ON THE $\Gamma$-CONVERGENCE OF LAPLACE–BELTRAMI OPERATORS IN THE PLANE

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Abstract. We show here that if $f_h$ is a sequence of mappings of finite distortion $K_h$, uniformly bounded in some exponential norm, weakly converging to $f$ in $W^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$, then the matrices $A(x, f_h)$ in the Beltrami operators associated to each $f_h$, $\Gamma$-converge, in the sense of De Giorgi, to the matrix $A(x, f)$ in the Beltrami operator associated to $f$.

1. Introduction

For $\Omega$ an open subset of $\mathbb{R}^2$ we shall study mappings $f = (f_1, f_2): \Omega \to \mathbb{R}^2$ in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^2)$. We say that $f$ has finite distortion if

$$|Df(x)|^2 \leq \mathcal{K}(x) J(x, f) \quad \text{a.e.}$$

(1.1)

Here $|Df(x)|$ stands for the Hilbert–Schmidt norm of the differential matrix $Df(x) \in \mathbb{R}^{2 \times 2}$ and $J(x, f) = \det Df(x)$. That is,

$$|Df(x)|^2 = \sum_{i,j=1}^{2} \left| \frac{\partial f_i}{\partial x_j} \right|^2 \quad \text{and} \quad J(x, f) = \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1}.$$

The function $\mathcal{K} = \mathcal{K}(x)$ is assumed to be measurable with values in the interval $[2, \infty)$. It will be advantageous to write $\mathcal{K}$ as

$$\mathcal{K}(x) = K(x) + \frac{1}{K(x)} \quad \text{where} \quad 1 \leq K(x) < \infty.$$

(1.2)

We refer to the smallest such $K(x)$ for which (1.1) holds as the distortion function of $f$. If $K(x)$ is bounded by a constant, say $1 \leq K(x) \leq K$ a.e., then we say that $f$ is $K$-quasiregular. An important quantity associated to a mapping with finite distortion is the so called distortion tensor $G(\cdot, f): \Omega \to \mathbb{R}^{2 \times 2}$, defined by

$$G(x, f) = \begin{cases} \frac{D^i f(x) D f(x)}{J(x, f)} & \text{if } J(x, f) \neq 0, \\ I & \text{if } J(x, f) = 0, \end{cases}$$

(1.3)


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where $D^t f(x)$ stands for the transposed differential.

The distortion inequality (1.1) reads as

\[(1.4) \quad \frac{\|\xi\|^2}{K(x)} \leq \langle G(x, f)\xi, \xi \rangle \leq K(x)\|\xi\|^2\]

and we have $\det G(x, f) = 1$ a.e.

The symmetric matrix function $G(\cdot, f)$ can be viewed as a Riemannian metric on $\Omega$, the pullback of the Euclidean structure via the mapping $f$. It is obvious that $f$ is conformal with respect to this new metric. This raises an important question: how does $G(\cdot, f)$ change with $f$? We are particularly concerned with the continuity property of the map $f \to G(\cdot, f)$, since many constructions in quasiconformal geometry and elliptic PDE’s rely on limiting processes. The natural convergence of the mapping $f_h: \Omega \to \mathbb{R}^2$ with finite distortion is that of the weak topology in $W^{1,2}(\Omega, \mathbb{R}^2)$. This, however, does not guarantee convergence of the matrices $G(x, f_h)$ to $G(x, f)$ in any familiar sense (compare with Example 6.1 here and also [LV]). Note that the condition $\det G(x, f_h) = 1$ is not necessarily preserved under the weak convergence of $G(x, f)$.

S. Spagnolo [S2] first realized that the proper way to overcome this difficulty is by considering the $\Gamma$-convergence of the inverse matrices

$$A(x, f) = G(x, f)^{-1}.$$ 

This matrix clearly verifies the bounds at (1.4) as well. See Section 3 for the definition of $\Gamma$-convergence.

Spagnolo’s result dealt with the special case of $K$-quasiregular mappings in which $A(x, f)$ were bounded and uniformly elliptic matrices. In that case $\Gamma$-convergence is equivalent to the $L^2$-convergence of solutions of the Dirichlet problem. More precisely, given a sequence $\{A_h\}$ of $2 \times 2$ matrices satisfying

$$\frac{\|\xi\|^2}{K} \leq \langle A_h(x)\xi, \xi \rangle \leq K\|\xi\|^2, \quad K \geq 1,$$

we consider the elliptic operators on a bounded open set $\Omega \subset \mathbb{R}^2$

$$\mathcal{L}_h = \text{div}[A_h(x)\nabla]: W^{1,2}_0(\Omega) \to W^{-1,2}(\Omega).$$

They are certainly invertible. Following [S1], we say that $\{A_h\}$ $\Gamma$-converges to $A$ if for every $\varphi \in W^{-1,2}(\Omega)$, $\mathcal{L}_h^{-1}(\varphi) \rightharpoonup \mathcal{L}^{-1}(\varphi)$ in $L^2(\Omega)$, where $\mathcal{L} = \text{div}[A(x)\nabla]$. Later these results were generalized to the $n$-dimensional case by [DD].

In the present paper we extend Spagnolo’s result to sequences of mappings with pointwise unbounded distortion. Our only assumption will be that the distortion functions stay bounded in the $\text{EXP}_\alpha$ class for a certain $\alpha > 1$, see Section 2, for the definitions.

The main result is as follows (see Section 5):
Theorem. Let \( f_h \) converge weakly in \( W^{1,2}(\Omega,\mathbb{R}^2) \) to a mapping \( f \), and suppose that their distortion functions \( K_h \) converge to \( K \) weakly in \( L^1(\Omega) \) and satisfy
\[
\int_{\Omega} \exp\left( \frac{K_h(x)}{\lambda} \right)^{\alpha} \, dx \leq c
\]
for some \( \alpha > 1 \), \( \lambda > 0 \) and \( c > 0 \). Then \( f \) has distortion \( K \) and
\[
A(x, f_h) \xrightarrow{\Gamma_\alpha} A(x, f).
\]

For the notion of \( \Gamma_\alpha \)-convergence, we refer to the definition in Section 3.

In Section 6 we will relate our results to some known convergence theorems for quasiregular mappings [GMRV], [IK], [Bo].

2. Some Orlicz spaces

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). An Orlicz function is a nonnegative continuously increasing function \( P: \mathbb{R}_+ \to \mathbb{R}_+ \), verifying \( P(0) = 0 \) and \( P(\infty) = \infty \).

The Orlicz space \( L^P(\Omega) \) consists of all measurable functions \( \varphi: \Omega \to \mathbb{R} \) such that
\[
\int_{\Omega} P(\lambda^{-1} |\varphi|) < \infty
\]
for some \( \lambda = \lambda(\varphi) > 0 \) (see [RR]).

For \( \alpha > 1 \), we denote by \( \text{EXP}_\alpha(\Omega) \) the Orlicz space with the defining function \( P(t) = \exp(t^\alpha) - 1 \). It consists of all measurable functions \( \varphi \) on \( \Omega \) such that
\[
\|\varphi\|_{\text{EXP}_\alpha(\Omega)} = \inf \left\{ \lambda > 0 : \frac{1}{|\Omega|} \int_{\Omega} \exp\left( \frac{|\varphi(x)|}{\lambda} \right)^\alpha \, dx \leq 2 \right\} < \infty.
\]

Here
\[
\frac{1}{|\Omega|} \int_{\Omega} \psi = \psi_{\Omega},
\]
and \( \|\varphi\|_{\text{EXP}_\alpha(\Omega)} \) provides a norm of \( \varphi \). Another space of interest to us will be the Zygmund space \( L^p \log^\beta L(\Omega) \), with \( p \geq 1 \) and \( \beta \geq 0 \), with the defining function \( P(t) = t^p \log^\beta(e + t) \). It consists of all measurable functions \( \varphi \) on \( \Omega \) such that
\[
\int_{\Omega} |\varphi|^p \log^\beta \left( e + \frac{|\varphi|}{|\varphi|_{\Omega}} \right) \, dx < \infty.
\]

Observe that both are Banach spaces and \( \text{EXP}_\alpha(\Omega) \) is the dual to \( L^1 \log^\beta L \), when \( \beta = 1/\alpha \).

The Luxemburg norm of a function \( \varphi \in L^p \log^\beta L(\Omega) \) is given by
\[
\|\varphi\|_{L^p \log^\beta L(\Omega)} = \inf \left\{ \lambda > 0 : \frac{1}{|\Omega|} \int_{\Omega} \left( \frac{|\varphi|}{\lambda} \right)^p \log^\beta \left( e + \frac{|\varphi|}{\lambda} \right) \, dx \leq 1 \right\}.
\]
Proposition 2.1 (Generalized Hölder inequality). Let $\alpha \geq 1$. Let $K(x) \in \text{EXP}_\alpha(\Omega)$, $\varphi \in L^2 \log^{1/\alpha} L$, and $\psi \in L^2 \log^{1/\alpha} L$. Then
\[
\left| \int_\Omega K(x)\varphi(x)\psi(x) \, dx \right| \leq c \left\| K \right\|_{\text{EXP}_\alpha} \|\varphi\|_{L^2 \log^{1/\alpha} L} \|\psi\|_{L^2 \log^{1/\alpha} L}.
\]

For $P(t) = t^2 \log^\beta(e + t)$ we denote by $W^{1,P}(\Omega)$ the Orlicz–Sobolev space of functions $\varphi \in L^2 \log^\beta L$ whose gradient belongs to the Zygmund space $L^2 \log^\beta L$. We supply this space with the norm
\[
\|\varphi\|_{W^{1,P}(\Omega)} = \|\varphi\|_{L^2 \log^\beta L(\Omega)} + \|\nabla \varphi\|_{L^2 \log^\beta L(\Omega)}.
\]

3. The $\Gamma$-convergence

We denote by $\mathbb{R}^{2 \times 2}$ the set of symmetric $2 \times 2$ matrices $A$, such that $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^2$. Consider measurable functions $A : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ on $\Omega \subset \mathbb{R}^2$ satisfying
\[
\frac{|\xi|^2}{K(x)} \leq \langle A(x)\xi, \xi \rangle \leq K(x)|\xi|^2
\]
for some $1 \leq K(x) < \infty$ a.e. The smallest $K(x)$, for which the above holds, denoted by $K_A(x)$, is called the distortion function of $A$.

The present paper is concerned with mappings whose distortion belongs to the exponential class $\text{EXP}_\alpha(\Omega)$, $1 < \alpha \leq \infty$. For the purpose of this work, we adopt the following variant of De Giorgi’s notion of $\Gamma$-convergence ([DF]).

Definition 3.1. Let $A$ and $A_h$ ($h = 1, 2, \ldots$) be matrix functions whose distortions $K_A$ and $K_{A_h}$ are uniformly bounded in the norm of $\text{EXP}_\alpha(\Omega)$. We say that $\{A_h\}$ $\Gamma_\alpha$-converges to $A$ if the following two conditions are verified:

1. The inequality
\[
\int_\Omega \langle A(x)\nabla u, \nabla u \rangle \, dx \leq \liminf_{h \rightarrow \infty} \int_\Omega \langle A_h(x)\nabla u_h, \nabla u_h \rangle \, dx
\]
holds whenever $|\nabla u_h|, |\nabla u| \in L^2 \log^{1/\alpha} L(\Omega)$ and $u_h \rightarrow u$ in $L^2 \log^{1/\alpha} L$.

2. For every $v \in L^2 \log^{1/\alpha} L(\Omega)$ with $|\nabla v| \in L^2 \log^{1/\alpha}(\Omega)$ there exists a sequence $v_h \in L^2 \log^{1/\alpha} L(\Omega)$ with $|\nabla v_h| \in L^2 \log^{1/\alpha} L$ such that $v_h \rightarrow v$ in $L^2 \log^{1/\alpha} L(\Omega)$ and
\[
\int_\Omega \langle A(x)\nabla v, \nabla v \rangle = \lim_{h} \int_\Omega \langle A_h\nabla v_h, \nabla v_h \rangle.
\]

Remark. The assumption that $K_A$ and $K_{A_h}$ belong to $\text{EXP}_\alpha(\Omega)$ is needed to guarantee that the above integrals are finite. This follows from the inequality
\[
\int_\Omega \langle A(x)\nabla u, \nabla u \rangle \, dx \leq \int_\Omega K_A(x)|\nabla u|^2 \, dx
\]
\[
\leq c \left\| K_A \right\|_{\text{EXP}_\alpha(\Omega)} \|\nabla u\|_{L^2 \log^{1/\alpha} L(\Omega)}^2.
\]
If one merely assumes that $K_A$ and $K_{Ah} \in L^1$ then one must be confined to Lipschitz functions. In this case we speak of $\Gamma$-convergence. We say that a sequence $A_h$ of matrix functions $A_h \in L^1(\Omega, R^{2\times2})$ $\Gamma$-converges to $A$ if:

1. Inequality (3.2) holds whenever $u, u_h \in \text{Lip}(\Omega)$ and $u_h \to u$ in $L^2(\Omega)$;
2. For every $v \in \text{Lip}(\Omega)$ one can find a sequence $v_h \in \text{Lip}(\Omega)$ converging to $v$ in $L^2(\Omega)$ satisfying (3.3).

Actually, by the general properties of $\Gamma$-convergence, conditions (1) and (2) remain true if we replace $\Omega$ by any of its open subsets.

We report here the fundamental compactness result concerning $\Gamma$-convergence [MS].

**Theorem 3.1.** Let $A_h$ be a sequence of symmetric $2 \times 2$ matrices satisfying

$$0 \leq \langle A_h(x)\xi, \xi \rangle \leq K_h(x)|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and } \xi \in R^2.$$

Assume that $K_h \rightharpoonup K$ weakly in $L^1(\Omega)$. Then there exists a subsequence $A_{hr}$, $\Gamma$-converging to a symmetric matrix $A$. Moreover, this matrix $A$ also satisfies

$$0 \leq \langle A(x)\xi, \xi \rangle \leq K(x)|\xi|^2.$$

In this connection it is appropriate to mention another important notion of convergence of matrix functions $A_h: \Omega \to R^{2\times2}$, the so-called $G$-convergence. For simplicity we confine ourselves to bounded domains and to sequences such that

$$1 \leq K_{Ah}(x) \leq K \quad \text{a.e. for } h = 1, 2, \ldots,$$

and

$$1 \leq K_A(x) \leq K \quad \text{a.e.}$$

We recall from the introduction the elliptic operators and their inverse

$$\mathcal{L}_h = \text{div} [A_h(x)\nabla]: W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega), \quad \mathcal{L}_{h}^{-1}: W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega),$$

$$\mathcal{L} = \text{div} [A(x)\nabla]: W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega), \quad \mathcal{L}^{-1}: W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega).$$

Following Spagnolo [S1], $\{A_h\} G$-converges to $A$ if $\mathcal{L}_{h}^{-1}(\varphi) \rightharpoonup \mathcal{L}^{-1}(\varphi)$ weakly in $W_0^{1,2}(\Omega)$, for every $\varphi \in W^{-1,2}(\Omega)$. We emphasize that under condition (3.5) all the above notions of convergence are equivalent, though we shall not pursue this matter here, see [MS].
4. Mappings of finite distortion and the Laplace–Beltrami operators

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^2 \) and \( f = (f^1, f^2) \in W^{1,2}(\Omega, \mathbb{R}^2) \) be a mapping of finite distortion \( K : \Omega \to [1, \infty) \), i.e. satisfying, for a.e. \( x \in \Omega \),
\[
|Df(x)|^2 \leq [K(x) + K^{-1}(x)]J(x, f),
\]
where \( J(x, f) \) is the Jacobian determinant of \( f \). The distortion tensor \( G(x, f) \) of \( f \) at \( x \) is defined in (1.3). It is easy to check that \( G \) is a symmetric matrix with \( \det G = 1 \) and that (1.4) is equivalent to (4.1). In fact, for any \( 2 \times 2 \)-matrix \( F \) with \( \det F > 0 \), we can consider
\[
G = \frac{F^t F}{\det F}.
\]
Then, obviously
\[
\det G = 1.
\]
Moreover, recalling the Hilbert–Schmidt norm of \( F \),
\[
|F|^2 = \text{tr} F^t F
\]
the distortion inequality
\[
|F|^2 \leq \left( K + \frac{1}{K} \right) \det F
\]
is equivalent to
\[
\text{tr} G \leq K + \frac{1}{K}.
\]
Let \( \lambda \) and \( 1/\lambda \) be the eigenvalues of \( G \). Then the last inequality means that
\[
\lambda + \frac{1}{\lambda} \leq K + \frac{1}{K};
\]
hence \( 1/K \leq \lambda \leq K \).

Now we consider the inverse matrix
\[
A(x, f) = G(x, f)^{-1}
\]
which obviously satisfies the ellipticity condition
\[
\frac{|\xi|^2}{K(x)} \leq \langle A(x, f)\xi, \xi \rangle \leq K(x)|\xi|^2.
\]
Connections between mappings of finite distortion and PDEs are established via the Laplace–Beltrami operator \( \mathcal{L} = \text{div} [A(x, f)\nabla] \). Note that the components \( f^i \) \((i = 1, 2)\) solve the equations
\[
\begin{cases}
\mathcal{L}[f^i] = 0, \\
\langle A(x, f)\nabla f^i, \nabla f^j \rangle = \delta_{ij} J(x, f),
\end{cases}
\]
see for example [BI] and [HKM]. Planar mappings with unbounded distortion have been recently studied by [D], [I˘S] and most recently by [MM], [BJ], [RSY], [IS]. In particular in [MM] the following higher integrability result, which will be useful to us, was established.
Theorem 4.1. If $f \in W^{1,2}(\Omega)$ satisfies (4.1) with $K \in \text{EXP}_\alpha(\Omega)$, for certain $\alpha > 1$, then $|Df|$ belongs to $L^2 \log^{1/\alpha} L(\Omega_1)$ for any $\Omega_1 \subset \subset \Omega$ and the following inequality holds:

$$(4.3) \quad \|Df\|_{L^2 \log^{1/\alpha} L(\Omega_1)} \leq c(\Omega_1)\|K\|_{\text{EXP}_\alpha(\Omega)}\|Df\|_{L^2(\Omega)}.$$ 

This is true in all dimensions, provided the exponent 2 is replaced by the dimension $n$.

In view of Hadamard’s inequality

$$\langle A(x,f)\nabla f^i, \nabla f^i \rangle = J(x,f) \leq \frac{1}{2}|Df(x)|^2,$$

we deduce by (4.3)

$$(4.4) \quad \|\langle A(x,f)\nabla f^i, \nabla f^i \rangle\|_{L^1 \log^{1/\alpha} L(\Omega_1)} \leq c(\Omega_1)\|K\|_{\text{EXP}_\alpha(\Omega)} \int_\Omega |Df|^2 \, dx.$$ 

We show here that the limit mapping $f$ of a weakly convergent sequence of mappings $f_h$ with finite distortion also has finite distortion. Our arguments are based on the weak continuity of the Jacobian determinant $[R]$, $[M\ddot{u}]$ and the concept of polyconvexity. General $n$-dimensional results of this type have been recently obtained by F.W. Gehring and T. Iwaniec in [GI]. They adopted slightly different definition of the distortion, which for $n = 2$ reduces to

$$|Df(x)|^2 \leq 2K(x)J(x,f).$$

Theorem 4.2. Let $f_h: \Omega \to \mathbb{R}^2$ be mappings of finite distortion $K_h(x)$:

$$(4.5) \quad |Df_h(x)|^2 \leq \left[ K_h(x) + \frac{1}{K_h(x)} \right] J(x,f_h).$$ 

Assume that $K_h$ are integrable and converge weakly to $K$ in $L^1(\Omega)$, while $f_h \rightharpoonup f$ weakly in $W^{1,2}(\Omega, \mathbb{R}^2)$. Then the above inequality remains valid for the limit map.

Proof. Let us first introduce some useful notation. Set $F = (B, E)$ where the vectors $B, E$ are defined by

$$E = \nabla f^1, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla f^2$$

and let

$$F^+ = \frac{1}{2}(E + B), \quad F^- = \frac{1}{2}(E - B).$$

It is obvious that

$$J(x,f) = \langle B, E \rangle = |F^+|^2 - |F^-|^2 := J(F), \quad |F|^2 = 2(|F^+|^2 + |F^-|^2).$$
Hence the distortion inequality
\[ |F|^2 \leq \left( K + \frac{1}{K} \right) J(F) \]
is easily seen to be equivalent to
\[ |F^-| \leq \frac{K - 1}{K + 1} |F^+| . \]
This, in turn, is equivalent to
\[ \frac{\|F_h\|^2}{J(F_h)} \leq K_h \]
with \( K_h \to K \) weakly in \( L^1 \). The desired conclusion
\[ \frac{\|F\|^2}{J(F)} \leq K \]
follows by applying the inequality
\[ \frac{\|F\|^2}{J(F)} \leq \frac{\|F_h\|^2}{J(F_h)} + \frac{2\|F\|}{J(F)} (\|F\| - \|F_h\|) - \frac{\|F\|^2}{J(F)^2} [J(F) - J(F_h)] \]
The latter is immediate from the convexity of the function \((x, y) \to x^2/y\). The well-known weak continuity property of the Jacobians [R], together with the lower semicontinuity of the norm \( \| \cdot \| \), imply (4.7). Here, for simplicity, we have assumed \( J(F) > 0 \) and \( J(F_h) > 0 \). To get rid of this redundant assumption one must replace \( J(F) \) by the expression \( J(F) + \varepsilon \|F\| \), \( J(F_h) \) and then pass to the limit as \( \varepsilon \to 0 \).

5. The convergence theorem

In this section we consider a sequence \( f_h = (f^1_h, f^2_h) \in W^{1,2}(\Omega, \mathbb{R}^2) \) of non-constant mappings with distortion \( 1 \leq K_h(x) < \infty \), that is
\[ |Df_h(x)|^2 \leq [K_h(x) + K_h^{-1}(x)] J(x, f_h) . \]
Our basic assumptions are:
(i) There exists \( \alpha > 1 \) and \( c_0 > 0 \) such that
\[ \|K_h\|_{\text{EXP}}(\Omega) \leq c_0 \quad \text{for } h = 1, 2, \ldots . \]
(ii) \( K_h \rightharpoonup K \) weakly in \( L^1(\Omega) \).
(iii) \( f_h \rightharpoonup f = (f^1, f^2) \) weakly in \( W^{1,2}(\Omega, \mathbb{R}^2) \).
By virtue of Theorem 3.1 there exists a subsequence \( A_r(x) = A(x, f_{h_r}) \), \( r = 1, 2, \ldots, \) such that

\[
A(x, f_{h_r}) \Gamma \to A(x)
\]

where \( A(x) \) is a symmetric matrix field satisfying

\[
0 \leq \langle A(x)\xi, \xi \rangle \leq K(x)|\xi|^2.
\]

Our aim here is to prove that \( A(x) \) can be identified with \( A(x, f) \), which is the inverse of the distortion tensor of \( f \):

\[
A(x, f) = [D^t f(x)Df(x)]^{-1}J(x, f).
\]

As a byproduct of our proof, we improve the lower bound at (5.3)

\[
K^{-1}(x)|\xi|^2 \leq \langle A(x)\xi, \xi \rangle
\]

and show that actually the entire sequence \( \{A(x, f_{h})\} \) \( \Gamma \)-converges to \( A(x, f) \).

**Theorem 5.1.** Under the above assumptions

\[
\int_{\Omega_1} \langle A(x)\nabla f^i, \nabla f^i \rangle \, dx = \lim_{r \to \infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla f_{h_r}^i, \nabla f_{h_r}^i \rangle \, dx
\]

on compact subdomains \( \Omega_1 \subset \Omega \), for \( i = 1, 2 \).

**Proof.** In fact, we have

\[
\int \langle A(x, f)\nabla u, \nabla u \rangle \leq \int K_h|\nabla u|^2 \, dx \leq c\|K_h\|_{\text{EXP,} \alpha(\Omega)}\|\nabla u\|^2_{L^2, \log^{1/\alpha} L(\Omega)}
\]

\[
\leq cC_0\|u\|^2_{W^{1,2}, \log^{1/\alpha} L(\Omega)}.
\]

It then follows that the functionals \( (\int_\Omega \langle A(x, f_h)\nabla u, \nabla u \rangle \, dx)^{1/2} \) are equilipschitz in \( W^{1, P}(\Omega) \) with \( P(t) = t^2 \log^{1/\alpha}(e + t) \), a legitimate reason for passing from \( \Gamma \)-convergence to the stronger one

\[
A(x, f_{h_r}) \Gamma \alpha \to A(x);
\]

see [MS] for details.

For \( i = 1, 2 \) fixed, set for simplicity \( u_r = f_{h_r}^i \) and \( u = f^i \). Note that \( u_r \to u \) in \( L^2 \log^{1/\alpha} L(\Omega_1) \). Let now \( (v_r) \) be a sequence in \( W^{1, P}(\Omega_1) \) such that \( v_r \to u \) in \( L^2 \log^{1/\alpha} L(\Omega_1) \) and

\[
\lim_{r \to \infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla v_r, \nabla v_r \rangle \, dx = \int_{\Omega_1} \langle A(x)\nabla u, \nabla u \rangle \, dx.
\]
Let \( \Omega' \) be an arbitrary compact subdomain of \( \Omega_1 \) and \( \varphi \in C_0^\infty(\Omega_1) \) be such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in \( \Omega' \); then for every \( t \in ]0, 1[ \)

\[
\int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle \ dx \\
\leq \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla (\varphi v_r + (1 - \varphi) u_r), \nabla (\varphi v_r + (1 - \varphi) u_r) \right\rangle \ dx \\
= \int_{\Omega_1} \left\langle A(x, f_{h_r}) \left\{ \frac{t}{t} (\nabla \varphi)(v_r - u_r) + \frac{1 - t}{1 - t} (\varphi \nabla v_r + (1 - \varphi) \nabla u_r) \right\} \right\rangle \ dx \\
\leq t \int_{\Omega_1} \left\langle A(x, f_{h_r}) \left\{ \frac{1}{t} (\nabla \varphi)(v_r - u_r) \right\} \right\rangle \ dx \\
+ (1 - t) \int_{\Omega_1} \left\langle A(x, f_{h_r}) \left\{ \frac{1}{1 - t} (\varphi \nabla v_r + (1 - \varphi) \nabla u_r) \right\} \right\rangle \ dx \\
\leq \frac{1}{t} \int_{\Omega_1} K |D\varphi|^2 |v_r - u_r|^2 \ dx + \frac{1}{1 - t} \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla v_r, \nabla v_r \right\rangle \varphi \ dx \\
+ \frac{1}{1 - t} \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle (1 - \varphi) \ dx.
\]

This yields

\[
(1 - t) \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle \ dx \leq \frac{1 - t}{t} c ||v_r - u_r||^2_{L^2 \log^{1/\alpha} L} \cdot ||D\varphi||^2_{L^\infty(\Omega_1)} \\
+ \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla v_r, \nabla v_r \right\rangle \varphi \ dx \\
+ \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle (1 - \varphi) \ dx.
\]

The final estimate reads as

\[
\int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla v_r, \nabla v_r \right\rangle \varphi \ dx \geq \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle (1 - t - 1 + \varphi) \ dx \\
- \frac{1 - t}{t} c ||D\varphi||^2_{L^\infty(\Omega_1)} \cdot ||v_r - u_r||^2_{L^2 \log^{1/\alpha} L}.
\]

Now, passing to the limit as \( r \to \infty \), we obtain

\[
\int_{\Omega_1} \left\langle A(x) \nabla u, \nabla u \right\rangle \ dx \geq \limsup_{r \to \infty} \int_{\Omega_1} \left\langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \right\rangle (\varphi - t) \ dx.
\]
We let the parameter $t$ go to zero

$$\int_{\Omega_1} \langle A(x) \nabla u, \nabla u \rangle \geq \limsup_{r \to \infty} \int_{\Omega_1} \langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \rangle \varphi \geq \liminf_{r \to \infty} \int_{\Omega'} \langle A(x, f_{h_r}) \nabla u_r, \nabla u_r \rangle \geq \int_{\Omega'} \langle A(x) \nabla u, \nabla u \rangle.$$  

Since $\Omega'$ was arbitrary, we get (5.5). □

Now we are in a position to rigorously state and prove our main result.

**Theorem 5.2.** Under the conditions (i), (ii), and (iii), the limit mapping $f$ is either constant or, if not, has finite distortion $K(x)$ and

$$\text{(5.8)} \quad A(x, f_h) \xrightarrow{\Gamma_{\alpha}} A(x, f).$$

**Proof.** That $f$ has finite distortion $K(x)$ was already established in Section 4. Since we wish to identify the $\Gamma_{\alpha}$-limit of $A(x, f_h)$, we can assume that in (5.2) and (5.5) the convergence of the entire sequence holds.

As in the proof of Theorem 5.1, set $u_h = f^i_h$, $u = f^i$, for $i = 1, 2$ and $A_h(x) = A(x, f_h)$.

For the compact subdomain $\Omega_1 \subset \Omega$ consider step functions

$$\text{(5.9)} \quad \varphi = \sum_{j=1}^{\nu} \lambda_j \chi_{B_j}, \quad \lambda_j \geq 0,$$

where $B_j$ are pairwise disjoint open subsets of $\Omega_1$ such that $|\Omega_1 \setminus \bigcup_{j=1}^{\nu} B_j| = 0$.

From (5.5) it follows that

$$\text{(5.10)} \quad \liminf_{h \to \infty} \int_{\Omega_1} \langle A_h(x) \nabla u_h, \nabla u_h \rangle \varphi \, dx \geq \int_{\Omega_1} \langle A(x) \nabla u, \nabla u \rangle \varphi \, dx.$$

Moreover, by an approximation, this also holds if $\varphi$ is a nonnegative continuous function on $\overline{\Omega}_1$.

Let us now prove more, namely, that (5.10) holds as equality for every continuous function $\varphi$ in $\overline{\Omega}_1$, not necessarily nonnegative.

Applying (4.4), we infer that the sequence $J(x, f_h) = \langle A_h(x) \nabla u_h, \nabla u_h \rangle$ admits a subsequence weakly converging in $L^1(\Omega_1)$ to a function $E(x)$. Thus

$$\text{(5.11)} \quad \lim_{r \to \infty} \int_{\Omega_1} \langle A_{h_r}(x) \nabla u_{h_r}, \nabla u_{h_r} \rangle \varphi(x) \, dx = \int_{\Omega_1} E(x) \varphi(x) \, dx$$

for any $\varphi \in C^0(\overline{\Omega}_1)$. By (5.10) it follows

$$\text{(5.12)} \quad \int_{\Omega_1} \langle A(x) \nabla u, \nabla u \rangle \varphi(x) \, dx \leq \int_{\Omega_1} E(x) \varphi(x) \, dx.$$
Let $S$ be a measurable subset of $\Omega_1$ and let $(\varphi_k) \subset C^0(\overline{\Omega}_1)$ be such that $\varphi_k(x) \rightarrow \chi_S(x)$ a.e. in $\Omega_1$. Then from the previous relation and the Lebesgue theorem it follows that

\begin{equation}
\int_S \langle A(x)\nabla u, \nabla u \rangle \leq \int_S E(x) \, dx.
\end{equation}

On the other hand we deduce from (5.11) and Theorem 5.1 that

\begin{equation}
\int_{\Omega_1} \langle A(x)\nabla u, \nabla u \rangle \, dx = \int_{\Omega_1} E(x) \, dx.
\end{equation}

Hence

\[ E(x) = \langle A(x)\nabla u, \nabla u \rangle \quad \text{a.e. in } \Omega_1. \]

Therefore, we have for the whole sequence

\begin{equation}
\lim_{h \to \infty} \int_{\Omega_1} \langle A(x, f_h)\nabla u_h, \nabla u_h \rangle \varphi \, dx = \int_{\Omega_1} \langle A(x)\nabla u, \nabla u \rangle \varphi \, dx
\end{equation}

for every $\varphi \in C^0(\overline{\Omega}_1)$.

Now we recall from (4.2) that

\begin{equation}
\langle A(x, f_h)\nabla f_h^i(x), \nabla f_h^j(x) \rangle = J(x, f_h)\delta_{ij} \quad \text{a.e. on } \Omega, \ i, j = 1, 2.
\end{equation}

By the symmetry of the matrix $A(x, f_h)$, (5.15), (5.16) and the weak continuity property of Jacobian ([R]) we have

\begin{equation}
\int_{\Omega_1} \langle A(x)\nabla f^i, \nabla f^j \rangle \varphi \, dx = \lim_{h \to \infty} \int_{\Omega_1} \langle A(x, f_h)\nabla f_h^i, \nabla f_h^j \rangle \varphi \, dx
\end{equation}

\[ = \lim_{h \to \infty} \int_{\Omega_1} J(x, f_h)\delta_{ij} \varphi \, dx = \int_{\Omega_1} J(x, f)\delta_{ij} \varphi \, dx, \]

where $\varphi \in C^0_0(\Omega_1), \ i, j = 1, 2$. Since $\varphi$ was arbitrary, it follows that

\begin{equation}
\langle A(x)\nabla f^i(x), \nabla f^j(x) \rangle = J(x, f)\delta_{ij} \quad \text{a.e. in } \Omega_1, \ i, j = 1, 2,
\end{equation}

and consequently, as $J(x, f)$ is a.e. positive,

\begin{equation}
A(x) = J(x, f)[Df(x)^t \cdot Df(x)]^{-1} \quad \text{a.e. in } \Omega_1.
\end{equation}

Since $\Omega_1$ was arbitrary, (5.18) holds a.e. in $\Omega$. Hence (5.8) holds. \(\square\)
6. The Bers–Bojarski theorem

For the sake of brevity we will now confine ourselves to the particular case $K(x) = K \geq 1$ and relate our results to some classical convergence theorems for quasiregular mappings.

Let $G(x, f)$ be defined as in (1.3). No natural continuity result can be traced for the map

$$f \rightarrow G(x, f)$$

of the type obtained in the present paper for the map

$$f \rightarrow A(x, f)$$

unless we consider a convergence $f_h \rightarrow f$ stronger than weak-$W^{1,2}$; see also [LV], [D].

**Example 6.1.** Let $\psi_h : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of bounded measurable functions such that $0 < K^{-1} \leq \psi_h(t) \leq K$ and

$$\psi_h \rightharpoonup 1, \quad \frac{1}{\psi_h} \rightharpoonup \frac{1}{c} \quad (c \neq 1),$$

in $\sigma(L^\infty, L^1)$; for example, let us choose

$$\psi_h(t) = 1 + \delta \frac{\sin ht}{|\sin ht|} \quad (0 < \delta < 1).$$

Then, the sequence of $K$-quasiregular mappings

$$f_h(x_1, x_2) = \left( \int_0^{x_1} \psi_h(t) dt, x_2 \right)$$

converges locally uniformly to the identity mapping $f(x_1, x_2) = (x_1, x_2)$.

It is immediate that the distortion tensor of $f_h$ is

$$G(x, f_h) = \begin{pmatrix} \psi_h(x_1) & 0 \\ 0 & (\psi_h(x_1))^{-1} \end{pmatrix}$$

and the distortion tensor of the limit $f$ is

$$G(x, f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
The sequence \( G(x, f_h) \) does not converge weakly nor does it \( \Gamma \)-converge to the identity matrix \( G(x, f) \). Actually

\[
G(x, f_h) \rightharpoonup \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{weakly in } L^1(\Omega, \mathbb{R}^{2 \times 2}).
\]

Moreover it can be proved that

\[
G(x, f_h) \xrightarrow{\Gamma} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}.
\]

Thus, of the two matrices \( A(x, f) \), \( G(x, f) \) only the first one exhibits a suitable continuity behaviour as a function of \( f \).

In the following we deduce by our results a well-known theorem of Bers–Bojarski for planar \( K \)-quasiregular mappings whose \( n \)-dimensional version has been recently proved in [GMRV] (see also [IK]). The result states that if \( f_h: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \) verify a.e. in \( \Omega \) \((K \geq 1)\)

\[
|Df_h(x)|^2 \leq \left(K + \frac{1}{K}\right)J(x, f_h);
\]

if \( f_h \to f \) locally uniformly and the distortion tensors \( G(x, f_h) \) defined as in (1.3) converge a.e. to \( G_0(x) \) then \( G_0(x) = G(x, f) \). Namely we have the following

**Theorem 6.1.** Let \( f_h \) be a sequence of mappings of finite distortion \( K \geq 1 \) on \( \Omega \) such that

(i) \( f_h \rightharpoonup f \ in W^{1,2}(\Omega) \),

(ii) \( G(x, f_h) \rightharpoonup G_0(x) \) a.e. in \( \Omega \).

Then

\[ G_0(x) = G(x, f) \quad \text{a.e. in } \Omega. \]

We start with

**Lemma 6.1.** Let \( A_h \) be a sequence of symmetric \( 2 \times 2 \) matrices satisfying

\[
\frac{|\xi|^2}{K} \leq \langle A_h(x)\xi, \xi \rangle \leq K|\xi|^2 \quad \text{for a.e. } x \in \Omega.
\]

If

\[
A_h^{-1} \rightharpoonup A_0^{-1} \quad \text{in } L^1(\Omega, \mathbb{R}^{2 \times 2})
\]

and

\[
(6.2) \quad A_h \rightharpoonup A
\]

then

\[ A = A_0. \]
Proof. It is easy to check that
\[ A_h - A_0 = A_h (A_0^{-1} - A_h^{-1}) A_0. \]
So by our assumptions we deduce
\[ A_h \to A_0 \quad \text{in } L^1(\Omega, \mathbb{R}^{2 \times 2}). \]
Since it is well known that strong $L^1$ convergence of coefficients matrices imply \( \Gamma \)-convergence \([S1]\), we get
\[ A_h \rightharpoonup A_0 \]
and therefore, by (6.2)
\[ A = A_0. \]

Proof of Theorem 6.1. Theorem 5.2 implies that \( A(x, f_h) \rightharpoonup A(x, f) \). By (ii) and Vitali’s theorem we deduce
\[ G(x, f_h) = A(x, f_h)^{-1} \rightharpoonup L^1, G_0(x) = A_0^{-1}(x) \]
so Lemma 6.1 implies \( A(x, f) = A_0(x) = G_0^{-1}(x) \) and this means \( A^{-1}(x, f) = G_0(x) \), that is \( G(x, f) = G_0(x) \).

Actually, $L^1$-convergence of the coefficient matrix $A_h$ to $A$ implies strong convergence in $W^{1,2}_{\text{loc}}$ of local solutions $u_h$ of the equation
\[ \text{div } A_h(x) \nabla u_h = 0 \]
to local solutions $u$ of
\[ \text{div } A(x) \nabla u = 0 \]
(see \([S1, \text{Theorem 5}]\)). So, in particular, under our assumptions we deduce $f^i_h \to f^i$ in $W^{1,2}_{\text{loc}}$, for $i = 1, 2$, due to the fact that $\text{div } A_h(x, f_h) \nabla f_h = 0$.

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References


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