_n(x_n) = Q_n(x_n) \to 0 \).

(b) implies (c): Since \( \tau \)-null sequences are bounded, the Taylor series of \( f \) converges uniformly over \( (x_n) \). Also, each \( k \)-homogeneous polynomial in this series is weakly compact, so (b) applies. ☐

We are now in a position to define the \( \tau \)-Dunford–Pettis property, given a sequential convergence \( \tau \).

**Definition.** We will say that a Banach space \( X \) has the \( \tau \)-Dunford–Pettis property if one—and therefore all—of the conditions in Theorem 2 hold.

Note that the classical Dunford–Pettis property is obtained when \( \tau \) is weak convergence. Other forms of the Dunford–Pettis property have been studied which conform to this scheme. Among them:

1. \( \tau = \omega_p \) produces the DPP \( p \) studied by Castillo and Sánchez [6].
2. \( \tau = P(\leq k X) \) corresponds to the polynomial Dunford–Pettis property studied by Farmer and Johnson [12].
3. Polynomial convergence gives rise to the polynomial Dunford–Pettis property as defined by Biström, Jaramillo and Lindström [2].

Next, we prove that in the definition of the \( \tau \)-Dunford–Pettis property, the class \( \mathcal{A} \) of holomorphic functions of bounded type (or the space of \( k \)-homogeneous polynomials, or that of linear functions) cannot be replaced by the class of all holomorphic functions over \( X \). Indeed, the resulting property is in general strictly stronger than the \( \tau \)-Dunford–Pettis property. We will use the following definition.

**Definition.** We will say a Banach space \( X \) has the \( * \)-Dunford–Pettis property if for all weakly null sequences \( (x_n) \) in \( X \) and all weak-\( * \) null sequences \( (x'_n) \) in \( X' \), \( x'_n(x_n) \) converges to 0.

This property implies the Dunford–Pettis property. Note that all Schur spaces have the \( * \)-Dunford–Pettis property. Also, if \( X \) has this property, all its complemented subspaces have it. For a Grothendieck space this property is equivalent to the classical Dunford–Pettis property, thus for instance \( l^\infty \) has the \( * \)-Dunford–Pettis property [8], and so does \( H^\infty \) (see Bourgain [4], [5]). The space \( l^1 \bigoplus l^\infty \) is an example of a space having the \( * \)-Dunford–Pettis property which is neither Schur nor Grothendieck. It is easily seen that \( X \) has the \( * \)-Dunford–Pettis property if and only if every weakly relatively compact subset of \( X \) is limited.

On the other hand, it is not difficult to check that \( * \)-Dunford–Pettis property coincides with the DP\( * \) property introduced in [3], that is, the fact that weak\( * \) and Mackey convergence coincide sequentially in \( X' \). We refer to [3] for further information about this property and its connection with differentiability.
A typical example of a space having the Dunford–Pettis property, but lacking the \(\ast\)-Dunford–Pettis property is \(c_0\). In such a space although a weakly compact holomorphic function \(F\) of bounded type will map weakly null sequences into sequences converging in norm to \(F(0)\), the same will not be true of all weakly compact holomorphic functions.

**Theorem 3.** If \(X\) has the Dunford–Pettis property but not the \(\ast\)-Dunford–Pettis property, there are scalar-valued holomorphic functions \(f\) on \(X\) which map a weakly null sequence into a sequence not norm convergent to \(f(0)\).

**Proof.** Take \((x_k)\) weakly null in \(X\) and \((x_k')\) weak-\(\ast\) null in \(X'\) such that \(x_k'(x_k)\) does not converge to 0. Passing to a subsequence and multiplying the \(x_k\)'s by suitable (bounded) scalars, we may suppose that \(x_k'(x_k) = 1\) for all \(k\). Now let \(0 < \varepsilon < 1\). We will extract subsequences \((x_{k_j})\) and \((x_{k_j}')\) inductively. Set \(k_1 = 1\), and having defined \(k_1, \ldots, k_{r-1}\), define \(k_r\) as follows: since \((x_{k_j}')\) is weak-\(\ast\) null,

\[
| x_{k_j}'(x_{k_{j-1}}) | < \frac{\varepsilon}{2 + \varepsilon}, \quad \text{if } k \geq n_r,
\]

and since \((x_k)\) is weakly null, take \(m_r > k_{r-1}\) and such that

\[
\sum_{j < r} | x_{k_j}'(x_k) |^j < \frac{\varepsilon}{2}, \quad \text{if } k \geq m_r.
\]

Define \(k_r = \max\{n_r, m_r\}\). Now construct \(f: X \rightarrow \mathbb{C}\) as \(f(x) = \sum_{j=1}^{\infty} x_{k_j}'(x)^j\).

Note that since \((x_{k_j}')\) is weak-\(\ast\) null and \(0 < \limsup_j \| x_{k_j}' \| < \infty\) this is an entire function on \(X\) which is not of bounded type. Finally, for each \(r \in \mathbb{N}\) we have

\[
|f(x_{k_r}) - 1| \leq \sum_{j < r} | x_{k_j}'(x_{k_r}) |^j + \sum_{j > r} | x_{k_j}'(x_{k_r}) |^j < \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} \left( \frac{\varepsilon}{2 + \varepsilon} \right)^j = \varepsilon,
\]

bearing in mind that \(k_r \geq m_r\) and that for \(j > r\), \(k_j \geq n_{r+1}\). Thus \(|f(x_{k_r}) - 1| < \varepsilon < 1\), and \(f(x_{k_r})\) cannot tend to 0 = \(f(0)\). \(\square\)

We show next that we can find spaces \(X\) without the \(\ast\)-Dunford–Pettis property in the following two situations: if \(X\) does not contain a copy of \(l^1\); and if \(X\) is not Schur, and the unit ball of \(X'\) is weak-\(\ast\) sequentially compact.

**Proposition 4.** If \(X\) has the \(\ast\)-Dunford–Pettis property, then \(X\) contains a copy of \(l^1\).

**Proof.** By the Josefson–Nissenzweig theorem [13] there is a weak-\(\ast\) null sequence \((y_n')\) in the unit sphere of \(X'\). Take a sequence \((y_n)\) in \(X\) with \(\| y_n \| \leq 2\)
and \( y'_n(y_n) = 1 \) for all \( n \). If \( X \) contains no copy of \( l^1 \), we may use the Rosenthal–Dor theorem [9, p. 201] to extract from this last sequence a subsequence (which we still call \((y_n)\)) which is weakly Cauchy. From this, we extract another subsequence inductively: set \( 0 < \varepsilon < 1 \) and \( n_1 = 1 \); and having defined \( n_1, \ldots, n_{r-1} \), choose \( n_r > n_{r-1} \) such that \( |y'_n(y_{n_{r-1}})| < \varepsilon \). Since \((y_n)\) is weakly Cauchy, \( x_k = y_{n_k} - y_{n_{k-1}} \) is weakly null. Setting \( x'_k = y'_{n_k} \) we have \((x_k)\) weakly null and \((x'_k)\) weak-\(*\) null such that

\[
|x'_k(x_k) - 1| = |y'_{n_k}(y_{n_{k-1}})| < \varepsilon < 1.
\]

Thus \( x'_k(x_k) \) cannot converge to zero. \( \Box \)

The following result is obtained in [3]. Our proof is different and we include it for the sake of completeness.

**Proposition 5.** If \( X \) has the \(*\)-Dunford–Pettis property, and the unit ball of \( X' \) is weak-\(*\) sequentially compact, then \( X \) is Schur.

**Proof.** We follow [15]. If \( X \) were not Schur, take a weakly null sequence \((y_n)\) contained in the unit sphere of \( X \). From \((y_n)\) we extract, by the Bessaga–Pełczyński selection theorem [9, p. 42], a basic sequence, which we still call \((y_n)\), and consider the coordinate functionals \( y'_n: S \to \mathbb{C} \) where \( S \) is the closed linear span of the \( y_n \)'s. These have norm not larger than twice the basis constant of \((y_n)\). Extend each \( y'_n \) to all of \( X \) by Hahn–Banach, preserving the norms (we still call them \( y'_n \)). Since this sequence is bounded and the unit ball of \( X' \) is weak-\(*\) sequentially compact, choose a weak-\(*\) convergent subsequence \((y'_{n_k})\) and call \( y' \) its weak-\(*\) limit. Now set \( x'_k = y'_{n_k} - y' \) and \( x_k = y_{n_k} \). The sequence \((x_k)\) is weakly null, \((x'_k)\) weak-\(*\) null, and for each \( k \) we have \( x'_k(x_k) = 1 \). \( \Box \)

Note that if \( X \) is weakly compactly generated, the unit ball of \( X' \) is weak-\(*\) sequentially compact [9, p. 228]. For further information about spaces with weak-\(*\) sequentially compact dual balls, we refer to [9, Chapter 13] and references therein. A hereditarily Dunford–Pettis space lacking the \(*\)-Dunford–Pettis property will contain a copy of \( c_0 \) [7, Proposition 2], but we do not know if in this case one can find a complemented copy.

Finally, we consider what happens when the convergence \( \tau \) is holomorphic convergence, that is: \((x_n)\) is \( H\)-convergent to \( x \) if \( f(x_n) \to f(x) \) for any \( f \in H(X) \). In [15] Petunin and Savkin prove that holomorphic convergence implies norm convergence in spaces which are weakly compactly generated. However, there are spaces such as \( l^\infty \) where holomorphic convergence does not imply norm convergence [1]. In spite of this fact, as we see below, the \( "H\)-Dunford–Pettis property" holds in all Banach spaces. We consider the strongest form of this property, by taking \( \mathcal{A} \) as the class of all holomorphic functions.
Theorem 6. In any Banach space $X$, the following hold.

(i) For all $Y$, all weakly compact $F \in H(X,Y)$, and all $H$-null sequences $(x_n)$, $F(x_n) \to F(0)$ in norm.

(ii) For all weakly null sequences $(f_n) \in H(X)$ and all $H$-null sequences $(x_n)$, $f_n(x_n) \to 0$.

Proof. (i): Let $F$ and $(x_n)$ be as above. By [17, 3.7], $F$ factors through a reflexive Banach space $G$, that is, there is a holomorphic function $g: X \to G$ and a continuous linear operator $T: G \to Y$ such that $F = T \circ g$. Since $g$ is holomorphic, the sequence $g(x_n)$ is $H$-null. But $G$ is reflexive, so by [15] $g(x_n)$ converges in norm to $g(0)$. Thus $F(x_n) = T(g(x_n))$ is norm convergent to $T(g(0)) = F(0)$.

We now prove that (i) implies (ii). The proof is similar to (c) implies ($c'$) above. We define $F: X \to c_0$ by $F(x) = (f_n(x))$. This function is holomorphic, and once again $F^*: l^1 \to H(X)$ given by $F^*(a) = a \circ F$ is weakly compact, but in this case, the weak compactness of the closed absolutely convex hull of $\{f_n : n \in \mathbb{N}\} \cup \{0\}$ is more involved. Consider the Nachbin–Coeuré topology $\tau_\delta$ on $H(X)$ (see [10, 3.16] for definition). Note that $\tau_\delta$ is bornological and coincides with $\tau_\omega$ on bounded sets of $H(X)$ (see [10, 3.19]). Thus, closed absolutely convex hulls in both topologies are the same. Now Krein’s theorem [18, IV.11.4] assures that this set is weakly compact (since it is complete for $\tau_\delta$, which is the Mackey topology). $\Box$

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