ON GEOMETRICAL NETS IN SPACE

By

William Rowan Hamilton

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NOTE ON THE TEXT

The paper *On Geometrical Nets in Space* by Sir William Rowan Hamilton was originally published in the seventh volume of the *Proceedings of the Royal Irish Academy*. The following typographical errors in that text have been corrected in this edition:

- in article [13.], equation (47), $F_1$ has been corrected to $F'$;
- in article [21.], equation (100), a missing overline has been placed over the 3rd occurrence of $\sigma$ on the line.
- in article [33.], ‘(’ has been corrected to ‘[’ in the ternary symbol for the plane $AA'D_1C_1B_2$;
- in article [56.], the final point in the list of points on the third typical line $B'C'$ was printed as $\Lambda^{IX}$, but this has been corrected to $\Lambda^{I\text{X}}$;
- in article [58.], the ternary symbol for the point $b_0$ on the line [011] was printed as $(T11)$, but has been corrected to $(1T1)$;
- in article [74.], the ternary symbol for the line $AA'$ was printed as [011], but this has been corrected to [01T];
- in article [100.], ‘remaked’ has been corrected to ‘remarked’;
- in article [123.]: the third point on the line of intersection of the planes $ABD$ and $A_1B_1D_1$ was printed as $\Lambda''$; but this has been corrected to $\Lambda''$.

Also, from article [97.] onwards, the roman superscripts on the points $A^{IV}$, $B^{IV}$, $C^{IV}$, etc. were printed in lower case, as $A^{iv}$, $B^{iv}$, $C^{iv}$, etc., but these superscripts have been changed to uppercase, in conformity with the notation established in the earlier articles. Similarly $p_0$, $p_1$, and $p_2$ in articles [1.] and [2.] were originally printed in normal size uppercase roman, but have been changed in this edition to ‘small capitals’, in conformity with the notation in the remainder of the paper.

David R. Wilkins
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ON GEOMETRICAL NETS IN SPACE.

Sir William Rowan Hamilton.

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[1.] When any five points of space, \(ABC\), are given, whereof no four are supposed to be complanar, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect each other: deriving thus a system of ten lines \(\Lambda_1\), ten planes \(\Pi_1\), and ten points \(p_1\), from the given system of five points \(p_0\), by what may be called a First Construction.

We may next propose to determine all the new and distinct lines \(\Lambda_2\), and planes \(\Pi_2\), which connect the ten derived points \(p_1\), with the five given points \(p_0\), and with each other, and may then inquire what new and distinct points \(p_2\) arise, as intersections* \(\Lambda \cdot \Pi\) of lines and planes already obtained: all such new lines, planes, and points being said to belong to a Second Construction. And then we might proceed, on the same plan, to a Third Construction, and to indefinitely many others following: building up thus what Professor Möbius, in his Barycentric Calculus,† has proposed to call a Geometric Net in Space.

[2.] In general, if \(n\) denote five or any greater number of independent points of space, the number of the derived points of the form \(\Lambda \cdot \Pi\), or \(AB \cdot CDE\), which can be obtained by what is relatively to them a First Construction, of the kind just now described, is easily seen to be the function,

\[
f(n) = \frac{n(n - 1)}{2} \cdot \frac{(n - 2)(n - 3)(n - 4)}{2 \cdot 3};
\]

so that \(f(5) = 10\), as above, but \(f(15) = 30030\). If then the fifteen points \(p_0, p_1\) were thus independent, or unconnected with each other, we might expect to find that the number of points \(p_2\) derived from them, at the next stage, should exceed thirty thousand. And although it was obvious that many reductions of this number must occur, on account of the dependence of the ten points \(p_1\) on the five points \(p_0\), yet when I happened to feel a curiosity, some time ago, to determine the precise number of those which have been above called Points of Second

* Intersections \(\Lambda \cdot \Lambda\) of line with line (when complanar) are included in this class \(\Lambda \cdot \Pi\); and intersections \(\Pi \cdot \Pi \cdot \Pi\) of three distinct planes, when not included at this stage, may be reserved for a subsequent construction, in which they naturally offer themselves, as of the standard form \(\Lambda \cdot \Pi\).

† Der Calcul Barycentrische, Leipzig, 1827, p. 291. Some first results connected with the subject were given, according to the writer’s recollection, in a Memoir by Carnot on Transversals, to which he cannot at present refer.
Construction, and to assign their chief geometrical relations to each other, and to the fifteen
former points, it must be confessed that I thought myself about to undertake the solution of
a rather formidable Problem. But the motive which led me to attack that problem, namely
the desire to try the efficiency of a certain system of Quinary Symbols, for points, lines, and
planes in space, which the Method of Vectors had led me to invent, inspired me with a hope,
which I trust that the result of the attempt has not altogether failed to justify. And, in the
present communication, I wish first to present some outline of what may be called perhaps a
Quinary Calculus, before proceeding to give, in the second place, some sketch of the results
of its application to the geometrical Net in Space.

Part I.—On a Quinary Calculus for Space.

3. Let $\{A, B, C, D, E\}$ be (as in [1.]) any five given points of space, whereof no four are
situated in any common plane; then, by decomposing $ED$ in the directions of $EA$, $EB$, $EC$, we
can always obtain an equation of the form,

$$a \cdot EA + b \cdot EB + c \cdot EC + d \cdot ED = 0,$$

(1)
in which the coefficients $abcd$ have determined ratios. And if we next introduce a fifth
coefficient $e$, such that

$$a + b + c + d + e = 0,$$

(2)
and add to (1) the identity

$$(a + b + c + d + e)OE = 0,$$

(3)
in which $O$ is any arbitrary point (or origin of vectors), we arrive at the following equivalent
but more symmetric form,

$$a \cdot OA + b \cdot OB + c \cdot OC + d \cdot OD + e \cdot OE = 0,$$

(4)
in which $abcde$ may be called the five (numerical) constants of the given system of five
points, $A \ldots E$, although only their ratios are important, and (as above) their sum is zero.

4. Let $P$ be any other point of space, and let $xyzwv$ be coefficients satisfying the
equation,

$$(x - v)a \cdot PA + (y - v)b \cdot PB + (z - v)c \cdot PC + (w - v)d \cdot PD = 0;$$

(5)
then, adding the identity,

$$v(a \cdot PA + b \cdot PB + c \cdot PC + d \cdot PD + e \cdot PE) = 0,$$

(6)
which results from (4), we obtain this other symmetric formula,

$$xa \cdot PA + yb \cdot PB + zc \cdot PC + wd \cdot PD + ve \cdot PE = 0,$$

(7)
which may also be thus written,

$$OP = \frac{xa \cdot OA + yb \cdot OB + zc \cdot OC + wd \cdot OD + ve \cdot OE}{xa + yb + zc + wd + ve},$$

(8)
o being again an arbitrary origin; and the five new and variable coefficients, \(xyzwv\), whereof the ratios of the differences determine the position of the point \(P\), when the five points \(A \ldots E\) are given, may be called the Quinary Coordinates of that Point \(P\), with respect to the given system of five points.

[5.] Under these conditions, we may agree to write, briefly,

\[ P = (x, y, z, w, v), \quad \text{or even} \quad P = (xyzwv), \tag{9} \]

whenever it seems that the omission of the commas will not give rise to any confusion; and may call this form a Quinary Symbol of the Point \(P\). But because (as above) only the ratios of the differences of the coefficients or coordinates are important, we may establish the following Formula of Quinary Congruence, between two equivalent Symbols of one common point,

\[ (x'y'z'w'v') \equiv (xyzwv), \tag{10} \]

if \(x' - v' : y' - v' : z' - v' : w' - v' = x - v : y - v : z - v : w - v\); \(\tag{11}\)

reserving the Quinary Equation,

\[ (x'y'z'w'v') = (xyzwv), \tag{12} \]

to imply the coexistence of the five separate and ordinary equations,

\[ x' = x, \quad y' = y, \quad z' = z, \quad w' = w, \quad v' = v. \tag{13} \]

We shall also adopt, as abridgments of notation, the formulæ,

\[ t(x, y, z, w, v) = (tx, ty, tz, tw, tv); \tag{14} \]

\[ (x' \ldots v') \pm (x \ldots v) = (x' \pm x, \ldots v' \pm v); \tag{15} \]

and shall find it convenient to employ occasionally what may be called the Quinary Unit Symbol,

\[ u = (11111); \tag{16} \]

although this symbol represents no determined point, because both the denominator and numerator of the expression (8) vanish, by (2) and (4), when the five coefficients \(xyzwv\) become each equal to unity.

[6.] With these notations, if \(Q\) and \(Q'\) be any other quinary symbols, and \(t\) and \(u\) any two coefficients, we shall have the congruence,

\[ Q' \equiv Q, \quad \text{if} \quad Q' = tQ + uv; \tag{17} \]

the two points \(P\) and \(P'\), which are denoted by these two symbols, in this case coinciding. Again the equation

\[ Q'' = tQ + t'Q' + uv, \tag{18} \]

is found to express that \(Q, Q', Q''\) are symbols of three collinear points; and the complanarity of four points, of which the symbols are \(Q, Q', Q'', Q''', \) is expressed by this other equation of the same form,

\[ Q''' = tQ + t'Q' + t''Q'' + uv. \tag{19} \]
If then a variable point $P$ be thus complanar with three given points, $P_0$, $P_1$, $P_2$, its coordinates must be connected with theirs, by five equations of the form,

$$x = t_0x_0 + t_1x_1 + t_2x_2 + u; \quad \ldots \quad v = t_0v_0 + t_1v_1 + t_2v_2 + u;$$

whence, by elimination of the four arbitrary coefficients $t_0$, $t_1$, $t_2$, $u$, a linear equation is obtained, of the form

$$lx + my + nz + rw + sv = 0,$$

with the general relation

$$l + m + n + r + s = 0$$

between its coefficients; and this equation may be said to be the Quinary Equation of the Plane $P_0P_1P_2$. The five new coefficients $lmnrs$ may be called the Quinary Coordinates of that Plane; and the plane itself may be denoted by the Quinary Symbol,

$$\Pi = [l, m, n, r, s], \quad \text{or briefly,} \quad \Pi = [lmnrs],$$

when the commas can be omitted without confusion.

If $R$, $R'$, $\ldots$ be symbols of this form, for planes $\Pi$, $\Pi'$, $\Pi''$, then the equation

$$R' = tR,$$

in which $t$ is an arbitrary coefficient, expresses that the two planes $\Pi$, $\Pi'$ coincide; the equation

$$R'' = tR + t'R'$$

expresses that the three planes $\Pi$, $\Pi'$, $\Pi''$ are collinear, or that the third passes through the line of intersection of the other two; and the equation

$$R''' = tR + t'R' + t''R''$$

expresses that the four planes $\Pi$, $\Pi'$, $\Pi''$, $\Pi'''$ are compunctual (or concurrent), or that the fourth passes through the point of intersection of the other three.

It is easy to conceive how problems respecting intersections of lines and planes can be resolved, on the foregoing principles. And if we define that a point $P$, or plane $\Pi$, is a Rational Point, or a Rational Plane of the System determined by the five given Points $A \ldots E$, or that it is rationally related to those five points, when its coordinates are equal (or proportional) to whole numbers, it is obvious, from the nature of the eliminations employed, that a plane which is determined as containing three rational points, or a point which is determined as the intersection of three rational planes, is itself, in the above sense, rational. We may also say that a right line $\Lambda$ is a Rational Line, when it is the line $P\longrightarrow P$ which connects two rational points, or the intersection $\Pi \cdot \Pi$ of two rational planes: and then the intersection of a rational line with a rational plane, or of two complanar and rational lines with each other, will be a rational point.
[9.] When any two points, \( p, p' \), or any two planes \( \Pi, \Pi' \), have symbols which differ only by the arrangement or (order) of the five coefficients or coordinates in each, those points, or those planes, may then be said to have one common type; or briefly, to be syntypical. For example, the five given points are thus syntypical, because (omitting commas, as in [5.]) their symbols are,

\[
A = (10000), \quad B = (01000), \quad C = (00100), \quad D = (00010), \quad E = (00001).
\] (27)

In general, any two syntypical points, or planes, admit of being derived from the five given points, by precisely similar processes of construction, the order only of the data being varied; and in the most general case, a single type includes 120 distinct points, or distinct planes, although this number may happen to be diminished, even when the coordinates are all unequal: for example, the type \((1\,2\,3\,4\,5)\) includes only sixty distinct points, because, by (17), we have in this case the congruence,

\[
(1\,2\,3\,4\,5) \equiv (5\,4\,3\,2\,1).
\] (28)

[10.] The anharmonic function of any group of four collinear points \( ABCD \) being denoted by the symbol \((ABC\,D)\), and defined by the equation,

\[
(ABC\,D) = \frac{AB}{BC} \cdot \frac{CD}{DA} = \frac{AB}{CB} : \frac{AD}{CD},
\] (29)

it will be found that if \( p_0 \ldots p_3 \) be thus any four collinear points, of which therefore, by (18), the quinary symbols \( q_0 \ldots q_3 \) are connected by two linear relations, of the forms,

\[
q_1 = t_0q_0 + t_2q_2 + wu, \quad q_3 = t'_0q_0 + t'_2q_2 + u'u,
\] (30)

then the anharmonic of this group of points is given by the formula,

\[
(p_0\,p_1\,p_2\,p_3) = \frac{t_2t'_0}{t'_0t_2},
\] (31)

of which the applications are numerous and important.

And in like manner, if \( \Pi_0 \ldots \Pi_3 \) be any four collinear planes, of which consequently, by (25), the symbols \( r_0 \ldots r_3 \) are connected by two other linear relations, such as

\[
r_1 = t_0r_0 + t_2r_2, \quad r_3 = t'_0r_0 + t'_2r_2,
\] (32)

we have then this other very useful formula of the same kind, of the anharmonic of this pencil of planes,

\[
(\Pi_0\,\Pi_1\,\Pi_2\,\Pi_3) = \frac{t_2t'_0}{t'_0t_2},
\] (33)

it being understood that the anharmonic function of such a pencil is the same as that of the group of points, in which its planes are cut by any rectilinear transversal: so that we may write generally, for any six points \( A \ldots F \), the formula,

\[
(\text{E.F.} \, ABCD) = (A' \, B' \, C' \, D'),
\] (34)

if any transversal \( GH \) cut the four planes \( EFA, \ldots EFD \) in the four points \( A', \ldots D' \); or in symbols, if

\[
A' = GH \cdot EFA, \quad \ldots \quad D' = GH \cdot EFD.
\] (35)
The expression of fractional form,

\[
\varphi(x y z w) = \frac{l' x + m' y + n' z + r' w + s' v}{l x + m y + n z + r w + s v} = \frac{f'}{f},
\]

(36)
in which the ten coefficients \( l \ldots s \) and \( l' \ldots s' \), are supposed to be given, and to be such (comp. (22)) that

\[
l + \cdots + s = 0, \quad \text{and} \quad l' + \cdots + s' = 0,
\]

(37) may represent the quotient of any two linear and homogeneous functions, \( f \) and \( f' \), of the coordinates \( x \ldots v \) of a variable point \( P \), or rather of the differences of those coordinates (comp. [5.]); and if we assign any particular or constant value, such as \( k \), to this quotient, or fractional function, \( \varphi \), the equation so obtained will represent (comp. (21)) a plane locus for that point, which plane \( \Pi \) will always pass through a given line \( \Lambda \), determined by equating separately the denominator and numerator of \( \varphi \) to zero. Hence the four equations,

\[
f = 0, \quad f' = f, \quad f' = 0, \quad f' = k f,
\]

(38) which answer to the four values,

\[
\varphi = \infty, \quad \varphi = 1, \quad \varphi = 0, \quad \varphi = k,
\]

(39) represent a pencil of four planes \( \Pi_0 \ldots \Pi_3 \), of which the quinary symbols (23) may be thus written:

\[
R_0 = [l m n r s]; \quad R_2 = [l' m' n' r' s']; \quad R_1 = R_2 - R_0; \quad R_3 = R_2 - kR_0;
\]

(40) and of which the anharmonic is consequently, by (33), the same quotient,

\[
(\Pi_0 \Pi_1 \Pi_2 \Pi_3) = (k = \varphi) = \frac{f'}{f},
\]

(41) as before. We have therefore this Theorem:—

“The Quotient of any two given homogeneous and linear Functions, of the Differences of the Quinary Coordinates of a variable Point in Space, can always be expressed as the Anharmonic of a Pencil of Planes, whereof three are given, while the fourth passes through the variable Point, and through a given Right Line, which is common to the three former Planes.”

[12.] For example, we find thus that

\[
\frac{x - v}{w - v} = (B C \cdot A E D P); \quad \frac{y - v}{w - v} = (C A \cdot B E D P); \quad \frac{z - v}{w - v} = (A B \cdot C E D P);
\]

(42)
and that

\[
\frac{x - v}{y - v} = (C D \cdot A E B P); \quad \frac{y - v}{z - v} = (A D \cdot B E C P); \quad \frac{z - v}{x - v} = (B D \cdot C E A P);
\]

(43)
the product of these three last anharmonics of pencils being therefore equal to positive unity, so that we have, for any six points of space, ABCDEF, the general equation,

\[(\text{AD BCEF}) \cdot (\text{BD CEA}F) \cdot (\text{CD AEB}F) = 1.\] (44)

If then we suppress the fifth coefficient, \(v\), in the quinary symbol (9) of a point \(p\), which comes to first substituting, as the congruence (10) permits, the differences \(x - v, y - v, z - v, w - v,\) and \(v - v\) or 0, for \(x, y, z, w,\) and \(v,\) and then writing simply \(x, \ldots w\) instead of \(x - v, \ldots w - v,\) and omitting the final zero, whereby the quinary symbol (0 0 0 0 1) for the fifth given point \(E\) (27) becomes first \((-1, -1, -1, -1, 0),\) or \((11110),\) and then is reduced to the quaternary unit symbol \((11111),\) we shall fall back on that system of anharmonic coordinates in space, of which some account was given in a former communication* to this Academy: the anharmonic (or quaternary) symbol of a plane \(\Pi\) being, in like manner, derived from the quinary symbol (23), by simply suppressing the fifth coefficient, or coordinate, \(s.\) Anharmonic coordinates, whether for point or for plane, are therefore included in quinary ones; but although they have some advantages of simplicity, it appears that their less perfect symmetry, of reference to the five given points \(A \ldots E,\) renders them less adapted to investigations respecting the Geometrical Net in Space, which is constructed with those five points as data: and that therefore they are less fit than quinary coordinates for the purposes of the present paper.

[13.] Retaining then the quinary form, we may next observe that although, when the five coefficients \(l \ldots s\) are given, as in [7.], and the coordinates \(x \ldots v\) of a point \(p\) are variable, the linear equation \(lx + \cdots sv = 0\) (21) may be said to be the Local Equation of a Plane, namely of the plane \([l \ldots s],\) considered as the locus of the point \((x \ldots v);\) yet if, on the contrary, we now regard \(x \ldots v\) as given, and \(l \ldots s\) as variable, the same linear equation (21) expresses the condition necessary, in order that a variable plane \([l \ldots s]\) may pass through a given point \((x \ldots v);\) and in this view, the formula (21) may be considered to be the Tangential Equation of that given Point. Thus the very simple equation,

\[l = 0,\] (45)

expresses the condition requisite for the plane \([l \ldots s]\) passing through the given point \((10000),\) or \(A\) (27); and it is, in that sense, the tangential equation of that point: while \(m = 0\) is, in like manner, the equation of \(B, &c.\) This being understood, if we suppose that \(F\) and \(F'\) denote two given, linear, and homogeneous functions of the coordinates \(l \ldots s\) of a variable plane \(\Pi,\) we may consider the four equations,

\[F = 0, \quad F' = F, \quad F' = 0, \quad F' = kF,\] (46)

as the tangential equations of four collinear points, \(P_0, P_1, P_2, P_3,\) whereof the three first are entirely given, but the fourth varies with the value of the coefficient \(k,\) although always remaining on the line \(A\) of the other three; and it is easy to deduce, from the formula (31), by reasonings analogous to those employed in [11.], the following anharmonic of the group:

\[(P_0 P_1 P_2 P_3) = k = \frac{F'}{F}.\] (47)

We have therefore this new Theorem, analogous to one lately stated:—

* See the Proceedings for the Session of 1859–60.
"The Quotient of any two given, homogeneous, and linear Functions, of the Quinary Coordinates of a variable Plane, may always be expressed as the Anharmonic of a Group of Points; whereof three are given and collinear, while the fourth is the Intersection of the variable Plane on which the other three are situated."

[14.] For example, if we wish in this way to interpret the quotient \( m : n \), of these two coordinates of a variable plane \( \Pi \), or \([lmnrs]\) (23), as denoting the anharmonic of a group of points, the first three points \( P_0, P_1, P_2 \) of that group (47) have here for their tangential equations,

\[
\begin{align*}
  n &= 0, \\
  m - n &= 0, \\
  m &= 0,
\end{align*}
\]

whereof the third has recently been seen [13.] to represent the given point \( B \), and the first represents in like manner another given point, namely \( C \), of the initial system: while the second represents the point \((0, 1, -1, 0, 0)\), or briefly \((01\overline{1}00)\), if, to save commas, we write \( \overline{1} \) for \(-1\). To construct this last point, let us write

\[
A' = (01100) \equiv (10011), \quad \text{and} \quad A'' = (01\overline{1}00);
\]

then, by (18), these two new points \( A' \) and \( A'' \) are each collinear with \( B, C \), or are on the line \( BC \); and they are, with respect to that line (or to its extreme points) harmonically conjugate to each other, because the formula (31) gives easily, by the first symbol for \( A' \), the harmonic equation,

\[
(BA'C A'') = -1;
\]

but also the second (or congruent) symbol for \( A' \) shows, by (19), that \( A' \) is in the plane \( ADE \); we may therefore write the formula of intersection,

\[
A' = BC \cdot ADE,
\]

whereby the point \( A' \) is entirely determined; and then the point \( A'' \), as being its harmonic conjugate with respect to \( B \) and \( C \), or as satisfying the equation (50), is to be considered as being itself a known point. We have thus assigned the three first points \( P_0, P_1, P_2 \), of the group (47), namely the points \( C, A'', B \); and if we denote by \( L \) the point \( BC \cdot \Pi \) in which the variable plane \( \Pi \), or \([l \ldots s]\), intersects the given line \( BC \), so that

\[
L = (0, n, -m, 0, 0), \quad \text{or briefly,} \quad L = (0n\overline{m}00),
\]

writing \( \overline{m} \) for \(-m\), then the fourth point \( P_3 \) is \( L \); and the required formula of interpretation for the quotient \( m : n \) becomes,

\[
\frac{m}{n} = (CA'' \cdot BL).
\]

In like manner, if we write

\[
B' = (10100), \quad C' = (11000), \quad B'' = (\overline{1}0100), \quad C'' = (1\overline{1}000),
\]

and

\[
M = (\overline{1}0100), \quad N = (m\overline{1}000),
\]

\[
8
\]
in which \( \bar{n} = -n \) and \( \bar{l} = -l \), so that \( M = CA \cdot \Pi, N = AB \cdot \Pi, \) and

\[
B' = CA \cdot BDE, \quad C' = AB \cdot CDE, \quad (C'B'A'B'') = (AC'B'C''') = -1,
\]

we shall have these two other formulæ of interpretation, analogous to (53),

\[
\frac{n}{l} = (AB''CM), \quad \frac{l}{m} = (BC''AN);
\]

and therefore,

\[
(AB''CM) \cdot (BC''AN) \cdot (CA''BL) = 1.
\]  

[15.] Again, if we denote by \( Q, R, S \) the intersections \( DA \cdot \Pi, DB \cdot \Pi, DC \cdot \Pi \), so that \( \overline{Q} = -r \); if also we introduce seven new points syntypical [9.] with the three points \( A'B'C' \), and seven others syntypical with \( A''B''C'' \), as follows:

\[
A_1 = (10001), \quad B_1 = (01001), \quad C_1 = (00101), \quad D_1 = (00011);
\]

\[
A_2 = (10010), \quad B_2 = (01010), \quad C_2 = (00110);
\]

\[
A'_1 = (1000\overline{T}), \quad B'_1 = (0100\overline{T}), \quad C'_1 = (0010\overline{T}), \quad D'_1 = (0001\overline{T});
\]

\[
A'_2 = (100\overline{T}0), \quad B'_2 = (010\overline{T}0), \quad C'_2 = (001\overline{T}0);
\]

so that, by principles already established, we shall have the seven relations of intersection,

\[
A_1 = EA \cdot BCD, \quad B_1 = EB \cdot CAD, \quad C_1 = EC \cdot ABD, \quad D_1 = ED \cdot ABC,
\]

\[
A_2 = DA \cdot BCE, \quad B_2 = DB \cdot CAE, \quad C_2 = DC \cdot ABE,
\]

and the seven harmonic relations,

\[
(EA_1 A A'_1) = (EB_1 B B'_1) = (EC_1 C C'_1) = (ED_1 D D'_1) = -1,
\]

\[
(DA_2 A A'_2) = (DB_2 B B'_2) = (DC_2 C C'_2) = -1,
\]

by means of which 14 last relations these 14 new points can all be geometrically constructed; we shall then be able to interpret, on the recent plan [13.], the three new quotients, \( l : r, m : r, n : r \), as anharmonics of groups, as follows:

\[
\frac{l}{r} = (DA'_2 A Q); \quad \frac{m}{r} = (DB'_2 B R); \quad \frac{n}{r} = (DC'_2 C S);
\]

with the analogous interpretations,

\[
\frac{l}{s} = (EA'_1 A X); \quad \frac{m}{s} = (EB'_1 B Y); \quad \frac{n}{s} = (EC'_1 C Z); \quad \frac{r}{s} = (ED'_1 D W),
\]

if \( X, Y, Z, W \) denote the intersections \( EA \cdot \Pi, EB \cdot \Pi, EC \cdot \Pi, ED \cdot \Pi \), so that

\[
X = (\overline{Q}000l), \quad Y = (0\overline{Q}00m), \quad Z = (00\overline{Q}0n), \quad W = (000\overline{Q}r),
\]

where \( \overline{s} = -s \).
As regards the notations employed, it may be observed that although we have often, as in (9) or (27), &c., "equated a point, or rather its literal symbol, A or P, &c., to the corresponding quinary symbol (1 0 0 0 0) or (x y z w v), &c., of that point, yet in some formulae, such as (17) (18) (19), in which we had occasion to treat of linear combinations of such quinary symbols, we substituted new letters, such as q, q', for P, P', &c., in order to avoid the apparent strangeness of writing such expressions* as $tP + t'P'$, &c. To economise symbols, however, we may agree to retain the literal symbols first used, for any system of given or derived points, but to enclose them in parentheses, when we wish to employ them as denoting quinary symbols in combination with each other; writing, at the same time, for the sake of uniformity (U) instead of u, as the quinary unit symbol (16). And thus, if we agree also that an equation between two unenclosed and literal symbols of points, p and p', shall be understood as expressing that the two points so denoted coincide, we may write anew those formulae (17) (18) (19) as follows:

\[ P' = P, \text{ if } (P') = t(P) + u(U); \]
\[ P'' \text{ on line } PP', \text{ if } (P'') = t(P) + t'(P') + u(U); \]
\[ P''' \text{ in plane } PP'P'', \text{ if } (P''') = t(P) + t'(P') + t''(P'') + u(U). \]

We may also occasionally denote a point in the given plane of A, B, C by the ternary symbol, $(x; y; z)$, or $(xyz)$; considered here as an abridgment of the quinary symbol $(x y z 0 0)$; and the right line which is the trace on that plane, of any other plane $\Pi$, or $[l m n r s]$ (23) may be denoted by this other ternary symbol, $[l; m; n]$, or $[lmn]$; these two last ternary symbols being connected by the relation,

\[ lx + my + nz = 0, \]

if the point $(x y z)$ be on the line $[lmn]$. And the point $P$ in which any other line $\Lambda$, not situated in the plane ABC, intersects that plane, may be said to be the trace of that line.

For example, the point $D_1$ is, by (64), the trace of the line DE; and if we write,

\[ A_0 = (\bar{T} 1 1), \quad B_0 = (1 \bar{T} 1), \quad C_0 = (1 1 \bar{T}), \]

then these three points are the respective traces of the three lines $A_1A_2$, $B_1B_2$, $C_1C_2$; because they are, by the notation (74), in the given plane, and we have, by (60) and (61), the three following symbolical equations of the form (72),

\[ (A_0) + (A_1) + (A_2) = (B_0) + (B_1) + (B_2) = (C_0) + (C_1) + (C_2) = (U), \]

* Expressions of this form occur continually in the Barycentric Calculus of Moebius, but with significations entirely different from those here proposed.
which expresses the three collineations, \( A_0A_1A_2, B_0B_1B_2, C_0C_1C_2 \).

We have also the three other collineations, \( AD_1A', BD_1B', CD_1C' \), because the quinary symbols (27) (49) (54) (60) give the equations,

\[
(A) + (A') + (D_1) = (B) + (B') + (D_1) = (C) + (C') + (D_1) = (U);
\]

and these three lines, \( AA'D_1, \&c. \), are the traces of the three planes \( ADE, BDE, CDE \), of which planes the respective equations (21), and quinary symbols (23), are

\[
y - z = 0, \quad z - x = 0, \quad x - y = 0, \quad (80)
\]

and

\[
[01\overline{0}0], \quad [\overline{0}100], \quad [1\overline{0}00]; \quad (81)
\]

so that the ternary symbols of the three last lines, regarded as their traces, are simply, by (75),

\[
[01\overline{1}], \quad [\overline{0}11], \quad [1\overline{1}0]. \quad (82)
\]

Accordingly, whether we consider the point \( A = (100) \), or \( A' = (011) \), or \( D_1 = (111) \), (this ternary symbol of \( D_1 \) being congruent to the former quinary symbol \( (00111) \) for that point (60),) we have in each case the relation \( y - z = 0 \) between its coordinates; and similarly for the two other lines.

[19.] As other examples, the four planes,

\[
A_1B_1C_1, \quad A_2B_2C_2, \quad A'_1B'_1C'_1, \quad A'_2B'_2C'_2; \quad (83)
\]

have for their quinary equations,

\[
x + y + z = 2w + v, \quad x + y + z = w + 2v, \quad x + y + z + v = 4w, \quad x + y + z + w = 4v; \quad (84)
\]

and for their quinary symbols,

\[
[111\overline{2}1], \quad [111\overline{2}1], \quad [111\overline{1}1], \quad [1111\overline{1}]; \quad (85)
\]

they have therefore a common trace, namely the line

\[
[111], \quad \text{or } A''B''C''; \quad (86)
\]

because, by (49) and (54), we may now write

\[
A'' = (01\overline{1}), \quad B'' = (\overline{0}11), \quad C'' = (1\overline{1}0), \quad (87)
\]

and the coordinates of each of these three last points satisfy the equation,

\[
x + y + z = 0. \quad (88)
\]

Accordingly we have, by (60) (61) (62) (63), the three following sets of symbolical equations of the form (72),

\[
\begin{align*}
(A'') &= (B_1) - (C_1) = (B_2) - (C_2) = (B'_1) - (C'_1) = (B'_2) - (C'_2), \\
(B'') &= (C_1) - (A_1) = (C_2) - (A_2) = (C'_1) - (A'_1) = (C'_2) - (A'_2), \\
(C'') &= (A_1) - (B_1) = (A_2) - (B_2) = (A'_1) - (B'_1) = (A'_2) - (B'_2),
\end{align*}
\]

we see that the point \( A'' \) is the common trace of the four lines, \( B_1C_1, B_2C_2, B'_1C'_1, B'_2C'_2; \) \( B'' \) of \( C_1A_1, C_2A_2, C'_1A'_1, C'_2A'_2; \) and \( C'' \) of \( A_1B_1, A_2B_2, A'_1B'_1, A'_2B'_2. \)
In all such cases as these, in which we have to consider a set of three points \( p \), or a set of three planes \( \Pi \), of which the first is geometrically derived from \( ABCDE \) according to the same rule of construction, as that according to which the second is derived from \( BCDAE \), and the third from \( CABDE \), we can symbolically derive the second from the first, and in like manner the third from the second, (or again the first from the third,) by writing, in each case, the first, first, and second coefficients, or coordinates, in the places of the first, second, and third, respectively. In symbols, we may express the law of successive derivation, of certain syntypical points or planes [9.] from one another, by the formulæ,

\[
\text{if } p(ABC) = (xyzwv), \text{ then } p(BCA) = (zxywv), \text{ and } p(CAB) = (yzxwv); \quad (90)
\]

and if

\[
\Pi(ABC) = [lmnrst], \text{ then } \Pi(BCA) = [nlmrst], \text{ and } \Pi(CAB) = [mnlrst]; \quad (91)
\]
as has been already exemplified in the systems (27), (60), (61), (62), (63), (77), (81), (87), for points or planes, and in (82) for lines, considered as traces of planes. In all these cases, therefore, we can, with perfect clearness and definiteness of signification, abridge the notation, by writing only the first (or indeed any one) of the three equations (90) or (91) and then appending an “&c.”; for the law which has been just stated will always enable us to recover (or deduce) the other two. We may therefore briefly but sufficiently express several of the foregoing results, by writing,

\[
\begin{align*}
A & = (100), \text{ &c.}; & A' & = (011), \text{ &c.}; & A'' & = (01\overline{T}), \text{ &c.}; & A_0 & = (\overline{T}11), \text{ &c.}; \\
A_1 & = (1001), \text{ &c.}; & A_2 & = (100\overline{T}), \text{ &c.}; & A'_1 & = (100\overline{T}), \text{ &c.}; & A'_2 & = (100\overline{T}), \text{ &c.};
\end{align*}
\]

\[
\text{Plane } ADE = [0\overline{T}100], \text{ &c.}; \quad \text{Line } AD_1A' = [0\overline{T}1], \text{ &c.}; \quad (93)
\]
to which we may add these other symbols of planes and lines, each supposed to be followed by an “&c.”:

\[
\begin{align*}
\text{plane } BCD & = [100\overline{T}]; & BCE & = [100\overline{T}]; & \text{trace } BC & = [100]; \\
\text{plane } DB'B_1C'C_1 & = [\overline{T}110\overline{T}]; & EB'B_2C'C_2 & = [\overline{T}11\overline{T}]; & \text{trace } B'C'C' & = [\overline{T}11]; \\
\text{plane } AB_1C_2C_1B_2 & = [011\overline{T}T]; & \text{trace } A'A'' & = [011];
\end{align*}
\]

this line \( AA'' \) passing also, by (77), through the two points \( B_0 \) and \( C_0 \):

\[
\begin{align*}
\text{plane } B_1C_1D_1 & = [\overline{2}11\overline{1}1]; & B_2C_2D_1 & = [\overline{2}11\overline{1}1]; & \text{trace } D_1A'' & = [\overline{2}11]; \\
\text{plane } A'B_1B_2 & = [\overline{2}\overline{T}111]; & \text{trace } A'B_0 & = [\overline{2}\overline{T}1]; \\
\text{plane } A'C_1C_2 & = [\overline{2}\overline{T}1\overline{1}1]; & \text{trace } A'C_0 & = [\overline{2}\overline{T}1];
\end{align*}
\]

where it may be noticed that the symbol for \( A'C_1C_2 \), or for \( A'C_0 \), may be deduced from that for \( A'B_1B_2 \), or for \( A'B_0 \), by simply interchanging the second and third coefficients, or coordinates. It is easy to see that the quinary symbol for the plane \( ABC \) itself is on the same plan \([0001\overline{T}]\), the equation of that plane being \( w = v \); and it will be remembered that, by [18.], the ternary symbol for the point \( D_1 \) in that plane is \((111)\).
A right Line $\Lambda$ in Space may be regarded in two principal views, as follows. Ist, it may be considered as the locus of a variable point $P$, collinear with two given points $P_0, P_1$; and in this view, the symbol
\[ t_0(P_0) + t_1(P_1), \]
for the variable point upon the line, may be regarded as a Local Symbol (or Point-Symbol) of the Line $\Lambda$ itself. Thus
\[ (0 \cdot t t'), \text{ or } (0 \cdot y z), \]
may either represent an arbitrary point on the line $bc$; or, as a local symbol, that line itself. Or IInd, we may consider a line $\Lambda$ as a hinge, round which a plane turns, so as to be always collinear [7.] with two given planes $\Pi_0, \Pi_1$ through the line; and then a symbol of the form
\[ t_0[\Pi_0] + t_1[\Pi_1], \]
which represents immediately the variable plane $\Pi$, may be regarded as being also a Tangential Symbol (or Plane-Symbol) for the line $\Lambda$. For example the line $bc$ may be thus represented, not only by the local symbol (99), but also by the tangential symbol,
\[ [\sigma \cdot 00 \cdot t u], \text{ if } \sigma = t + u, \text{ and } \overline{\sigma} = -\sigma. \]
In fact, this last symbol can be derived, by linear combinations, from the symbols (94) for the two planes $bcd, bce$, which intersect in the line $bc$; and if any particular value be assigned to the ratio $t : u$, a particular plane through that line results. But it is time to apply these general principles to the Geometrical Net in Space.


The data of the Geometrical Net are, by [1.] the five points $A B C D E$, or $P_0$; of which the quinary symbols (27) have been assigned, and shown to be syntypical [9.]; and also the ternary symbols (92) of the three first of them. Of these the symbol
\[ \Lambda = (100) \]
may be taken as the type; and the point $\Lambda$ itself may be said to be a First Typical Point.

The derived lines $\Lambda_1$ of First Construction [1.], are the ten following,
\[ BC, \& c.; \quad DA, \& c.; \quad EA, \& c.; \quad \text{and DE;} \]
the “&c.” being interpreted as in [20.]; and each line $\Lambda_1$ connecting, by this construction, two points $P_0$. Among these the line $bc$ may be selected as a First Typical Line; and its symbols [21.], namely,
\[ (0 \cdot y z), \text{ and } [\sigma \cdot 00 \cdot t u], \]
whereof the former represents this line $bc$ considered as the locus of a variable point, while the latter represents the same line considered as the hinge of a variable plane, may be taken as types (the point-type and the plane-type) of the group of the ten lines $\Lambda_1$. 

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[24.] The derived planes $\Pi_1$ of first construction are in like manner ten; namely

$$\text{ADE, &c.; BCE, &c.; BCD, &c.; and ABC,}$$

each obtained by connecting three points $p_0$. Of these the last has, by [20.] the quinary symbol,

$$\text{ABC} = [0001\overline{1}]$$

which may be taken as a type of the group $\Pi_1$; and the plane ABC itself may be called a First Typical Plane. As a verification, we see that when we make $\sigma = t + u = 0$, in the second symbol [23.], and divide by $t$, we are led to the recent symbol for ABC, as one of the planes which pass through the line BC.

[25.] The derived points $p_1$, of the same first construction, which are all, by [1.], of the form $\Lambda_1 \cdot \Pi_1$, are in like manner ten; namely the intersections,

$$\text{BC \cdot ADE, &c.; DA \cdot BCE, &c.; EA \cdot BCD, &c.; and DE \cdot ABC,}$$

which have been denoted in [14.] and [15.] by the letters, or literal symbols,

$$A', \ &c.; \ A_2, \ &c.; \ A_1, \ &c.; \ \text{and} \ D_1,$$

and for which quinary symbols (49) (54) (60) (61) have been assigned. Of these ten points four, namely $A'$, $B'$, $C'$, $D_1$, are situated in the plane ABC, and have accordingly been represented [20.] by ternary symbols also: and we may take the particular symbol of this sort,

$$A' = (011),$$

as a type of this group $P_1$; understanding, however, that the full or quinary type is to be recovered from this ternary type, by restoring the two omitted zeros; so that we have, more fully,

$$A' = (01100) \equiv (10011).$$

And the point $A'$ itself may be considered as a Second Typical Point.

[26.] We have thus denoted, by literal and by quinary symbols, whereof some have been abridged to ternary ones [17.], and have been also represented by types [9.], not only the five given points $p_0$, but all the ten lines $\Lambda_1$, ten planes $\Pi_1$, and ten points $p_1$, of what has been called, in [1.], the First Construction. And it is evident that we have, at this stage, ten triangles $T_1$, namely the ten,

$$\text{ADE, &c.; BCE, &c.; BCD, &c.; and ABC,}$$

whereof each is contained in a plane $\Pi_1$; and also five pyramids $R_1$, each bounded by four of these triangles, namely, the pyramids,

$$\text{BCDE, CADE, ABDE, ABCD,}$$

which may be called the pyramids $A, B, C, D, E$; each marked by the literal symbol of that one of the five points $p_0$, which is not a corner of the pyramid.
[27.] It may be remarked, that ten arbitrary lines in space intersect, generally, ten arbitrary planes, in one hundred points; but that this number of intersections \( \Lambda_1 \cdot \Pi_1 \) is here reduced to fifteen, whereof only ten are new; because each of the five points \( p_0 \) counts as twelve, since in each of those points four lines cut (each) three planes, while each of the ten planes contains three lines; so that thirty binary combinations are not cases of intersection, and sixty such cases conduct only to the five old (or given) points. This sort of arithmetical verification of the accuracy of an enumeration of derived points, or lines, or planes, will be found useful in more complex cases, although it was not necessary here.

[28.] Proceeding to a Second Construction [1.], we may begin by determining the lines \( \Lambda_2 \), whereof each connects some two (at least) of the fifteen points \( p_0, p_1 \), but not any two of the five points \( p_0 \), since otherwise it would be a line \( \Lambda_1 \). If the fifteen points to be connected were independent, they would give, generally, by their binary combinations, 105 lines; but the ten collineations of construction,

\[
\text{BCA'}, \ &\&; \text{DAA}_2, \ &\&; \text{EAA}_1, \ &\&; \text{and EDD}_1,
\]

show that 30 of these combinations are to be rejected, as giving only the ten old lines. The remaining number, 75, is still farther reduced by the consideration that we have (comp. (79)) the fifteen derived collineations,

\[
\text{AA}'D_1, \ &\&; \text{AB}_1C_2, \ &\&; \text{AC}_1B_2, \ &\&; \text{DA}'A_1, \ &\&; \text{EA}'A_2, \ &\&;
\]

which represent only fifteen new lines, of a group which we shall denote by \( \Lambda_2,1 \), but count (comp. [27.]) as 45 binary combinations of the 15 points. There remain only 30 such combinations to be considered; and these give in fact a second group, \( \Lambda_2,2 \), consisting of thirty lines of second construction: namely the thirty edges of the five new pyramids \( R_2 \),

\[
\text{C}'B'A_2A_1, \ \text{A}'C'B_2B_1, \ \text{B'}A'C_2C_1, \ \text{A}_2B_2C_2D_1, \ \text{A}_1B_1C_1D_1,
\]

which are respectively inscribed in the five former pyramids \( R_1 \) [26.], and are homologous to them, the five given points \( A \ldots E \) being the respective centres of homology; for example, \( C' = AB \cdot CDE, \ &\&; \text{etc.} \) The corresponding planes of homology will present themselves somewhat later, in connexion with the points \( p_2 \).

[29.] On the whole, then, there are only forty-five distinct lines of second construction \( \Lambda_2 \); and these naturally divide themselves into two groups, of 15 lines \( \Lambda_2,1 \), and 30 lines \( \Lambda_2,2 \), as above. Each line of the first group \( \Lambda_2,1 \) connects one point \( p_0 \) with two points \( p_1 \); as each line \( \Lambda_1 \) had connected one point \( p_1 \) with two points \( p_0 \), but no line of the second group \( \Lambda_2,2 \) connects, at this stage of the construction, more than two points, which are both points \( p_1 \). Through no point \( p_0 \), therefore, can we draw any line \( \Lambda_2,2 \); but through each point \( p_0 \) we can draw three lines \( \Lambda_2,1 \); and each of these is determined as the intersection of two planes \( \Pi_1 \) through that point, or as crossing two opposite edges of that pyramid \( \Pi_1 \), which has not the point \( p_0 \) for a corner (comp. [26.]): for example, AA'D_1 is the intersection of ABC, ADE,
and crosses the lines BC, DE. And besides being, as in [28.], the edges of certain other and inscribed pyramids R₂, the 30 lines A₂,₂ are also the sides of ten new triangles T₂, namely,

\[ D_1A_1A_2, \text{ &c.}; C_1B_1A', \text{ &c.}; C_2B_2A', \text{ &c.}; \text{ and } A'B'C', \]

situated in the ten planes \( \Pi_1 \), and inscribed in the ten old triangles \( T_1 \), to which also they are homologous; the corresponding centres of homology being the ten points \( P_1 \), in the same order,

\[ A', \text{ &c.}; A_2, \text{ &c.}; A_1, \text{ &c.}; \text{ and } D_1, \text{ as before.} \]

The axes of homology of these ten pairs of triangles \( T_1, T_2 \), will offer themselves a little later, in connexion with points \( P_2 \).

[30.] All this may be considered as evident from geometry alone, at least with the assistance of literal symbols, such as those used above. But to deduce the same things by calculation, with quinary symbols and types, on the plan of the present Paper, we may observe that the symbolical equation,

\[ (10000) + (01100) + (00011) = (11111), \]

considered as a type of all equations of the same form, proves by (18) or (72) that each point \( P_1 \) can, in three different ways, be combined with another point \( P_1 \), so that their joining line shall pass through a point \( P_0 \); and that thus the group of the 15 lines \( L_2,1 \) arises, of which the line \( AA'D_1 \) is a specimen, and may be called a Second Typical Line (the first such line having been BC, by [23.]). The complete quinary symbol of a point on this line is \( (tuvv) \), which is however congruent to one of the form \( (tuu00) \), and may therefore be abridged to the ternary symbol \( (tuu) \), or \( (xyy) \); and the quinary symbol of a plane through the same line is of the form \( [0mrr] \) or \( [0ttu] \); we may therefore, by [21.] (comp. [23.]) consider the two expressions,

\[ (xyy), \text{ and } [0ttu], \]

as being not only local and tangential symbols for the particular (or typical) line \( AA'D_1 \) itself, but also local and tangential types for the group \( L_2,1 \); or as the point-type, and the plane-type, of that group.

[31.] The two points \( P_1 \), of which the quinary symbols have been thus combined in [30.], had no common coordinate different from zero; but there remains to be considered the case, in which two points of that group have such a coordinate: for example, when the points have for their symbols,

\[ (10100) \text{ and } (11000), \text{ or } (101) \text{ and } (110). \]

The point-symbol and plane-symbol of the line \( L_2 \) connecting these two points \( P_1 \) are easily seen to be (with the same significations of \( \sigma \) and \( \overline{\sigma} \) as before),

\[ (\sigma tu00), \text{ or } (\sigma tu), \text{ and } [tttu]; \]

but no choice of the arbitrary ratio, \( t : u \), with \( \sigma = t + u \), will reduce the symbol \( (\sigma tu) \) to denote any one of the 15 points \( P_0, P_1 \), except the two points \( P_1 \) (in this case B’ and C’), by
joining which the line is obtained; considering therefore the last two symbols as types, we see that they represent a second group, consisting of thirty lines \( \Lambda_2,2 \); but that there can be no third group, of lines \( \Lambda_2 \) of second construction. The particular line \( B'C' \), which the symbols in the present paragraph represent may be taken as typical of this second group; and may be called (comp. [23.] and [30.]) a Third Typical Line of the System, or Net, determined by the five given points \( A \ldots E \). And the pyramids \( R_1, R_2 \), and triangles \( T_1, T_2 \), of first and second constructions, of which the literal symbols have been assigned in [26.] [28.] [29.], might also have easily been suggested and studied, by quinary symbols and types alone.

[32.] As regards the Planes \( \Pi_2 \) of Second Construction [1.], it is easily seen that no such plane contains any two points \( P_0 \), or any one line \( \Lambda_1 \); for example, the first typical line \( B'C' \) [23.] contains the point \( A' \); and if we connect it with any one of the four points \( A, B', C', D_1 \), we only get a plane \( \Pi_1 \), namely \( ABC \); if with \( D, A_1, B_2 \), or \( C_2 \), we get another plane \( \Pi_1 \), namely \( BCD \); and if with any one of the four remaining points \( E, A_2, B_1, C_1 \), the plane \( BCE \) is obtained. Accordingly the general symbol \( \overline{\tau 0 t u \tau} \), in [23.], for a plane through the line \( BC \), gives \( \sigma = 0 \), or \( t = 0 \), or \( u = 0 \), when we seek to particularize it, by the first, the second, or the third of these three sets of conditions respectively.

[33.] But if we take the symbol \( \overline{0 t t u \tau} \), in [30.], for a plane through the second typical line \( AA'D_1 \), and seek to particularize this symbol by the condition of passing through some one of the eight points \( P_1 \) which are not situated upon it, we are conducted to the following results. The points \( B', C' \) give \( t = 0 \), and the points \( A_1, A_2 \) give \( u = 0 \); these points therefore give only two planes \( \Pi_1 \), namely the two planes \( ABC \) and \( ADE \), of which the line \( \Lambda_2 \) is the intersection. But the points \( B_1, C_2 \) give \( t = u \), and the points \( C_1, B_2 \) give \( t = -u \); these points therefore give two planes of a new group \( \Pi_2,1 \), namely (comp. [20.]) the two following:

\[
\text{plane } AA'D_1B_1C_2 = \overline{011T1T}; \quad \text{plane } AA'D_1C_1B_2 = \overline{01TT1};
\]

which are of the same type as the plane (96), namely,

\[
\text{plane } AB_1C_2C_1B_2 = \overline{011TT}.
\]

There are fifteen such planes \( \Pi_2,1 \), as the type sufficiently shows; each passes through one point \( P_0 \), and contains two lines \( \Lambda_2,1 \), containing also four lines \( \Lambda_2,2 \); as, for instance, the last-mentioned plane \( AB_1C_2C_1B_2 \), which we shall call (comp. [24.]) the Second Typical Plane, contains the two lines \( AB_1C_2, AC_1B_2 \) [28.], and the four lines \( B_1C_1, C_1C_2, C_2B_2, B_2B_1 \); that is to say, the two diagonals and the four sides of the quadrilateral \( B_1C_1C_2B_2 \), of which the plane \( \Pi_2,1 \) passes through \( A \).

[34.] We have now exhausted all the planes \( \Pi_2 \) which contain any point \( P_0 \); but there exists a second group of planes, \( \Pi_2,2 \), each of which is determined as connecting three points \( P_1 \), although passing through no point \( P_0 \). Thus if we take the third typical line \( B'C' \) [31.], and the symbol \( \overline{ttt u \tau} \) for a plane through it, we get indeed \( t = 0 \), or a plane \( \Pi_1 \), namely, \( ABC \), if we oblige the plane through \( B'C' \) to contain \( A \), or \( B \), or \( C \), or \( A' \), or \( D_1 \); and we get \( u = 0 \), or \( \overline{T110T} \), or a plane \( \Pi_2,1 \), namely \( DB'B_1C'C_1 \), as in (95), if we oblige it to contain...
D, or B₁, or C₁; while we get \( \sigma = 0 \), or \([T110]\), or \(EB'B₂C'C₂\), again as in (95), if we oblige it to contain E, or B₂, or C₂. But there remain the two points \( \Lambda₁ \) and \( \Lambda₂ \), determining the two new planes \( B'C'A₁ \) and \( B'C'A₂ \), for the former of which we have \( t + \sigma = 0 \), or \( u = -2t \), \( \sigma = -t \), and therefore have the symbol \([T112]\); while for the latter we have \( u = t \), \( \sigma = 2t \), and therefore the syntypical symbol \([T111]\). There are twenty planes of this group \( \Pi₂₂ \), as may be at once concluded from inspection of the type; among which (comp. [19.]) we shall select the following,

plane \( \Lambda₁B₁C₁ = [1112] \),

and call this a Third Typical Plane. And it is evident that these 20 planes \( \Pi₂₂ \) are the twenty faces of the five inscribed pyramids \( R₂ \) [28.], of which the edges have been seen to be the thirty lines \( \Lambda₂₂ \). On the whole, then, there are only thirty-five planes \( \Pi₂ \) of second construction; which thus divide themselves into two groups, of fifteen and twenty, respectively.

[35.] To verify arithmetically (comp. [27.], [28.]) the completeness of the foregoing enumeration of the planes \( \Pi₂ \), we may proceed as follows. In general fifteen independent points would determine 455 planes, by their ternary combinations; but the 25 collineations [28.], which give only the lines \( \Lambda₁, \Lambda₂₁ \) account for 25 such combinations, leaving only 430 to be accounted for, by so many triangles. Now each plane \( \Pi₁ \) contains three points \( p₀ \), and four points \( p₁ \), connected by six collineations; it contains therefore 29 (= 35 − 6) triangles, and thus the ten planes \( \Pi₁ \) account for 290 triangles, leaving only 140, situated in planes \( \Pi₂ \). But each of the 15 planes \( \Pi₂₁ \) contains one point \( p₀ \), and four points \( p₁ \), connected by two collineations; it contains therefore 8 (= 10 − 2) triangles, and thus 120 are accounted for, leaving only 20 ternary combinations to be represented, by triangles in other planes \( \Pi₂ \). And these accordingly have presented themselves, as the twenty faces \( \Pi₂₂ \) of the five inscribed pyramids \( R₂ \). It must be mentioned, that the enumeration and classification of the foregoing lines and planes had been completely performed by MÖBIUS, although with an entirely different notation and analysis.

[36.] It is much more difficult, however, or at least without the aid of types it would be so, to enumerate and classify what we have called in [1.] the Points \( P₂ \) of Second Construction; and to assign their chief geometrical relations, to each other, and to the five given and ten (formerly) derived points, \( p₀ \) and \( p₁ \). In fact, it is obvious that these new points \( P₂ \), being (by their definition) all the intersections of lines \( \Lambda₁ \) or \( \Lambda₂ \) with planes \( \Pi₁ \) or \( \Pi₂ \), which have not already occurred, as points \( p₀ \) or \( p₁ \), may be expected to be (comp. [2.]) considerably more numerous, than either the lines or the planes themselves.

[37.] The total number of derived lines and planes, so far, is exactly one hundred; namely 55 lines \( \Lambda \), and 45 planes \( \Pi \), of first and second constructions. Their binary combinations, of the form \( \Lambda\Pi \), are therefore 2475 in number; but as it is not difficult to prove that there are 240 distinct cases of coincidence of line with plane (or of a plane containing a line), we must subtract this from the former number, and thus there remain only 2235 cases of intersection, of the kind which we have proposed to consider. Every one, however, of these 2235 cases, must be accounted for, either as a given point \( p₀ \), or as a derived point \( p₁ \) of first construction, or finally as one of those new points \( P₂ \), of which we have proposed to

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accomplish the enumeration, and to determine the natural groups, as represented by their respective types.

[38.] We saw, in [27.], that each point $P_0$, as for instance the point $\Lambda$, represents twelve intersections of the form $\Lambda_1 \cdot \Pi_1$; and it is easy to prove that the same point $P_0$ represents twelve other intersections of the form $\Lambda_1 \cdot \Pi_2,1$; twelve of the form $\Lambda_2,1 \cdot \Pi_1$; and three of the form $\Lambda_1 \cdot \Pi_2,1$; but none of any other form. It represents therefore, in the whole, a system of 39 intersections, included in the general form $\Lambda \cdot \Pi$; and we must, for this reason, subtract 195 ($= 5 \times 39$) from 2235, leaving 2040 other cases of intersection of line with plane, to be accounted for by the old and new derived points, $P_1$ and $P_2$.

[39.] An analysis of the same kind shows, that each of the ten points of first construction, as for example the typical point $\Lambda'$ [25.], represents one intersection of the form $\Lambda_1 \cdot \Pi_1$; six of the form, $\Lambda_1 \cdot \Pi_2,1$; six of the form $\Lambda_1 \cdot \Pi_2,2$; six of the form $\Lambda_2,1 \cdot \Pi_1$; twelve of the form $\Lambda_2,1 \cdot \Pi_2,1$; eighteen of the form $\Lambda_2,1 \cdot \Pi_2,2$: eighteen of the form $\Lambda_2,2 \cdot \Pi_1$; twenty-four, of the form $\Lambda_2,2 \cdot \Pi_2,1$; and twenty-four others, of the remaining form $\Lambda_2,2 \cdot \Pi_2,2$. It represents, therefore, in all, 115 intersections $\Lambda \cdot \Pi$; and there remain only 890 ($= 2040 - 1150$) cases of intersection to be accounted for, or represented, by the points $P_2$ of which we are in search. But all these 890 cases of intersection must be accounted for by such new points, if the investigation is to be considered as complete.

[40.] A first, but important, and well-known group of such points $P_2$, consists of the ten points (already considered in Part I. of this Paper),

$$\Lambda''', \&c.; \quad \Lambda'_2, \&c.; \quad \Lambda'_1, \&c.; \quad \text{and} \quad D'_1,$$

namely, the harmonic conjugates of the ten points $P_1$, with respect to the ten lines $\Lambda_1$, which we shall call collectively the points, or the group, $P_{2,1}$; and among which we shall select the point

$$\Lambda'' = (01 \Gamma),$$

as a Third Typical Point of the Net. In fact, it is what we have called a point $P_2$, because, without belonging to either of the two former groups $P_0 P_1$, it is an intersection $\Lambda_1 \cdot \Pi_{2,2}$; or rather, it represents six such intersections, of the line $BC$ with planes of second construction, and of the second group: namely, with two such through $B'C'$, two through $B_2C_2$, and two through $B_1C_1$, being pairs of faces [28.] of three pyramids $R_2$, inscribed in those three pyramids $R_1$, which have been distinguished, in [26.] by the letters $A, D, E$. The same point $\Lambda''$ is also the intersection of the same line $BC$ with three planes $\Pi_{2,1}$; namely, with the three which connect, two by two, the three lines $B'C'$, $B_2C_2$, $B_1C_1$, and contain the three points $A, D, E$. It is also, in six ways, the intersection of one or other of these three last lines $\Lambda_{2,2}$ with a plane $\Pi_1$; in three ways, with a plane $\Pi_{2,1}$; and in twelve ways, with a plane $\Pi_{2,2}$; so that a single point $P_{2,1}$ represents thirty intersections of the form $\Lambda \cdot \Pi$; and the group of the ten such points represents 300 such intersections. We have therefore only to account for 590 ($= 890 - 300$) intersections $\Lambda \cdot \Pi$, by other groups $P_{2,2}, \&c.,$ of points of second construction.

[41.] A second group, $P_{2,2}$ of such points $P_2$ has already presented itself, in the case of the traces $A_0, B_0, C_0$ [18.] of the lines $\Lambda_1A_2, B_1B_2, C_1C_2$, on the plane $ABC$. The ternary
symbol of the point $a_0$ has been found (77) (92) to be $(1\,1\,1)$, its quinary symbol is therefore $(\overline{1}\,1\,1\,0\,0)$, which is congruent (10) with $(2\,0\,0\,1\,1)$; hence in the full, or quinary sense [9.], this point $a_0$ is syntypical with the following other point, in the same plane $ABC$,

$$a''' = (2\,1\,1),$$

which we shall call a Fourth Typical Point, and shall consider as representing the group $p_{2,2}$; this group consisting of thirty such points, namely of two on each of the 15 lines $\Lambda_{2,1}$.

[42.] Each of these thirty points $p_{2,2}$ represents seven intersections of line with plane; namely two of each of the three forms $\Lambda_{2,1} \cdot \Pi_{2,1}$, $\Lambda_{2,1} \cdot \Pi_{2,2}$, $\Lambda_{2,2} \cdot \Pi_{2,1}$, and one of the form $\Lambda_{2,2} \cdot \Pi_{1}$. For example, the typical point $a'''$, which is the intersection of the two lines $A\Lambda'D_1$ and $B'C'$, is at the same time the intersection of the former line $\Lambda_{2,1}$ with each of the four planes $\Pi_2$ which contain the latter line $\Lambda_{2,2}$; being also the intersection of this last line $B'C'$ with a plane $\Pi_1$, namely $ADE$, and with two planes $\Pi_{2,1}$ which contain the first line $A\Lambda'D_1$. The group $p_{2,2}$ represents therefore 210 intersections $\Lambda \cdot \Pi$; and there remain only 380 ($= 590 - 210$) intersections of this standard form, to be accounted for by other groups of second construction, such as $p_{2,3}$ &c.

[43.] In investigating such groups, we need only seek for typical points; and because every such point is on a line of one of the three forms $\Lambda_1$, $\Lambda_{2,1}$, $\Lambda_{2,2}$, we may confine ourselves to the three typical lines,

$$BC, \quad AA'D_1, \quad B'C'; \quad \text{or} \quad (0\,tu), \quad (tu\,u), \quad (\sigma\,t\,u);$$

in which, as before, $\sigma = t + u$, and in which the ratio of $t$ to $u$ is to be determined. And because a line in the plane $ABC$ intersects any other plane in the point in which it intersects the line which is the trace of the latter plane upon the former, we need only, for the present purpose, consider these lines, or traces: whereof there are, by what has been already seen, seven distinct ternary types, namely the following:

$$[1\,0\,0], \quad [0\,1\bar{T}], \quad [\bar{T}\,1\,1], \quad [1\,1\,1], \quad [0\,1\,1], \quad [\bar{D}\,1\,1], \quad [\bar{D}\,1\bar{T}];$$

which answer to the seven typical traces of planes, $BC, \quad AA'D_1, \quad B'C', \quad A''B''C'', \quad AA'', \quad D_1A'', \quad A'C_0$.

There are 22 ($= 3 + 3 + 3 + 1 + 3 + 3 + 6$) such lines, answering to 44 ($= 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 1 \cdot 2 + 3 \cdot 1 + 3 \cdot 2 + 6 \cdot 1$) planes; namely to all the 45 planes $\Pi_1$, $\Pi_2$, except the particular plane $ABC$, on which the traces are thus taken. And we have now to combine these seven types of lines, with the three symbols of points, $(0\,tu)$, $(tu\,u)$, $(\sigma\,t\,u)$, according to the general law $lx + my + nz = 0$ (76).
[44.] The line $BC$ is itself one of the three traces of the first type; and it intersects the twelve other traces, of the five first types, only in points which have been already considered. The line $AA'D_1$ is, in like manner, a trace of the second type; and it gives no new point, by its intersections with the eight other traces, of the three first types; but its intersection with the common trace $A''B''C''$, of the two planes $A_1B_1C_1$ and $A_2B_2C_2$ [19.], which is the only line of the fourth type, gives what we shall call a Fifth Typical Point, namely the following:

$$A^{IV} = (211);$$
or more fully, $$A^{IV} = (21100) \equiv (30011).$$

This last quinary symbol shows that the point $A^{IV}$ is syntypical with this other point in the plane $ABC$,

$$A^{IV}_1 = (31100) = (311);$$

so that this plane contains six points $p_{2,3}$, which (in the quinary sense) belong to one common group, although their two ternary types are different. In fact, the point $A^{IV}_1$ is the common intersection of the line $AA'D_1$ with the two planes $[12T11]$ and $[1T211]$, or $B'C_1C_2$ and $C'B_1B_2$, as the point $A^{IV}$ is the common intersection of the same line with the two planes $[1112T]$ $[1T12T]$, or $A_1B_1C_1$ and $A_2B_2C_2$, as above.

[45.] There are thirty distinct points $p_{2,3}$, of this third group of second construction; and each represents two (but only two) intersections, which are both of the form $\Lambda_2 \cdot \Pi_2$. The group therefore represents a system of 60 intersections $\Lambda \cdot \Pi$; and there remain only 320 ($= 380 - 60$) such intersections to be accounted for by other points, or groups, such as $p_{2,4}$, &c. It will be found that we have now exhausted all the points, or groups, of second construction, which are situated on lines $\Lambda_2$; but that two other groups of points $p_2$ may be determined on lines $\Lambda_1$, by combining the typical line $BC$ with the two last sets of traces [43.] as follows.

[46.] Combining thus $BC$ with $D_1C''$ and $D_1B''$, or with the traces $[112]$ and $[121]$, we get the two following points, of a fourth group of second construction,

$$A^V = (021); \quad A^V_1 = (012);$$

whereof the former may be taken as a Sixth Typical Point. There are twenty points of this group $p_{2,4}$, whereof each represents three intersections, of the form $\Lambda_1 \cdot \Pi_2$; for example, the typical point $A^V$ is the common intersection of the line $BC$ with the three planes $c'A_1A_2$, $D_1A_1B_1$, $D_1A_2B_2$; the group therefore represents sixty intersections $\Lambda \cdot \Pi$, and there remain 260 ($= 320 - 60$) to be accounted for.

[47.] Again, combining $BC$ with $C'B_0$, and with $B'C_0$, or with $[1T2]$ and $[12T]$, we get the two following other points, belonging to a fifth group of second construction,

$$A^{V1} = (02T); \quad A^{V1}_1 = (0T2);$$

whereof the first may be said to be a Seventh Typical Point. There are twenty points of this new group $p_{2,5}$, whereof each represents only one intersection, of the form $\Lambda_1 \cdot \Pi_2$; for example, $A^{V1} = BC \cdot C'B_1B_2$. We are therefore to subtract 20 from the recent number 260; and thus there remain still 240 intersections to be accounted for, by new points $p_2$ upon the lines $\Lambda_2$; since the lines $\Lambda_1$ as well as $\Lambda_2$ have been exhausted, as on examination will easily appear.

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[48.] The line $b'c'$ intersects the traces $BB''$ and $CC''$ of the fifth type [43.] in the two following points, of a sixth group of second construction,

$$A^{\text{VII}} = (12\overline{T}); \quad A_1^{\text{VII}} = (1\overline{T}2);$$

whereof the former may be called an Eighth Typical Point. There are sixty points of this new group, $p_{2,6}$, whereof each represents one intersection, of the form $A_{2,2} \cdot \Pi_{2,1}$; for example $A^{\text{VII}}$ is the intersection of the line $b'c'$ with the plane $BC_1A_2A_1C_2$; there remain therefore 180 ($= 240 - 60$) intersections $\Lambda \cdot \Pi$ to be still accounted for, by other points $p_2$, on the same set of lines $\Lambda_{2,2}$.

[49.] The traces $D_1B''$, $D_1C''$, which belong to the sixth type [43.] intersect the line $b'c'$ in two new points, namely

$$A^{\text{VIII}} = (321); \quad A_1^{\text{VIII}} = (312);$$

which belong to a seventh group $p_{2,7}$, of second construction, and of which the former may be regarded as a Ninth Typical Point. There are sixty points of this group, namely two on each of the 30 lines $\Lambda_{2,2}$; and each is the intersection of one such line with two distinct planes $\Pi_{2,2}$; their group therefore represents a system of 120 such intersections; and only 60 ($= 180 - 120$) intersections remain to be accounted for, by other points of this last form, $\Lambda_{2,2} \cdot \Pi_{2,2}$.

[50.] Accordingly, when we combine the line $b'c'$ with the traces $A'C_0$, $A'B_0$, which are of the seventh type [43.], we obtain, for the intersections of that line $\Lambda_{2,2}$ with two new planes $\Pi_{2,2}$, namely with $A'C_1C_2$ and $A'B_1B_2$ (98), two new points, belonging to a new or eighth group $p_{2,8}$ of second construction, namely,

$$A^{\text{IX}} = (23\overline{T}); \quad A_1^{\text{IX}} = (2\overline{T}3);$$

whereof the former may be selected, as a Tenth (and, for our purpose, last) Typical Point: for the sixty points of this last group represent each one intersection, and thus account for all the intersections which lately remained [49.], after all the preceding groups had been exhausted.

[51.] We are now therefore enabled to assert that the proposed Enumeration of the Points $p_2$ of Second Construction, and the proposed Classification of such Points in Groups, have both been completely effected. For the number of such groups $p_{2,1}, \ldots, p_{2,8}$ has been seen to be eight, represented by the 8 typical points, $A'' \ldots A^{\text{IX}}$; which, along with the first given point $A$, and the first derived point $A'$, make up a system of ten types, as follows:

$$A = (100); \quad A' = (011); \quad A'' = (01\overline{T}); \quad A''' = (211); \quad A^{\text{IV}} = (\overline{2}11);$$

$$A^\text{V} = (021); \quad A^{\text{VI}} = (02\overline{T}); \quad A^{\text{VII}} = (12\overline{T}); \quad A^{\text{VIII}} = (321); \quad A^{\text{IX}} = (23\overline{T});$$

and the number of the points $p_2$ is $(10 + 30 + 30 + 20 + 20 + 60 + 60 + 60 =) 290$; so that, when combined with the points $p_1$, they make up a system of exactly three hundred points, $p_1, p_2$, derived from the five points $p_0$. 22
[52.] It is to be remembered that the three other ternary types,
\[ D_1 = (111), \quad A_0 = (\overline{111}), \quad A_{IV} = (311), \]
have been seen to represent points which are, in the quinary sense, syntypical with \( A', A''', A_{IV} \), and therefore belong to the same three groups, \( p_1, p_{2,2}, p_{2,3} \); all these three points being in the plane \( ABC \), and on the line \( AA'D_1 \). And it is evident that the five other points,
\[ A_1^V = (012); \quad A_1^{VI} = (0\overline{1}2); \quad A_1^{VII} = (1\overline{1}2); \quad A_1^{VIII} = (1312); \quad A_1^{IX} = (2\overline{1}3), \]
belong (as has been seen) to the same five last groups of second construction, as the five points above selected as typical thereof, namely the points \( A^{V} \ldots A^{IX} \), and are situated on the same two typical lines \( B'C' \) and \( B'C'' \). The transition from \( A' \) to \( B', C' \), or from \( A'' \) to \( B'', C'' \), &c., is very easily made, by a rule already stated [20.]; and therefore it is unnecessary to write down here the symbols for these derived points, \( B', B'', &c. \), or \( C', C'', &c. \). But we must now proceed, in the remainder of this Paper, to investigate some of the chief geometrical relations which connect the points, lines, and planes of the \( Net \), so far as they have been hitherto determined: namely to the end of the Second Construction.

PART III.—Applications to the \( Net \), continued: Enumeration and Classification of the Collinearities of the Fifty-Two Points in a Plane of First Construction.

[53.] The plane \( ABC \) has been seen to contain, besides the three points \( P_0 \) which determine it, four points \( p_1 \), namely \( A', B', C' \), and \( D_1 \); and it contains forty-five points \( p_2 \), namely the three points \( A'', B'', C'' \) of the group \( p_{2,1} \), and six points of each of the seven remaining groups of second construction. This plane \( P_1 \) contains therefore fifty-two points \( p_0, p_1, p_2 \); and we propose to examine, in the first place, the various relations of collinearity which connect these different points among themselves: intending afterwards to investigate their principal harmonic and involutionary relations.

[54.] The points on the first typical line \( BC \) [23.] are, in number, eight; their literal symbols being, by what precedes,
\[ B, \ C, \ A', \ A'', \ A^V, \ A_1^V, \ A_{IV}, \ A_1^{VI}; \]
the ternary symbols corresponding to which have been shown to be,
\[ (010), \ (001), \ (011), \ (01\overline{1}), \ (021), \ (012), \ (02\overline{1}), \ (0\overline{1}2). \]
In fact, that these eight points are all on the line \( BC \), is evident on mere inspection of their symbols, which are of the common form,
\[ (0\ y\ z) \quad [23]. \]

[55.] The points on the second typical line, \( AA' \) [30.], are in number seven: their literal symbols being,
\[ A, \ A', \ D_1, \ A''', \ A_0, \ A_{IV}, \ A_1^{IV}; \]
and their ternary symbols being,
\[ (100), \ (011), \ (111), \ (211), \ (\overline{1}11), \ (\overline{2}11), \ (311). \]
In fact, each of these seven symbols is evidently of the form \( (t\ u\ u) \), or \( (x\ y\ y) \) [30.].
The points on the third typical line, $b'c'$ [31.], are in number ten; namely the points,
$$B', C', A'', A'''', A^{VII}, A'^{VII}, A^{VIII}, A'^{VIII}, A^IX, A'^IX,$$
of which the ternary symbols are,
$$(101), (110), (01\overline{1}), (211), (12\overline{1}), (1\overline{1}2), (321), (312), (23\overline{1}), (2\overline{1}3);$$
each of these ten symbols being of the form $(\sigma tu)$ [31.], with $\sigma = t + u$, as before.

These three typical lines, in the plane $abc$, which may be denoted by the ternary symbols, $[100], [01\overline{1}], [\overline{1}11]$, and represent a system of nine lines $\Lambda_1, \Lambda_2$ in that plane $\Pi_1$, are also three typical traces [43.] of other planes thereon; and the remaining traces of such planes are in number thirteen, represented by four other lines, as types: of which lines, considered as such traces, the ternary symbols have been found [43.] to be,
$$[111], [011], [\overline{2}11], [\overline{3}1\overline{1}];$$
answering to the literal symbols,
$$A''B''C'', \ AA'', \ D_1A'', \ A'C_0,$$
and serving as abridged expressions for the four equations of ternary form,
$$x + y + z = 0, \ y + z = 0, \ 2x = y + z, \ 2x = y - z.$$

Each of these four last lines passes through six points; thus the trace $[111]$ passes through the points $(01\overline{1}) (10\overline{1}) (1\overline{1}1) (1\overline{2}1) (11\overline{2})$, or through $A''B''C''A^{IV}B^{IV}C^{IV}$; $[011]$ through $(100) (01\overline{1}) (11\overline{1}) (1\overline{1}1) (2\overline{1}1) (21\overline{1})$, or $A A''B_0C_0C^{VII}B^{VII}$; $[\overline{2}11]$ through $(111) (01\overline{1}) (102) (120) (213) (231)$, or $D_1A''B^{V}C^{VIII}B^{VIII}$; and $[\overline{3}1\overline{1}]$ through $(011) (11\overline{1}) (131) (120) (\overline{1}02) (23\overline{1})$, or $A'C_0B^{IV}_1C^{V}_1B^{VI}A^{IX}$; the correctness of the ternary symbols being evident on inspection, if the law $lx + my + nz = 0$ (76) be remembered: and the literal symbols being thence at once deduced, by [51.] and [52.].

So far, then, that is when we attend only to the twenty-two traces [43.] of planes $\Pi_1, \Pi_2$ on the plane $abc$, we find a system of three collineations of eight points; three of seven points; three of ten points; and thirteen of six points each. Each collineation of the first of these four systems counts as 28 binary combinations of the 52 points in the plane [53.]; each of the second system counts as 21 such combinations; each of the third system as 45; and each of the fourth as 15. We therefore account, in this way, for $84 + 63 + 135 + 195 = 477$ binary combinations; but the total number is $26 \cdot 51 = 1326$; there remain then 849 to be accounted for, by lines $\Lambda_3$ which are not traces, of any one of the foregoing groups.

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In seeking for such new lines, it is natural to consider first those which pass through one or other of the three given points \(A, B, C\); and the types of such are found to be the five following, each representing a new group of six lines \(\Lambda_3\):

\[
[021]; [021]; [031]; [032]; [031].
\]

As symbols, these answer respectively to the five new lines:

\[
\begin{align*}
& (100) (112) (012) (111) (311), & \text{or } A C^{IV} A_1^{VI} A_1^{VII} C^{IX}; \\
& (100) (113) (213), & \text{or } A C'' A_1^{V} B^{VII} A_1^{VIII}; \\
& (100) (123) (123), & \text{or } A C^{IV} C^{VII}; \\
& (100) (213), & \text{or } A C^{VIII} B^{IX}; \\
& (100) (112) (111) (111) (111), & \text{or } A A_1^{IX}. \\
\end{align*}
\]

We have thus twelve lines \(\Lambda_3\), each connecting a point \(P_0\), with four points \(P_2\), and counting as ten binary combinations; twelve other lines, each connecting a point \(P_0\) with two points \(P_2\), and counting as three such combinations; and six lines, each of which connects a point \(P_0\) with one point \(P_2\), and counts as only one combination. In this manner, then, we account for \(120 + 36 + 6 = 162\), out of the 849 which had remained in [59.]; but there still remain 687 combinations to be accounted for, by new lines of third construction, which pass through no given point.

Considering next the new lines which connect a point of the first construction, with one or more points of the second, we find these five new types,

\[
[311]; [122]; [123]; [133]; [134];
\]

which as symbols denote the five lines,

\[
\begin{align*}
& (011) (121) (112); (011) (201) (210); (111) (210) (121); \\
& (011) (312); (111) (132); \\
\end{align*}
\]

\[
\text{or } A' B^{IV} A_1^{VII}; A' B_1^{V} C^{VI}; D_1 C^{VI} C_1^{VII}; A' A_1^{VIII}; \text{ and } D_1 C_1^{IX};
\]

but as types represent each a group of six lines. We thus get 18 new lines, each passing through one point \(P_1\), and 2 points \(P_2\); and 12 other lines, each connecting a point \(P_1\) with only one point \(P_2\). And these thirty lines \(\Lambda_3\) account for \(54 + 12 = 66\) binary combinations of points; leaving however 621 such combinations to be accounted for, by new lines \(\Lambda_3\), of which each must connect at least two points \(P_2\), without passing through any point \(P_0\) or \(P_1\), and without being any one of the traces already considered.

The symbol \([233]\), which denotes a line passing through two points \(P_2\), namely, \((011)\) and \((311)\), or \(A''\) and \(A_1^{IV}\), but through no other point, represents, when considered as a type, a group of three such lines; and 40 other types, as for example \([134]\), which as a symbol denotes the line \((111) (132)\), or \(A_0 B^{VIII}\), are found to exist, representing each a group of six lines, whereof each connects in like manner two points \(P_2\), but only those two points. We have thus a system of 243 new lines, which represent only so many binary combinations: and there remain 378 such combinations to be accounted for, by new lines \(\Lambda_3\), whereof each must connect at least three points \(P_2\).

25
[63.] For lines connecting three such points, and no more, it is found that there are twenty types; whereof eight, as for instance the type [311], which as a symbol denotes the line (011) (121) (112), or \( A''B''C'' \), represent each a group of three such lines; while each of the twelve others, like [123], which as a symbol denotes the line (111) (121) (210), or \( A_0B''C'' \), represents a group of six lines. We have thus 96 new lines, counting as 288 binary combinations: but we must still account for 90 other combinations, by new lines \( \Lambda_3 \), connecting each more than three points \( p_2 \).

[64.] Accordingly, we find three new types of lines, which alone remain, when all those which have been above exhibited, or alluded* to are set aside: namely

\[
[124]; [\overline{1}24]; [112].
\]

And those represent, respectively, groups of six, of six, and of three new lines, and therefore on the whole a system of fifteen new lines, each passing through four points \( p_2 \), and consequently counting as six combinations; for example, as symbols, they denote the three following lines:

\[
(210) (\overline{2}11) (021) (231), \quad \text{or } c^V A^{IV} A^V B_1^{VIII};
\]
\[
(210) (2\overline{1}1) (02\overline{1}) (23\overline{1}), \quad \text{or } c^V C^{VII} A^{VI} A^IX;
\]
\[
(20\overline{1}) (1\overline{1}0) (02\overline{1}) (11\overline{1}), \quad \text{or } B_1^{VI} C'' A^V C_0.
\]

But \( 6 \cdot 15 = 90 \); we are therefore entitled to say, that all the 1326 binary combinations [59.], of the 52 points \( p_0, p_1, p_2 \) [53.] in the plane ABC, have now been fully accounted for.

[65.] Collecting the results, respecting the collineations in the plane ABC, it has been found that there are 261 lines \( \Lambda_3 \), whereof each connects two, but only two, of the the 52 points in that plane; and that these lines, which at the present stage of the construction are not properly cases of collinearity at all, are represented by a system of 44 ternary types.

[66.] There are 126 other lines \( \Lambda_3 \), each connecting three (but only three) points; they are represented by a system of 25 types; and account for 378 binary combinations.

[67.] There are 15 lines \( \Lambda_3 \), each connecting four points \( p_2 \); they are represented by a system of 3 types, and account for 90 combinations.

[68.] There are 12 lines \( \Lambda_3 \), each connecting one point \( p_0 \) with four points \( p_2 \); they are represented by 2 types, and represent 120 combinations.

[69.] There are 13 other lines \( \Lambda_3 \), namely the traces of planes \( \Pi_1 \) or \( \Pi_2 \), whereof each connects six points, namely a point \( p_0 \) or \( p_1 \) with five points \( p_2 \), or else six points \( p_2 \) with each other; they are represented by 4 types, and account for 195 combinations.

* It has been thought that it could not be interesting to set down all the types of lines, above referred to; especially as those which relate to lines not passing through at least four points give rise, at the present stage of the construction, to no theorems of harmonic (or anharmonic) ratio.
[70.] There are 3 lines $\Lambda_{2,2}$, each connecting two points $p_1$ with eight points $p_2$; they have one common type, and represent 135 combinations.

[71.] There are, in like manner, 3 lines $\Lambda_{2,1}$, each connecting one point $p_0$ with two points $p_1$, and with four points $p_2$, but having only one common type; and they represent 63 combinations.

[72.] Finally, there are (in the same plane) 3 lines $\Lambda_1$, each connecting two points $p_0$ with one point $p_1$, and with four points $p_2$, but having only one common type; and they account for 84 combinations: with the arithmetical verification, that

$$261 + 378 + 90 + 195 + 135 + 63 + 84 = 1326 = 26.51;$$

which proves that the enumeration is complete.

[73.] The total number of distinct lines, above obtained, is $261 + 126 + 15 + 12 + 13 + 3 + 3 + 3 = 436$; and the total number of their ternary types is 81. But if we set aside (as conducting to no general metric relations) all those lines which contain fewer than four points, there then remain only forty-nine lines, and only twelve types, to be discussed, with reference to harmonic (or anharmonic) relations, of the points upon those lines.

[74.] For the purpose of studying completely all such relations, it will therefore be permitted to confine ourselves to the three first typical lines, $bc$, $aa'$, $b'c'$, or $[100]$, $[01\overline{1}]$, $[\overline{1}11]$; the four other typical traces, $A''B''C''$, $AA''$, $D_1A''$, $A'C_0$, or $[111]$, $[01\overline{1}]$, $[\overline{1}11]$, $[\overline{1}\overline{1}]$; and five new typical lines $\Lambda_3$, connecting each at least four points: namely the two lines, $[02\overline{1}]$ and $[02\overline{1}]$, of $[60.]$, whereof each connects the given point $A$ with four points $p_2$; and the three lines $[1\overline{2}4]$, $[\overline{1}24]$, $[112]$, of $[64.]$, of which each connects four other points $p_2$ among themselves, but does not pass through any point $p_0$ or $p_1$.

**Part IV.**—Applications to the Net, continued: Harmonic and Involutionary Relations, of the Points situated on the Twelve Typical Lines, in a Plane of First Construction.

[75.] Commencing here with the examination of the last typical lines, because they contain only four points each, let us adopt, as temporary symbols, of the literal kind, the ten following:

$$a = (210), \quad b = (\overline{2}11), \quad c = (021), \quad d = (231);$$
$$b' = (2\overline{1}1), \quad c' = (02\overline{1}), \quad d' = (23\overline{1});$$
$$a'' = (20\overline{1}), \quad b'' = (1\overline{1}0), \quad d'' = (11\overline{1});$$

instead of the more systematic but less simple symbols, $c^V A^{IV} A^V B_1^{VII} C^{VII} A^{VI} A^{IX} B_1^{VI} C^{VII} C_0$.

[76.] The three lines referred to [64.], are then the following:

$$abcd; \quad ab'c'd'; \quad a''b''c'd''.$$
And because we have (comp. [16.]) the six symbolical relations,

\[(c) - (a) = (b); \quad (c) + (a) = (d);\]
\[(a) - (c') = (b'); \quad (a) + (c') = (d');\]
\[(a'') - (c') = 2(b''); \quad (a'') + (c') = 2(d''),\]

it results (31) that the three harmonic equations exist:

\[(a b c d) = (a b' c' d') = (a'' b'' c' d'') = -1.\]

We have therefore this Theorem:

“Each of the 150 lines \( \Lambda_3 \), which connect four points \( P_2 \), in any one of the ten planes \( \Pi_1 \), and pass through no other of the 305 points \( P_0, P_1, P_2 \), is harmonically divided.”

[77.] As verifications, the three right lines \( b b', c c', d d' \) concur in the point \( c; b d', c c', d b' \), in \( b; a a'', b b'', d d'' \), in \( \Lambda; c \); and \( a a'', b d'', d b'' \), in a point \( P_3 \), namely in \((41\overline{1})\): the existence of which four concurrences of lines was to be expected, from a known principle of homography, as consequences of the harmonic relations [76.]. It is worth noticing, however, how simply these concurrences are here expressed, by the ternary symbols of the points, according to the law (18); or, if we choose, by the corresponding symbols of the lines, with the analogous law (25): for example, the three last concurrent lines, \( a a'', c c'', d d'' \), in \( a_0 \); and \( a a', c c', d d' \), in an \( a_3 \), name \( y \) (411): the existence of which four concurrences of lines was to be expected, from a known principle of homography, as consequences of the harmonic relations [76.].

[78.] To examine, in like manner, the analogous relations of arrangement, on the two new typical lines [60.], namely \( [011] \) and \( [02\overline{1}] \), whereof each connects the given point \( \Lambda \) with four points of second construction, let us write as eight new temporary symbols of the literal kind, more convenient than the former symbols, \( c^{IV} A^{VI} A^{VII} c^{IX} b^{VII} b^{V} A^{V} c^{000} a^{VIII} \), the following:

\[b = (11\overline{2}), \quad c = (01\overline{2}), \quad d = (1\overline{1}\overline{2});\]
\[\beta = (\overline{1}12), \quad \gamma = (012), \quad \delta = (112), \quad \epsilon = (312),\]

so that the two lines in question are,

\[\Lambda a b c d e, \quad \Lambda \beta \gamma \delta \epsilon.\]

We have thus the eight following new symbolical relations, \( \Lambda \) being still \( = (100) \):

\[(\Lambda) - (c) = (b), \quad (\Lambda) + (c) = (d); \quad (e) - (b) = 2(d), \quad (e) + (b) = 4(\Lambda);\]
\[(\gamma) - (\Lambda) = (\beta), \quad (\gamma) + (\Lambda) = (\delta); \quad (e) + (\beta) = 2(\delta), \quad (e) - (\beta) = 4(\Lambda);\]

whence result at once the four harmonic relations,

\[(\Lambda a b c d e) = (\Lambda b d e) = (\Lambda \beta \gamma \delta) = (\Lambda \beta \delta \epsilon) = -1.\]

These two lines from \( \Lambda \) are therefore homographically divided, the point \( \Lambda \) corresponding to itself, and \( b \) to \( \beta \), \&c.; and accordingly the four right lines, \( b \beta, c \gamma, d \delta, e \epsilon \), which connect corresponding points, concur in one common point, which is easily found to be \( B \). And other verifications, by such concurrences, can be assigned with little trouble.
It may assist the conception of the common law of arrangement, of the five points on each of the two typical lines last considered, to suppose that the joining line \( b \beta \) is thrown off, by projection, to infinity; or, what comes to the same thing, that the two points \( b \) and \( \beta \), themselves, are thus made infinitely distant. For thus the harmonic equations [78.] will simply express that, in this projected state of the figure, the four points, \( d, e, \delta, \epsilon \), bisect respectively the four intervals, \( a c, a \delta, a \gamma, a \epsilon \); whence it is easy to construct a diagram, not necessary here to be exhibited. The consideration of the two other lines through the same given point \( \lambda \), which have \([012][0\overline{1}2]\) for their symbols, and belong to the same two types as the two last, would offer to our notice a pencil of four rays, which has some interesting properties, especially as regards its intersections with other pencils, but which we cannot here delay to describe.

It may, however, be worth while to state here, as a consequence from the preceding discussion, this other Theorem:

“The 120 lines \( \Lambda_3 \) in the ten planes \( \Pi_1 \), whereof each connects a point \( p_0 \) with four points \( p_2 \), and with no other of the 305 points, although not all syntypical, are all homographically divided.”

Proceeding to consider the arrangements of those six typical lines [58.] which contain each six points, we find that whether we write, as new temporary and literal symbols,

\[
\begin{align*}
    a &= (0 1 \overline{1}), & b &= (\overline{1} 0 1), & c &= (1 2 \overline{1}), & a' &= (\overline{2} 1 1), & b' &= (1 \overline{2} 1), & c' &= (1 1 \overline{2}), \\
    a &= (0 1 1), & b &= (1 \overline{1} 1), & c &= (1 2 0), & a' &= (2 3 \overline{1}), & b' &= (1 3 1), & c' &= (\overline{1} 0 2),
\end{align*}
\]

or

the six points \( abc a' b' c' \) being in the one case on the line \([111]\), and in the other case on the line \([\overline{2}1\overline{1}]\), we have in each case the three harmonic equations:

\[
(cab a') = (abc b') = (bca c') = -1.
\]

We may then at once infer this Theorem:

“The 70 lines \( \Lambda_3 \), in the ten planes \( \Pi_1 \), which are represented by the fourth and seventh typical traces of planes on the plane \( ABC \), although not all syntypical (or generated by similar processes of construction), are all homographically divided.”

This common mode of their division may deserve, however, a somewhat closer examination, its consequences being not without interest. When any six collinear points, \( a \ldots c' \), are connected by the three equations [81.], we are permitted to suppose that their symbols are so prepared (if necessary), by coefficients,* as to give,

\[
(a) + (b) + (c) = 0;
\]

* For example, in the second case [81.], we should change the symbols for \( c \) and \( b' \) to their negatives, before employing the formule of [82.].
\[ (a') = (b) - (c), \quad (b') = (c) - (a), \quad (c') = (a) - (b); \]

and therefore,

\[ (a') + (b') + (c') = 0, \]
\[ 3(a) = (c') - (b'), \quad 3(b) = (a') - (c'), \quad 3(c) = (b') - (a'). \]

Whenever, then, the three harmonic equations \[81\] exist, for a system of six collinear points, \( a \ldots c' \), the three other harmonic equations, formed by interchanging accented and unaccented letters,

\[ (c' a' b' a) = (a' b' c' b) = (b' c' a' c) = -1, \]

are also satisfied; and the three pairs (or segments),

\[ aa', \quad bb', \quad cc', \]

which connect corresponding points, compose an involution.*

[83.] Under the same conditions, the two points \( a \) and \( a' \) are harmonically conjugate to each other, not only with respect to \( b \) and \( c \), but also with respect to \( b' \) and \( c' \); they are therefore the double points (or foci) of that other involution which is determined by the two pairs of points, \( bc, b'c' \). In like manner, \( b, b' \) are the double points of the involution, determined by the two pairs, or segments \( ca, c'a' \); and \( c, c' \) are the double points of the involution determined by \( ab, a'b' \).

[84.] From any one of the three last involutions [83.], we could return, by known principles, to the involution [82.]; we can also infer from them that the three new pairs of points (or segments of the common line), \( aa', bc', cb' \); the three pairs, or segments, \( bb', ca', ac' \); and the three others, \( cc', ab', ba' \), form three other involutions, making seven distinct involutions of the six points, so far: in three of which, as we have seen in [83.] two of those six points are their own conjugates.

[85.] For these and other reasons we propose to say, that when any three collinear points (as \( a, b, c \)) are assumed (or given), and three other points on the same line are derived from them, by the condition that each shall be the harmonic conjugate of one, with respect to the other two, then these two sets of points are two Triads of Points in Involution. And it is easy to extend this definition so as to include cases of two triads of complanar and co-initial lines, or of collinear planes, which shall be, in the same general but (as it is supposed) new sense, in involution with each other: every such involution of triads including, by what precedes, a system of seven involutions of the old or usual kind.

* Compare p. 127 of the Géométrie Supérieure (Paris, 1852). In general, the reader is supposed to be acquainted with the chapter (chap. ix.) of that excellent work of M. Chasles, which treats of Involution.
For example, because the two triads of points, \( A''B''C'' \) and \( A'^{IV}B'^{IV}C'^{IV} \), are thus in involution, by the equations [81.] applied to the fourth typical trace [43.], it follows that the two pencils, each of three rays, \( D_1 \cdot A''B''C'' \), and \( D_1 \cdot ABC \), are triads of lines, in involution with each other; and that, for a similar reason, the two triads of planes, all passing through the line \( D_1 \cdot E\)A, \( D\)E, \( D\)E, \( D\)E and \( D\)E, \( D\)E, \( D\)E, \( D\)E, \( D\)E, \( D\)E, \( D\)E, \( D\)E, are, in the sense above explained, in involution. In fact, when the point \( D_1 \) is thus taken as a vertex of the pencils in the plane \( ABC \), the three harmonic equations of the first case [81.], namely,

\[
(C'' A'' B'' A^{IV}) = (A'' B'' C'' B^{IV}) = (B'' C'' A'' C^{IV}) = -1,
\]

or rather the three reciprocal equations (comp. [82.]),

\[
(C^{IV} A^{IV} B^{IV} A'') = (A^{IV} B^{IV} C^{IV} B'') = (B^{IV} C^{IV} A^{IV} C'') = -1,
\]

correspond simply to the elementary equations, (50), (56),

\[
(C A' B A'') = (A B' C B'') = (B C' A C'') = -1,
\]

which may be employed to define the three important points \( A'', B'', C'', (87) \), of the first group of second construction [40.], as being the (well known) harmonic conjugates of the points \( A', B', C' \) of first construction, with respect to the three lines of the same first construction, \( BC, CA, AB \), on which those points are situated.

[87.] The equations [82.], which connect the symbols \((a) \ldots (c')\) of the six points, give, by easy eliminations, these other equations of the same kind:

\[
(b') = (b) + 2(c); \quad -(c') = 2(b) + (c);
\]

we have therefore, by (31), the following anharmonic of the group \( b, b', c, c' \):

\[
(bb' c c') = +4;
\]

and other easy calculations of the same sort given, in like manner, the equal anharmonics,

\[
(ec' a a') = +4; \quad (a a' b b') = +4.
\]

But in general, for any four collinear points, \( a, b, c, d \), the definition (29) of the symbol \((a b c d)\) gives easily the relation,

\[
(a b c d) + (a c b d) = 1;
\]

and hence, or immediately by calculations such as those recently used, we have this other set of anharmonics, with a new common value:

\[
(bb' c c') = (c a' c' a') = (a b a' b') = -3;
\]

the negative character of which shows, by the same definition (29), that the segment (or interval) \( aa' \), for example, is cut internally by one of the two points \( b, b' \), or by one of the two points \( c, c' \), and externally by the other; with similar results for each of the two other segments, \( bb', cc' \).
which cannot conveniently be stated here.

Involutions, determined by the three right angles, that one of each of the three involutions of segments mentioned in [84.], namely \(aa', bc', cb'; bb', ca', ac'; cc', ab', ba'\), has real* double points; and the double points of the three other involutions, determined by the three pairs of segments, \(bc, b'c'; ca, c'a'; ab, a'b'\), are likewise real, and have been assigned [83.]; namely, in each of these three last cases, the two remaining points of the system.

Now, in general, when the foci (or double points) of an involution of collinear segments, \(aa', bb', cc'\), are quite imaginary, so that conjugate points, \(a, a', b, b', \&c.\), fall at opposite sides of the central point \(o\), it is known, and may indeed be considered as evident, that if an ordinate \(OP\) be erected, equal to the constant geometrical mean between the two distances \(OA, OA'\), or \(OB, OB'\), \&c., then, all the segments \(aa', bb', \&c.\), subtend right angles, at the extremity \(P\) of this ordinate. It follows, then, by what has been proved in [82.], and [88.], and by the first case of [81.], that each of the three segments \(A''AIV, B''BIV, C''CIV\), of the fourth typical trace [43.] subtends a right angle at some one point, \(P\), in the plane \(ABC\), or rather generally at each of two such points: and in like manner, by the second case [81.], that each of the three other segments, \(A'AIIX, C_0B_1IV, C_1^VBVI\), of the seventh typical trace, subtends a right angle, at each of two other points, \(P, P'\), in the same plane.

These results, by their nature, like all the foregoing results of the present Paper, are quite independent of the assumed arrangement of the five given (or initial) points of space \(A \ldots E\), and are unaffected by projection, or perspective. In saying this, it is not meant, of course, that one right angle will generally be projected into another; or that the new point \(P\), at which the three new segments \(A''AIV, B''BIV, C''CIV\), or \(A'AIIX, C_0B_1IV, C_1^VBVI\), subtend right angles, will be itself (what may be called) the projection of the old point \(P\) [89.], which was so related to the three old segments, denoted by the same literal symbols, when the arrangement (or configuration) of the five initial points is varied, by a process analogous to projection. We can only assert that there will always, in every state of the Figure, or of the Net, be some point \(P\), possessing the above-mentioned property: or rather that there will be a circle of such points in space, having for its axis the line to which the three segments belong.

To fix a little more definitely the conceptions, let \(A, B, C, D\) be supposed, for the moment, to be the corners of a regular pyramid, with \(E\) for its mean point, or centre of gravity. With this arrangement of the five given points \(P_0, \ldots P_5\), six of the derived points \(P_1\), namely \(A', B', C', A_2, B_2, C_2\), bisect the six edges \(BC, CA, AB, DA, DB, DC\), of the given pyramid; and the four other points \(P_1\), namely \(A_1, B_1, C_1, D_1\), are the mean points of the four faces, opposite to \(A, B, C, D\). Siz of the ten points \(P_2, \ldots P_{11}\), namely \(A'', B'', C'', A_2', B_2', C_2'\), are now infinitely distant; and the line \(A''B''C''AIVBIVCIV\) to which three of the lately mentioned segments belong, becomes the line at infinity in the plane \(ABC\): which might seem, at first

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* The determination of these double points gives rise naturally to some new theorems, which cannot conveniently be stated here.
sight, to render difficult, with respect at least to them, the verification of a recent theorem [89.]. That theorem is, however, verified in a very simple manner, by observing that, with the arrangement here conceived, the three angles \( \Lambda''D_1A^I \), \( B''D_1B^I \), \( C''D_1C^I \), which those infinite and infinitely distant segments may be imagined to subtend at the point \( D_1 \), are all right angles; \( D_1A'' \), for example, being parallel to the side \( BC \) of the triangle \( ABC \), which is now an equilateral one; while \( D_1A^I \) is perpendicular to the same side, because it is drawn from the mean point \( D_1 \), and passes through the opposite corner, \( A \). As another verification of the theorem [89.], it will be found that, with the arrangement here supposed, the segments \( \Lambda'\Lambda^IX \), \( C_0B^I \), \( C^I\Lambda^VI \), of the seventh trace [43.], subtend right angles at the given point \( B \).

[92.] The involution of the three segments [82.] is only one of the consequences of the three harmonic equations [81.], or of what we have called in [85.] the Involution of the two Triads, \( abc \) and \( a'b'c' \). We can therefore infer more, respecting the geometrical relations of the six points, even in the general state of the whole Figure, or \( NLT \), than merely that those three segments subtend right angles, as above, at every point of one real circle, which has its centre on the common line, and its plane perpendicular thereto. The order of succession of the six points being supposed to be the following, \( ac'ba'cb' \), from which it can only differ, if at all, by changes not important to the argument, let \( P \) be, as in [90.], a point such that the angles \( aPa' \), \( bPb' \), \( cPc' \) are right. Then, because the three pencils,

\[ P \cdot ac'bc, \quad P \cdot c'ba'b', \quad \text{and} \quad P \cdot ba'ca, \]

are all harmonic pencils by [81.], it follows that (with the supposed order of the points) the lines \( PC' \) and \( PC \) are respectively the internal and external bisectors of the angle \( aPb; Pb \) and \( Pb' \), of the angle \( c'Pa' \); and \( Pa' \), \( Pa \), of \( bPc: \) the line \( PC \) bisecting also the angle \( a'Pb' \) internally. Hence it is easy to infer the following continued equation between angles (which is supposed to be new):

\[ aPC' = c'Pb = bPa' = a'PC = cPb' = \frac{\pi}{6}; \]

and therefore we may enunciate this Theorem:—“When six collinear points form a system of two triads in involution, their five successive intervals subtend angles each equal to the third part of a right angle, at every point of a certain circle, of which the axis is their common line.”

For example, with the particular arrangement [91.] of the five initial points \( A \ldots E \), it is found that the five successive portions, \( C_0\Lambda^IX \), \( \Lambda^IXC^IY \), \( C^I\Lambda^VI \), \( B^I\Lambda^V \), \( \Lambda'\Lambda^VI \), of the seventh trace, subtend each an angle of thirty degrees, at the given point \( B \); and the six lines \( D_1A'' \), \( D_1C^IV \), \( D_1B'' \), \( D_1A^IV \), \( D_1C'' \), \( D_1B^IV \), if suitably distinguished from their own opposites, succeed each other at angular intervals, of the same common amount.

[93.] In general, if three equally inclined diameters of a circle, forming a regular and six-rayed star, be taken as a given triad of lines [85.], the triad in involution therewith is represented by that other star of the same kind, of which the diameters bisect the angles between those of the former star: so that if we consider any six successive rays of the compound or twelve-rayed star, which results from the combination of these two, their successive angles are evidently each equal to thirty degrees. But now we see further, that if a star of this last
kind be cut in six points by an arbitrary transversal in its plane; and if these six points of section be in any manner put into perspective, by any new pencil and transversal: the six new points, thus obtained, as forming still two triads in involution, must admit of having their five successive intervals seen, from every point of some new circle, under angles still equal each to the same third part of a right angle.

[94.] We have not yet considered the arrangement of the six points on either the fifth or the sixth typical trace [43.]; but it is easy to do this as follows. Let \( a b c \alpha \beta \gamma \) denote, as new temporary symbols, either the six points of the fifth trace (comp. [58.]),

\[
\begin{align*}
I. \quad a &= (100), \quad b = (1T1), \quad c = (11T), \quad \alpha = (01T), \quad \beta = (21T), \quad \gamma = (2T1); \\
\text{or these six other points, belonging to the sixth trace}, \\
II. \quad a &= (111), \quad b = (102), \quad c = (120), \quad \alpha = (01T), \quad \beta = (231), \quad \gamma = (213); \\
\end{align*}
\]

we shall then have, in each case, the three harmonic equations,

\[
(bac\alpha) = (c\beta a\alpha) = (a\gamma b\alpha) = -1.
\]

In each case, therefore, we may consider ourselves as first deriving from three points a fourth, as the harmonic conjugate of the first with respect to the other two; and then deriving a fifth point, and a sixth, as the harmonic conjugates of that fourth point, with respect, on the one hand, to the third and first points; and on the other hand, to the first and second points of the system.

[95.] Having regard merely to this common law, we may enunciate (comp. [80.] [81.]) this theorem:

"The sixty lines, in the ten planes of first construction, represented by the fourth and fifth typical traces of the planes on the plane ABC, although not all syntypical, are all homographically divided."

And this common mode of their division is such, that if the fourth point be thrown off to infinity, the first point bisects the interval between the second and third; the fifth point bisects the interval between third and first; and the sixth point bisects the interval between first and second: so that, on the whole, we have a finite line, \( bc \), quadrisection in the points \( \gamma, a, \beta, \) and cut at infinity in \( \alpha \); whereas if, on either the fourth or the seventh trace, one of the six points, but only one, had been thus made infinitely distant, the five others would have presented the figure of a finite right line, bisected and trisected. With the equations [94.], if \( a \), instead of \( \alpha \), be projected to infinity, it is then the line \( \beta\gamma \) which is quadrisection, namely, in the points \( c, \alpha, b \). In general, with these last equations, the first set of three points, \( abc \), can be derived from the second set, \( \alpha\beta\gamma \), by the same rule [94.], as that by which the second set has been derived from the first: so that there is a sense in which these two sets may be said to be reciprocal triads, although they are not triads in involution, according to the definition [85].

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[96.] It may be added that, on either the fifth or the sixth trace, the two points which we have called first and fourth, are the double points of a new involution, determined by the two pairs, second and third, fifth and sixth; or, with the recent notations [94.], that \( a \alpha \) are the foci of the involution \( bc, \beta \gamma \); because the three last harmonic equations conduct to this fourth equation,

\[
(\beta a\gamma\alpha) = -1.
\]

[97.] And, as regards the homography of the divisions on the same two traces, if we denote, for the sake of distinction, the six points on the sixth trace by \( a_0 \cdots \gamma_0 \), then (because \( 0 \gamma \)) the five lines \( aa_0, bb_0, cc_0, 0\gamma_0 \), or (comp. [58.]) the five lines

\[
A'D_1, B_0B_1^V, C_0C_1^V, B_1^VI B_1^{VII}, C^{VII}C^{VIII},
\]

ought to concur in some one point: which accordingly it is easy to see that they do, namely in the point \( \Lambda' \); in fact, with the recent signification of \( a; \cdots \) and \( a_0; \cdots \), we have the symbolic equations,

\[
(a') - (a) = (b') - (b) = (c') - (c) = (011) = (a');
\]

and

\[
(b') - (b) = (\gamma') - (\gamma) = (022) = 2(a').
\]

[98.] The two sets of six points, on these two traces, with one point common, are thus the points in which a certain six-rayed pencil, with \( \Lambda' \) for vertex, is cut by the two traces as transversals; the symbols of the six rays being the following:

\[
\Lambda'AD_1 = [01T]; \quad \Lambda'B_0B_1^V = [2T1]; \quad \Lambda'C_0C_1^V = [21T];
\]

\[
\Lambda'\Lambda'' = [100]; \quad \Lambda'B_1^{VI}B_1^{VIII} = [1T1]; \quad \Lambda'C^{VII}C^{VIII} = [11T].
\]

And from a mere inspection of these symbols, we can infer (comp. (33)) that the first and fourth rays are the common harmonic conjugates of the two pairs, second and third, fifth and sixth; or that they are the double rays of the involution, which those two pairs of rays determine: the theorem [96.] being thus, in a new way, confirmed.

[99.] We have now discussed the arrangements of the points on those nine typical lines \( \Lambda_3 \), whereof each passes through not less than four, nor more than six, of the 52 points in the plane \( ABC \); but we have still three other typical lines to consider, namely the lines \( \Lambda_1 \) and \( \Lambda_2 \), of which each passes through at least seven points. Taking first, for this purpose, the typical line \( \Lambda_{2,1} \), namely \( AA' \), which contains only seven points, whereof the ternary symbols have been assigned in [55.], and the literal symbols there given may be retained, we shall, for the moment, reserve the consideration of the two points \( P_{2,3} \); but shall introduce a new and auxiliary point \( P_{3,1} \) on the same line, which may be thus denoted:

\[
A^X = (122) = AA' \cdot BC'' \cdot CB''
\]

and which may be said to represent or typify a first group of third construction, containing fifteen points, one on each of the fifteen lines \( \Lambda_{2,1} \); although, in the present Paper, we can only allude to such new points \( P_3 \), and cannot here attempt to enumerate, or even to classify them.
We have thus again six points, at this stage, to consider, namely the points \( A, A', D_1, A''', A_0, A^X \); and their symbols easily show that they are connected by the three following harmonic equations,

\[
(A A' D_1 A''') = (A D_1 A' A_0) = (A' A D_1 A^X) = -1;
\]

from which it follows, by [85.], that the two triads of points, \( AA'D_1 \) and \( A^X A''' A_0 \),

are triads in involution: with, of course, all the properties which have been proved, in recent paragraphs of this Paper, to belong generally to any two such triads. As a verification, it may be mentioned that, with the particular arrangement [91.] of the five initial points \( A \ldots E \), if we determine two new points \( P, P' \), of third construction, by the formulæ,

\[
P = (214) = BC''' \cdot CA''', \quad P' = (241) = CB''' \cdot BA''',
\]

it can be proved that each of the five successive intervals (comp. [92.]) between the six points,

\( A, A''', D_1, A^X, A', A_0, \)

subtends the third part of a right angle at each of these two new auxiliary points, \( P \) and \( P' \). But with other initial configurations, the coordinates of these two new vertices would be different, because they are connected with angles, which are not generally projective [90.]; although, as has been already remarked, there would always be some new points \( P, \) or rather a circle of such, possessing the property in question.

We may however enunciate generally, and without reference to any such particular arrangement of the five initial points, this Theorem:—

"On any one of the fifteen lines \( \Lambda_{2,1} \), of second construction, and first group, the given point \( P_0 \), and the two derived points of first construction \( P_1 \), compose a triad, the triad in involution to which [85.] consists of the point \( P_{3,1} \), of third construction and first group, and of the two points \( P_{2,2} \), of second construction and second group, upon that line;" with seven involutions of segments (comp. [84.]) included under this general relation.

For example, on the line \( AA' \), the three segments \( AA^X, A'A''' \), \( D_1 A_0 \) form always an involution of the ordinary kind, with its double points imaginary; the three other sets of segments, \( AA^X, A'A_0, D_1 A''' \); \( A'A''' \), \( AA_0, D_1 A^X \); and \( D_1 A_0, AA''' \), \( A'A^X \), form each an involution, with real double points; the points \( A, A^X \) are the real foci of a fifth involution, determined by the two pairs of segments \( A'D_1 \), and \( A'''A_0 \); the points \( A', A''' \) are, in like manner, the real double points of that sixth involution, which the two other pairs, \( A, D_1 \), and \( A_0, A^X \), determine: and finally, \( D_1 \) and \( A_0 \) are such points, for the seventh involution, determined by \( AA', A'''A^X \).
Introducing now the consideration of the two lately reserved points \( P_{2,3} \) [99.] of second construction and third group [45.], upon the typical line \( \Lambda_{2,1} \), we may derive them from the point \( P_0 \), the two points \( P_1 \), and the two points \( P_{2,2} \) upon that line \( AA' \), by the two following harmonic equations:

\[
(A A'' A' A^{'IV}) = (A A_0 D_1 A_1'') = -1;
\]

or by these two others,

\[
(A A' A_0 A^{IV}) = (A D_1 A'' A^{IV}) = -1,
\]

which may indeed by inferred from the two former, with the help of the relations between the six points previously considered: for, in general, if \( abc, a'b'c' \) be collinear triads in involution, and if \( d \) and \( d' \) be the harmonic conjugates of \( b' \) and \( c' \), with respect to the two pairs, \( ab, ac \), they are also the harmonic conjugates of \( b \) and \( c \), with respect to the two other pairs, \( ac', ab' \); or in symbols,

\[
(a b c' d) = (a c b' d') = -1, \quad \text{if} \quad (a b' b d) = (a c' c d') = -1,
\]

when the three harmonic equations [81.] exist. We have also, generally, under these conditions, the equation

\[
(a d a' d') = -1;
\]

for example, on the line \( AA' \), we have

\[
(A A^{IV} A^X A_1^{IV}) = -1.
\]

It is scarcely worth while to remark that the 15 lines \( \Lambda_{2,1} \) of the net, as being all syntypical, are all homographically divided; although it may just be noticed, as a verification, that the six lines,

\[
BC, \quad B'C', \quad B''C'', \quad B_0C_0, \quad B^{IV}C^{IV}, \quad B_1^{IV}C_1^{IV},
\]

which connect corresponding points on the two other lines of the same group in the given plane, namely \( BB'D_1 \) and \( CC'D_1 \), concur in one point \( A'' \). But it may not be without interest to observe, that \( A^X \) is the common harmonic conjugate of \( A \), with respect to each of the three pairs, \( A'D_1, A''A_0, A^{IV}A_1^{IV} \); which three pairs,* or segments, form thus an involution, with \( A \) and \( A^X \) for its double points. We have therefore this Theorem:—

"On each of the fifteen lines \( \Lambda_{2,1} \), the three pairs of derived points, of first and second constructions, namely the pair \( P_1 \), the pair \( P_{2,2} \), and the pair \( P_{2,3} \), compose an involution, one double point of which is the given point \( P_0 \); the other double point being the point \( P_{3,1} \), of third construction and first group, upon the line."

* That the two first of these three pairs belong to an involution, with those two double points, was seen in [101.].
[104.] We have thus discussed the arrangements of the points \( P_0, P_1, P_2 \), on each of the ten typical lines which connect not fewer than four, and not more than seven of them; but there are still two other typical lines to be considered, belonging to the groups \( \Lambda_1 \) and \( \Lambda_2; \) whereof one, as \( BC \), passes through eight points [54.]; and the other, as \( B'C' \), has ten points upon it [56.]. Beginning with the first, we easily find that the two sets of points \( \Lambda'BC \) and \( \Lambda''A_1V \), are triads in involution [85.]; the latter set being thus deducible from the former: while the two other points upon the line may be determined by the condition that they satisfy this other involution of two triads, \( \Lambda'BC, \Lambda'A_1V \). With the initial arrangement [91.], the line \( \Lambda^V \Lambda^V \) is trisected in \( B \) and \( C \), and its middle part \( BC \) is likewise trisected in \( A^V \) and \( \Lambda^V \); while each line is bisected in \( \Lambda' \), and cut at infinity in \( \Lambda'' \). And in general we may enunciate these two Theorems:—

I. “On every line of first construction, the point \( P_1 \) and the two points \( P_0 \) form a triad, the triad in involution with which consists of the point \( P_{2,1} \), and the two points \( P_{2,4} \).”

II. “On every such line \( \Lambda_1 \), the triad formed by the point \( P_{2,1} \), and the two points \( P_0 \), is in involution with a triad which consists of the point \( P_1 \) and the two points \( P_{2,5} \).”

[105.] Besides these two involutions of triads, we have two distinct involutions of the ordinary kind, into each of which all the eight points enter; two being double points in each. For we have these two other Theorems, deducible, indeed from the two former, but perhaps deserving to be separately stated:—

III. “On every line of first construction, the two given points are foci of an involution of six points, in which the points \( P_1, P_{2,1} \), are one pair of conjugates, while the two other pairs are of the common form, \( P_{2,4}, P_{2,5} \).” For example, \( \Lambda^V, \Lambda^V \) are such a pair, on the line \( BC \).

IV. “On every such line \( \Lambda_1 \), the points \( P_1, P_{2,1} \), are the double points of a second involution of six points, obtained by pairing the two points of each of the three other groups.”

[106.] Finally, as regards the remaining typical line \( B'C' \), which connects two points \( P_1 \), and passes through eight points \( P_2 \), if we reserve for a moment the consideration of the last pair, \( P_{2,8} \), or \( \Lambda'^X \) and \( \Lambda_1'^X \), we have a system of eight points upon that line, homographic with the recent system of eight points on the line \( BC \); being indeed the intersections of the line \( B'C' \) with the eight-rayed pencil, \( A . \Lambda'B'C'A'^V \Lambda_1'^V, \Lambda_{1,2} \), when taken in the order \( \Lambda''C'B', \Lambda'A'^V \Lambda'^V \Lambda_1'^V \Lambda_1'^V \). No description of the arrangement of these latter points is therefore at this stage required; but as regards the pencil, it may be remarked that, by [104.], the first, 2nd, and 3rd rays form a triad of lines, in involution [85.] with the triad formed by the 4th, 5th and 6th; and that the triad of the 2nd, 3rd and 4th rays is, in the same new sense, in involution with the triad of the 7th, 8th, and 1st: from which double involution of triads, the five last rays may be derived, if the three first are given. We have also by [105.] a double involution of the rays, considered as paired with each other, or with themselves: thus the second and third rays are the double rays of an involution (or the usual kind), in which the first is conjugate to the fourth, the fifth to the seventh, and the sixth to the eighth; while the first and fourth rays are the double rays of another involution, in which the second and third, the fifth and sixth, and the seventh and eighth are conjugate.
[107.] It only remains to assign the arrangement of the two last points of second construction, \( p_{2,8} \), with respect to the other points \( p_1, p_2 \), on a line \( \Lambda_{2,2} \), or to some three of them; or to show how \( A^\text{IX} \) and \( A^\text{IX}_1 \) can be derived,\* for example, from \( b', c' \), and \( A'' \); which derivation may easily be effected, on the plan already described for the fifth and sixth typical traces. In fact, if we denote the six points \( \lambda'' c' b' \lambda'''' A^\text{IX}_1 A^\text{IX} \) by \( a b c \alpha \beta \gamma \), we have the three harmonic equations of [94.]; and if, by one of the modes of perspective, or projection, mentioned in [95.], which answers to the initial arrangement [91.], we throw off the first point \( \lambda'' \) to infinity, the finite line \( A^\text{IX} A^\text{IX}_1 \) is then quadrisected: being itself bisected at \( \lambda'''' \), while \( c' \) and \( b' \) bisect its halves. In general, we shall have again the equations [94.], if we otherwise represent the six lately mentioned points on \( B' c' \) by \( \alpha \beta \gamma a b c; \) and thus it is seen that those six points are always homographic, in every state of the figure, or net, with the six points \( \lambda'' B^\text{VIII}_1 C^\text{VII}_1 A B_0 C_0 \) on the fifth trace \( A A'' \), and with the six points \( \lambda'''' B^\text{VIII}_1 C^\text{VIII}_1 D_1 B^V C^V_1 \) on the sixth trace \( D_1 A'' \); in fact they are, if taken in a suitable order, the points in which the six-rayed pencil [98.], with \( \lambda' \) for vertex, is cut by the line \( B' c' \).

[108.] We have thus shown for each of the twelve typical lines [74.], in the plane \( ABC \), how all the points but three, upon that line, may be derived from those three by a system of harmonic equations, not necessarily employing any point \( p_3 \), or other foreign\† merely auxiliary point: although it appeared that something was gained, in respect to elegance and clearness, by introducing, on the line \( AA' \), such a point \( \lambda^x \) [99.]; or by considering generally, on any one of the fifteen lines \( \Lambda_{2,1} \), a point \( p_{3,1} \) of third construction, belonging to what may perhaps deserve to be regarded as a first group [103.] of the points \( p_3 \), in any future extension [1.] of the results of the present Paper.

PART V.—Applications to the Net, continued: Distribution of the Given or Derived Points, in a Plane of Second Construction, and of First or Second Group.

[109.] It will be necessary to be much more concise, in our remarks on the distribution of the net-points in planes of second construction; but a few general remarks may here be offered, from which it will appear that each plane \( \Pi_{2,1} \) contains forty-seven of the 305 points \( p_0, p_1, p_2 \); and that each plane \( \Pi_{2,2} \) contains forty-three of those points; with many cases of collineation for each.

[110.] We saw in [33.], that each plane \( \Pi_{2,1} \) contains two lines \( \Lambda_{2,1} \), which intersect in a point \( p_0 \), and may be regarded as the diagonals of a quadrilateral, of which the four sides are lines \( \Lambda_{2,2} \). It contains, therefore, as has been seen, one point \( p_0 \), and four points \( p_1 \); but it is found to contain also 42 points \( p_2 \), arranged in six groups, as follows.

\* This point \( A^\text{IX} \) may also, by [81.], be determined on the seventh trace, or seventh typical line [74.], as the harmonic conjugate of \( \lambda' \), with respect to \( C_0 \) and \( C^V_1 \).

\† This non-requirement of foreign points is the only remarkable thing here: for the anharmonic function of every group of four collinear net-points is necessarily rational; and whenever \( (a b c d) = \) any positive or negative quotient of whole numbers, it is always possible to deduce the fourth point \( d \) from the three points \( a, b, c \), by some system of auxiliary points, derived successively from them through some system of harmonic equations.
[111.] There are 2 points \( p_{2,1} \), namely the intersections of opposite sides of the quadrilateral; thus, in what we have called the second typical plane [33.], the sides \( B_1C_1, C_2B_2 \) intersect in the point \( A'' \); and the sides \( C_1C_2, B_2B_1 \) in \( D_1' \) (62).

[112.] The plane contains also 8 points \( p_{2,2} \); namely, two on each of the two diagonals, and one on each of the four sides; and it contains 4 points \( p_{2,3} \), namely two on each diagonal: but it contains no point of either of the two groups \( p_{2,4}, p_{2,5} \), as a comparison of their types sufficiently proves, or as may be inferred from the laws of their construction [46.] [47.].

[113.] The same plane contains 12 points \( p_{2,6} \); namely two on each side of the quadrilateral; and four others, in which the plane is intersected by four lines \( \Lambda_{2,2} \); as the types sufficiently prove. But to show, geometrically, why there should be only four such intersections, conducting thus to new points \( p_{2,6} \) in the plane, let the five inscribed pyramids [28.] be denoted by the symbols \( A' \ldots E' \); then the six edges of the pyramid \( A' \) are found to intersect the present plane \( \Pi_{2,1} \) in points already considered, namely in the two points \( p_{2,1} \), of meetings of opposite sides, and in those four points \( p_{2,2} \), which are situated on the diagonals of the quadrilateral; they give therefore no new points. Also, each side of the same quadrilateral is an edge of one of the four other pyramids \( B' \ldots E' \); but there remains, for each such pyramid, an opposite edge: and these are the four lines, out of the plane, which intersect it in the four points \( p_{2,6} \), additional to the eight points \( p_{2,6} \), which are ranged, two by two, upon the sides. There are thus twelve points of the group \( p_{2,6} \), in any one plane \( \Pi_{2,1} \); and we have now exhausted the intersections of that plane with the lines \( \Lambda_{2,2} \); and also, as it will be found, with the lines \( \Lambda_{2,1} \), and \( \Lambda_1 \).

[114.] But there remain eight points \( p_{2,7} \), and eight points \( p_{2,8} \), in the plane now considered; namely two of each group, on each of the four sides of the quadrilateral. There are, therefore, 16 such points; which, with the 12 points \( p_{2,6} \); the 4 points \( p_{2,3} \); the 8 points \( p_{2,2} \); the 2 points \( p_{2,1} \); the 4 points \( p_1 \); and the one point \( p_0 \), make up (as has been said in [109.]) a system of 47 points, given or derived, in any one of the fifteen planes \( \Pi_{2,1} \).

It may be remarked that with the initial arrangement [91.] of the five given points, the four points \( B'C'B_2C_2 \), in a new plane \( \Pi_{2,1} \), are corners of a square, which has the point \( E \) for its centre; and that thus the Figure, of the 47 points in such a plane, may be thrown into a clear and elegant perspective.

[115.] As regards the distribution in a plane \( \Pi_{2,2} \), such as the third typical plane [34.], it may here be sufficient to observe, that besides containing three lines \( \Lambda_{2,2} \), namely the sides of a triangular face [34.] of one of the five inscribed pyramids [28.], and three points \( p_1 \), which are the corners of that triangle, and serve to determine the plane [1.], it contains also forty points \( p_2 \), which are arranged in groups, as follows. Each of the four first groups, of second construction, \( p_{2,1}, \ldots, p_{2,4} \), gives three points to the plane; the fifth group, \( p_{2,5} \), furnishes only one point; and the sixth, seventh and eighth groups, \( p_{2,6}, \ldots, p_{2,8} \), supply six, twelve, and nine points, respectively. Of these 40 points \( p_2 \), twenty-four are ranged, eight by eight, on the three sides of the triangle, as was to be expected from [56.]; and the existence of at least 27 points, \( p_1, p_2 \), in a plane \( \Pi_{2,2} \), might thus have been at once foreseen. But we have also to consider the traces, on that plane, of the 52 lines, \( \Lambda_1, \Lambda_2 \), which are not contained therein.
Of these lines, it is found that 36 intersect the sides of the triangle, and give therefore no new points. But the sixteen other lines intersect the plane, in so many new and distinct points; and thus the total number \([109.]\), of forty-three derived points, \(P_1, P_2\), in a plane \(\Pi_{2,2}\), which contains no given point \(P_0\), is made up.

[116.] Without attempting here to enumerate the cases of collineation, in either of the two typical planes \(\Pi_2\), we may just remark, that while the traces of four of the planes \(\Pi_1\) on the typical plane \(\Pi_{2,1}\) are the four sides, and the traces of four others are the diagonals, of the quadrilateral already mentioned, the trace of a ninth plane \(\Pi_1\), namely \(ABC\), on that plane \(\Pi_{2,1}\) has been already considered, as the trace \(AA''\) of the latter on the former; but that the trace of the tenth plane \(\Pi_1\), namely \(ADE\), or \([01\overline{1}00]\), on \(AB_1C_2C_1B_2\), or on \([011\overline{1}\overline{1}]\), is a new line, \(AD'\); which passes thus through one point \(P_0\) and one point \(P_{2,1}\), and also through two points \(P_{2,2}\), namely \((01120)\) and \((01102)\), and through two points \(P_{2,6}\), namely \((2001\overline{1})\) and \((200\overline{1}1)\): being, however, syntypical with the formerly considered trace \(AA''\), and therefore leading to no new harmonic or anharmonic relations.

[117.] As a specimen of a case of collineation which conducts to such new relations, let us take the four following points \(P_2\), in the second typical plane,

\[
a = (01120), \quad b = (00211), \quad c = (0203\overline{1}), \quad d = (0\overline{1}302),
\]

whereof the two first are points \(P_{2,2}\), and the two last are points \(P_{2,8}\); and of which the symbols satisfy the equations,

\[
(c) = 2(a) - (b), \quad (d) = -(a) + 2(b); \text{ whence } (a d b c) = 4.
\]

These four points, therefore, with which it is found that no other given or derived point of the system \(P_0, P_1, P_2\) is collinear, do not form a harmonic group; and consequently we cannot construct the fourth point, \(d\), when the three other points, \(a, b, c\), are given, by means of harmonic relations alone (comp. [108.]), unless we introduce some auxiliary point, or points, \(e, \ldots\), which shall be at lowest of the third construction. But if we write

\[
e = (12020) \equiv (01\overline{1}1\overline{1}), \quad f = (\overline{1}0220) \equiv (01331),
\]

so that \(e\) is a point \(P_{3,1}\) [99.], while \(f\) may be said to be a point \(P_{3,2}\), we find that these two new or auxiliary points, \(e, f\), are the double points of the involution, determined by the two pairs, \(ab, cd\); because we have the two harmonic equations,

\[
(a e b f) = (c e d f) = -1.
\]

And because we have also,

\[
(c a b e) = (a b d e) = -1,
\]

we need only employ the one auxiliary point \(e\), considered as the harmonic conjugate of \(a\), with respect to \(b\) and \(c\); and then determine the fourth point \(d\), as the harmonic conjugate of \(a\), with respect to \(b\) and \(e\). It may be added that \(abe\) and \(def\) are triads in involution [85.]; so that if \(e\) be projected to infinity, the finite line \(cd\) is trisected at \(a\) and \(b\).
[118.] Although we have not proposed, in the present Paper, to enumerate, or even to classify, any points, lines, or planes, beyond what we have called the Second Construction [1.], yet some such points, lines, and planes have offered themselves naturally to our consideration: and we intend, in this Sixth Part, to consider a few others, chiefly in connexion with relations of homology, of triangles or pyramids which have been already mentioned.

[119.] It was remarked in [29.], that the thirty lines $\Lambda_{2,2}$ are the sides of ten triangles $T_{2}$, of second construction, which are certain inscribed homologues of ten other triangles $T_{1}$, of first construction [26.]; the ten corresponding centres of homology being the ten points $P_{1}$. For example, the triangle $A'B'C'$ is inscribed in $ABC$, and is homologous thereto, the point $D_{1}$ being their centre of homology; because we have the three relations of intersection,

$$A' = D_{1}A \cdot BC, \&c.;$$

or because, $A'$ being a point of $BC, \&c.$, the three joining lines $AA', \&c.$, concur in the point $D_{1}$.

[120.] Proceeding to determine the axis of this homology, or the right line which is the locus of the points of intersection of corresponding sides, we easily see that it is the line $A''B''C''$; because we had $A'' = BC \cdot B'C', \&c.$ And because an analogous result must take place in each of the ten planes $\Pi_{1}$, we see that the ten points $P_{2,1}$ are ranged, three by three, on ten lines $\Lambda_{3,1}$, in the ten planes $\Pi_{1}$; namely on the axes of homology of the ten pairs of triangles $T_{1}, T_{2}$, in those ten planes: which axes are the lines,

$$D'_{1}A'_1A'_2, \&c.; \quad C'_1B'_1A'', \&c.; \quad C'_2B'_2A'', \&c.; \quad \text{and} \quad A''B''C'';$$

each point $P_{2,1}$ being thus common to three of them, because it is common to those three planes $\Pi_{1}$, which contain the line $A_{1}$ whereupon it is situated. Each point $P_{2,1}$ is also the common intersection of this last line with three lines $\Lambda_{2,2}$; we have for example, the formula of concurrence,

$$A'' = BC \cdot B'C' \cdot B_{1}C_{1} \cdot B_{2}C_{2}.$$

[121.] The line $A''B''C''$ was seen to be the common trace of two planes $\Pi_{2,2}$, namely of $A_{1}B_{1}C_{1}$ and $A_{2}B_{2}C_{2}$, on the plane $\Pi_{1}$, namely $ABC$, in which it is situated; and a similar result must evidently hold good for each of the ten lines $\Lambda_{3,1}$. But we may add that the three triangles $ABC, A_{1}B_{1}C_{1}, A_{2}B_{2}C_{2}$, in the plane of each of which the line $A''B''C''$ is contained, are homologous, two by two, and have this line for the common axis of homology of each of their three pairs; having however three distinct centres of homology, namely $D'_{1}$ for second and third, $D$ for third and first, and $E$ for first and second: with (as we need not again repeat) analogous results for the other lines $\Lambda_{3,1}$, of which group we here take the line $A''B''C''$ as typical. It may be remarked that the four centres, recently determined, are collinear, and compose an harmonic group; and that the inscribed triangle $A'B'C'$ is also homologous with each of the two triangles $A_{1}B_{1}C_{1}, A_{2}B_{2}C_{2}$, although not complanar with either; the line $A''B''C''$ being still the common axis of homology; while the two centres, of these last two homologies, are the two given points, $D$ and $E$.
The six points $P_{2,2}$, in the plane $ABC$, have been seen to range themselves, according to their two ternary types [41.], into two sets of three, which are the corners of two new triangles; one of these, namely $\Lambda''B''C''$, being an inscribed homologue of $\Lambda'B'C'$; while the other, namely $A_0B_0C_0$, is an exscribed homologue of $ABC$; and these two triangles are also homologous to each other: the line $\Lambda''B''C''$ being still the common axis, and the point $D_1$ being the common centre of homology. And the same thing holds good for any one of these four triangles, $A_0B_0C_0, ABC, A'B'C', A''B''C''$, in the plane $\Pi_1$ here considered, as compared with the triangle $\Lambda IV B IV C IV$, whereof the corners are those three points $P_{2,3}$, which are not ranged on the line $\Lambda''B''C''$, as the three other points $P_{2,3}$, namely $\Lambda IV, B IV, C IV$, have been seen to be.

It was remarked in [28.], that each of the five pyramids $R_2$ is not only inscribed in the corresponding pyramid $R_1$ [26.], but it is also homologous therewith; the centre of their homology being a point $P_6$; thus the point $E$ is such a centre, for the two pyramids $ABCD$ and $A_1B_1C_1D_1$, or for those which we have lettered as $E$ and $E'$ [26.]. The planes $BCD$, $B_1C_1D_1$, of two corresponding faces, intersect in the line $C'2B'2A''$; the planes $CAD$, $C_1A_1D_1$, in $A'_2C'_2B''$; the planes $ABD$, $A_1B_1D_1$, in $B'_2A'_2C''$; and the planes $ABC$, $A_1B_1C_1$, in $A''B''C''$. Hence it is easy to infer that these six points $P_{2,1}$, namely

\[ A'', B'', C'', A'_2, B'_2, C'_2, \]

are all situated in one plane, which is the plane of homology of the two pyramids $E$ and $E'$, and which we shall denote by $[E]$; its quinary symbol being

\[ [E] = [1 1 1 1 4], \]

which may also serve as a type of the group $[\Lambda]$ ... $[E]$. And in fact, the quinary symbols of the six points all satisfy the equation (comp. [19.],

\[ x + y + z + w = 4v. \]

It may be noted that the two planes of homology, $[D]$ and $[E]$, have the line $\Lambda''B''C''$ for their common trace on the plane $ABC$; and that the traces of the three other planes of the same group, $[\Lambda], [B], [C]$, which have

\[ [411], [1 4 1], [1 1 4], \]

for their ternary symbols, pass respectively through the points $\Lambda X, B X, C X$, (comp. [99.]), and coincide with the lines $B^1 IV C^1 IV, \&c.$, or with the sides of the last mentioned triangle [122.]. And it follows from [123.], that the ten points $P_{2,1}$ are ranged six by six, and that the ten lines $\Lambda_{3,1}$ are ranged four by four, in five planes $\Pi_{3,1}$; namely in the five planes $[\Lambda] ... [E]$ of homology of pyramids. But these last laws of arrangement, of points and lines, must be considered as included in results which have been comparatively long known, respecting transversal* lines and planes in space.

* Compare the second note to [1.].
[125.] Instead of inscribing a pyramid $E'$ in the pyramid $E$, we may propose to exscribe to the latter a new pyramid $A'B'C'D'$, or $e'$, which shall be homologous with it, the given point $E$ being still the centre of homology. In other words, the four new planes $B'C'D'$, ..., $A'B'C'$, or $E_A$, $E_B$, $E_C$, $E_D$, are to pass through the four given points $A$, $B$, $C$, $D$; and the four new lines $AA'$, $BB'$, $CC'$, $DD'$ are to concur in the fifth given point $E$. The solution of this problem is found to be expressed by the following quinary symbols for the four sought planes:

$$[E_A] = [0 1 1 1 3], \ldots [E_0] = [1 1 1 0 3].$$

In fact, the pyramid $E'$, with these four planes for faces is evidently exscribed to the pyramid $ABCD$, or $E$; and because its corners may be represented by these other quinary symbols,

$$A'^1 = (3 0 0 0 1), \ldots D'^1 = (0 0 0 3 1),$$

the condition of concurrence is satisfied. We may remark that the plane $[E]$ of [123.] is the plane of homology of the two last pyramids $E$ and $E'$; and that this exscribed pyramid $E'$ is homologous also to the inscribed pyramid $E'$, the point $E$ being still the centre, and the plane $[E]$ the plane of their homology.

[126.] It may be remarked that the common trace of the two planes $E_D$ and $D_E$, on the plane $ABC$, is the line $A''B''C''$; to construct, then, the exscribed pyramid $E'$, we may construct the plane $E_D$ of one of its faces, by connecting the point $D$ with the line $A''B''C''$; and similarly for the rest. Or if we wish to determine separately the new point, or corner, $D'$, which corresponds to the given point $D$, we may do so, by the anharmonic equation,

$$(D_D_1 E D') = 3;$$

for which may be substituted* the system of the two following harmonic equations:

$$ (D_D_1 E F) = (D_D' D_1 F) = -1;$$

where $F$ is an auxiliary point, namely $D'_1$.

**PART VII.—** *On the Homography and Rationality of Nets in Space; and on a Connexion of such Nets with Surfaces of the Second Order.*

[127.] In general, all geometric nets in space are homographic figures; corresponding points, lines, and planes, being those which have the same or (congruent) quinary symbols, in whatever manner we may pass from one to another system of five initial points, $A$ ... $E$; whereof it is still supposed that no four are complanar. All points, lines, and planes of any such Net are evidently rational, in the sense [8] already defined, with respect to the initial system; and conversely it is not difficult to prove that every rational point, line, or plane, in space, is a net-point, net-line, or net-plane, whatever that initial system of five points may be. It follows that although no irrational point, line, or plane, can possibly belong to the net,

* Compare the note to [108].
with respect to which it is thus irrational, yet it can be indefinitely approached to, by points, lines, or planes which do so belong: a remarkable and interesting theorem, which appears to have been first discovered by M"obius:* to whom indeed, as has been already said, the conception of the net is due, but whose analysis differs essentially from that employed in the present Paper.

[128.] As regards the passage from one net in space to another, let the quinary symbols of some five given points $P_1 \ldots P_5$, whereof no four are in one plane, be with respect to the given initial system $A \ldots E$ the following:—

$$P_1 = (x_1 \ldots v_1), \quad \ldots \quad P_5 = (x_5 \ldots v_5);$$

and let $a' \ldots e'$ and $u'$ be six coefficients, determined so as to satisfy the quinary equation [5.],

$$a'(P_1) + b'(P_2) + c'(P_3) + d'(P_4) + e'(P_5) = -u'(U),$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \cdots + e'x_5 = \cdots = a'v_1 + \cdots + e'v_5 = -u'.$$

Let $P'$ be any sixth point of space, such that

$$(P') = xa'(P_1) + yb'(P_2) + zc'(P_3) + wd'(P_4) + ve'(P_5) + u(U);$$

then this sixth point $P'$ can be derived from the five points $P_1 \ldots P_5$ by the same constructions, as those by which the point $P = (x y z w v)$ is derived from the five given points $A B C D E$. For example, if we take the five points,

$$A_1 = (10001), \quad B_1 = (01001), \quad C_1 = (00101), \quad D_1 = (00011), \quad E = (00001),$$

we have the symbolic equation,

$$(A_1) + (B_1) + (C_1) + (D_1) - 3(E) = (U);$$

if then we write $v' = x + y + z + w - 3v$, the point $(x y z w v')$ is derived from $A_1 B_1 C_1 D_1 E$, by the same constructions as $(x y z w v)$ from $A B C D E$. In particular, $D$ is related to $A_1 B_1 C_1 D_1 E$, as the point $P = (00031)$ is related to $A B C D E$; but this point $P$ satisfies the anharmonic equation, $(D D_1 E P) = +3$; if then $E_1 = D_1 E \cdot A_1 B_1 C_1 = (0001102)$, we must have the corresponding equation $(D_3 E_1 E D) = +3$: which is accordingly found to exist and furnishes a construction for exscribing a pyramid $ABCD$ to a given pyramid $A_1 B_1 C_1 D_1$, with which it is to be homologous, and to have a given point $E$ for the centre of their homology, agreeing with the construction assigned in [126.] for a similar problem of exscription. And in general, from any five given points of a net, whereof no four are complanar, we can (as was first shown by M"obius) return, by linear constructions, to the five initial points $A \ldots E$; and therefore can, in this way, reconstruct the net.

* See page 295 of the Barycentric Calculus. As regards the theory of homographic figures, chapter XXV. of the Géométrie Supérieure of M. Chasles may be consulted with advantage. But with respect to anharmonic ratio, generally, it must be remarked that Professor M"obius was thoroughly familiar with its theory and practice, when he published in 1827; although he called it by the longer but perhaps more expressive name of Doppelschnittsverhältniss (ratio bissectionalis). It may be added that he denotes by $(A, C, B, D)$, what I write as $(A B C D)$. 

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[129.] If we content ourselves with quaternary (or anharmonic) coordinates [12.], or suppose (as we may) that \( v = 0 \), the equation of a surface of the second order takes the form,

\[
0 = f(xyzw) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 + 2(\epsilon yz + \zeta zx + \eta xy) + 2w(\theta x + iy + \kappa z);
\]

and if the ten coefficients \( \alpha \ldots \kappa \), or their ratios, be determined by the condition that the surface shall pass through nine given net-points, those coefficients may then be replaced by whole numbers, and the surface may be said to be rationally related to the given net, or to the initial system \( A \ldots E \), or briefly to be (comp. [8.]) a Rational Surface. For example, if the nine points be \( ABCDEC'C'A'C_2A_2 \), so that, besides passing through \( E \), the surface has the gauche quadrilateral \( ABCD \) superscribed upon it, the equation is

\[
I \ldots 0 = f = xz - yw;
\]

and if they be \( a, b, a', b', A_2, B_2, A_1, A'' = (1 2 \bar{T} 0) \), and \( F = (1 2 0 \bar{T}) \), so that this new point \( F \), like \( A'' \), belongs to the group \( P_{2,6} \); the equation of the surface is then found to be,

\[
\Pi \ldots 0 = f = w^2 + z^2 - (w + z)(x + y) - 2xy.
\]

[130.] In general, whether the surface of the second order be rational or not, it results from the principles of a former communication that any point \( p = (xyzw) \) of space is the pole of the plane \( \Pi = [XYZW] \), if \( X \ldots W \) be the derivatives,

\[
X = D_xf, \quad Y = D_yf, \quad Z = D_zf, \quad W = D_wf;
\]

hence, in particular, the pole of the plane \([E] \) of homology of the three pyramids \( E, E', E'' \), [26.] [113.] [125.], of which plane the quaternary symbol [12.] is [1 1 1 1], is the point \( K \) determined by the equations,

\[
X = Y = Z = W, \quad \text{or} \quad D_xf = D_yf = D_zf = D_wf;
\]

and if the point \( E \) be the mean point of the pyramid \( ABCD \), the plane \([E] \) is then infinitely distant, and this point \( K \) is the centre of the surface.

[131.] For example, in the case of the 1st surface [129.], this pole \( K \) is the point \((1 \bar{T} 1 \bar{T}) \equiv (2 0 2 0 1) \), which belongs to the group \( P_{3,1} \); and because it is on the plane \([E] \), that plane touches the surface in that point: so that when the point \( E \) is the mean point of the pyramid \( ABCD \), the surface becomes a ruled paraboloid. In the case of the II\textsuperscript{nd} surface [129.], the pole \( K \) of \([E] \) is always the point \((1 1 0 0) \), or \( C' \); this point \( C' \) becomes therefore the centre of the surface, when \( E \) is the mean point of the pyramid; and the five following lines,

\[
AB, \quad A'B''_1, \quad B''A''_1, \quad A_2F, \quad \text{and} \quad B_2G,
\]

where \( G \) is the new point \((2 1 0 \bar{1}) \) of the group \( P_{2,6} \), which are always chords through \( C' \), become in that case diameters. It may be added that, with the initial arrangement [91.], the surface last considered becomes the sphere, which is described with \( AB \) for diameter; and that it always passes through the auxiliary point \( P \), of third construction, which was mentioned in [100.].

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We have then here an example, of a surface of the second order, which was determined so as to pass through nine net-points

A, B, A', B', A2, B2, A1, A'VII, and F,

but which has been subsequently found to pass also through at least four other points of the net, namely

B1, B'VII, G, and P.

This is, however, only a very particular case of a much more general Theorem, with the enunciation of which I shall conclude the present Paper, regretting sincerely that it has already extended to a length, so much exceeding the usual limits of communications designed for the Proceedings* of the Academy, but hoping that some at least of its processes and results will be thought not wholly uninteresting:—

“If a Surface of the Second Order be determined by the condition of passing through nine given points of a Geometrical Net in Space, it passes through indefinitely many others: and every Point upon the Surface, which is not a point of the Net, can be included within a Geodetic Triangle on that surface, of which the corners are net-points, and of which the sides can be made as small as we may desire.”

In fact, the surface is a rational one [129.], or the coefficients of its equation may be made whole numbers; and therefore every rational line [8.], from any one net point, or rational point, upon it, if not happening to touch the surface, is easily proved to meet it again, in another rational point: whence, with the aid of a lately mentioned principle [127.], the theorem evidently follows.

* Some of the early formulæ of this Paper are unavoidably repeated from a communication of the preceding Session (1859–60), but with extended significations, as connected now with a quinary calculus. And in a not yet published volume, entitled “Elements of Quaternions,” the subject of Nets in Space is incidentally discussed, as an illustration of the Method of Vectors. But it will be found that the present Paper is far from being a mere reprint of the Section on Nets, in the unpublished work thus referred to: many new theorems having been introduced, and the plan of the treatment generally being different, although the notations have, on the whole, been retained. Besides it was thought that Members of the Academy might like to see the subject treated, in their Proceedings, without any express reference to quaternions: with which indeed the nets have not any necessary connexion.