Notes for MA2341 - Advanced Mechanics I

John Bulava

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1 Introduction

The foundation of classical mechanics is **Newton's Laws**:

1. In the absence of external forces ($\mathbf{F}_{ext} = 0$), objects in motion stay in motion, objects at rest stay at rest. This is somethiems called called the principle of inertia.

2. The rate of change of momentum of an object is equal to the sum of the forces acting on that object:

$$\mathbf{F}_{tot} = m\mathbf{a} = \dot{\mathbf{p}} = m\ddot{\mathbf{r}}.$$

3. The force that object 1 exerts on object 2 is equal in magnitude and opposite in direction to the force that object 2 exerts on object 1:

$$\mathbf{F}_{21} = -\mathbf{F}_{12}.$$

The fundamental goal of classical mechanics is to obtain the motion of an object, i.e. the position as a function of time $\mathbf{r}(t)$. In the case of systems with more than one particle or systems in which the motion is constrained in some way, we often employ **generalized coordinates** $\{q_i(t)\}$.

The **simple pendulum**, which consists of a mass m suspended by a rigid weightless rod of length ℓ in a gravitational field, has a single generalized coordinate, θ, which is the angle of the rod with respect to the vertical direction. This is an example of a **constraint**, where the two possible directions which are normally present for motion in a plane are reduced to a single **degree of freedom**, θ, because of the rod. This constraint requires a force to maintain, which in this case is the tension in the rod.

Typically obtaining $q_i(t)$ takes two steps. First, second-order differential equations involving the $q_i(t)$ must be determined. Then together with initial conditions these equations must be solved in some way. This could be done analytically by integrating them, numerically, or by making some approximations.

One way to obtain the second order differential equations for the $q_i(t)$ is by using Newton's second law. In practice this is often difficult in the presence of constraints. An alternative is to use the **Euler-Lagrange equations**. First define the **Lagrangian**

$$L = T - V, \tag{1}$$

where T is the kinetic energy of the system and V the potential energy. A differential equation for $q_i(t)$ is then obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, \ldots n. \tag{2}$$

Note that for a system with n degrees of freedom there are n such equations.

The advantage of the Euler-Lagrange formalism can be illustrated by solving the simple pendulum. Using Newton's second law, we note that there are two forces acting on the mass, the force of gravity and the tension force due to the rod. We express the coordinates of the mass in terms of θ

$$x = \ell \sin\theta, \qquad y = -\ell\cos\theta \tag{3}$$

and note that the tension force points in the negative radial direction $\mathbf{T} = -T\hat{r}$, so that its components may be written as

$$T_x = -T\sin\theta, \qquad T_y = T\cos\theta. \tag{4}$$

Applying Newton's second law in the y-direction gives

$$T\cos\theta - mg = m\ell(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta), \tag{5}$$

while applying it in the x-direction gives

$$-T\sin\theta = mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{6}$$

We now need to solve for T. Multiplying Eq. 5 by $\sin\theta$ gives

$$-T\cos\theta\sin\theta + mg\sin\theta = -m\ell(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\cos\theta\sin\theta) \tag{7}$$

while multiplying Eq. 6 by $\cos\theta$ gives

$$-T\cos\theta\sin\theta = m\ell(\ddot{\theta}\cos^2\theta - \dot{\theta}^2\sin\theta\cos\theta). \tag{8}$$

Combining these equations gives the answer

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta. \tag{9}$$

We now obtian the same equation of motion using the Euler-Lagrange equation. The kinetic energy is just given by the rotational kinetic energy of the mass, and the potential energy by its gravitational potential energy. Therefore we have

$$L = T - V = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta \tag{10}$$

Now directly applying the Euler-Lagrange equation for θ gives the answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\ell^2\dot{\theta}) + mg\ell\sin\theta = 0 \tag{11}$$

$$\Rightarrow m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0 \tag{12}$$

$$\Rightarrow \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0. \tag{13}$$

In this case it wasn't too bad to use Newton's second law. However, if we consider the **double pendulum**, in which a second weightless rod is attached to the mass and a send mass attached to the end of the new rod, Newton's second law quickly becomes unwieldy. Note that in this system there will be two unknowns that must be eliminated, corresponding to the tension in both rods.

2 Background

We now focus on some necessary background material. Recall that **angular momentum** is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{14}$$

This can be expressed using indices as

$$L_i = \sum_{jk} \epsilon_{ijk} r_j p_k. \tag{15}$$

We have introduced the **Levi-Cevita symbol**

$$\epsilon_{ijk} = \begin{cases} 0, & i = j,\ j = k, \text{ or } i = k \\ 1, & ijk \text{ is an even permutation of } 123 \\ -1, & ijk \text{ is an odd permutation of } 123. \end{cases} \tag{16}$$

Note that cyclic permuations are always even. We shall also employ the **Einstein summation convention** in which we assume (unless otherwise specified) that repeated indices are summed over.

Familiar properties of the cross product can be easily verified. The anti-symmetry of the Levi-Cevita symbol under the interchange of any two indicies results in the anti-commutativity of the cross product

$$(\mathbf{B} \times \mathbf{A})_i = \epsilon_{ijk} B_j A_k = -\epsilon_{ikj} A_k B_j = -(\mathbf{A} \times \mathbf{B})_i \tag{17}$$

The symmetry of ϵ_{ijk} under cyclic permutations results in the scalar triple product identity

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}). \tag{18}$$

A useful identity is

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{19}$$

which can be used to prove

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{C} \cdot \mathbf{B}). \tag{20}$$

Using angular momentum, we can define **torque**

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{21}$$

Note that

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{N}, \tag{22}$$

where we have used $\dot{\mathbf{r}} \times \mathbf{p} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} = 0$ and $\mathbf{F} = \dot{\mathbf{p}}$.

We can also define **work**. The work done by a force $\mathbf{F}$ on an object as it moves from point 1 to point 2 is

$$W_{12} = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{23}$$

Noting that $\mathbf{r} = \mathbf{r}(t)$ we can write $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t$. Using this and Newton's second law gives

$$W_{12} = m \int_1^2 \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \mathrm{d}t = m \int_1^2 \mathbf{v} \cdot \mathrm{d}\mathbf{v} = \frac{m}{2} v^2 \bigg|_1^2 = T_2 - T_1 \tag{24}$$

This is the **work-energy theorem**. Let's examine the work done by gravity while an object is lifted up to a certain height and then allowed to fall down to the starting point. The quantity $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is negative on the way up, as the force of gravity is antiparallel to the direction of motion, and positive on the way down. Therefore, the total work done by gravity is zero. This is in fact a more general property: the work done by gravity is **path-independent**, i.e. it only depends on the endpoints but not on the path which was taken between them.

Furthemore, gravity is a **conservative force**, as it can be expressed as

$$\mathbf{F} = -\boldsymbol{\nabla} V, \tag{25}$$

where $V = V(\mathbf{r})$ is some scalar function of the position. For conservative forces

$$W_{12} = -\int_1^2 (\boldsymbol{\nabla} V) \cdot \mathrm{d}\mathbf{r} = V_1 - V_2 = -\Delta V \tag{26}$$

the work done along a closed path is zero

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \tag{27}$$

Combining Eq.'s 24 and 26 results in the statement of energy conservation for conservative forces

$$W_{12} = T_2 - T_1 = V_1 - V_2 \qquad \Rightarrow \qquad T_2 + V_2 = T_1 + V_1 \tag{28}$$

Some common examples of conservative forces are the gravitational force between two massive objects

$$\mathbf{F}_{12}^G = \frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12} \qquad \Rightarrow \qquad V = -\frac{G m_1 m_2}{r_{12}}, \tag{29}$$

(where $\mathbf{r}_{12}$ is the vector from object 1 to object 2, and G is Newton's constant) and the electrostatic force between two charged objects

$$\mathbf{F}^E_{12} = -\frac{k_e q_1 q_2}{r_{12}^2}\hat{r}_{12} \qquad \Rightarrow \qquad V = \frac{k_e q_1 q_2}{r_{12}}. \tag{30}$$

A pecilular example is the magnetic force given by $\mathbf{F}_{mag} = q\mathbf{v}\times\mathbf{B}$. The work done by the magnetic force is

$$W_{12} = \int_1^2 (\mathbf{F}_{mag}\cdot\mathbf{v})\mathrm{d}t = 0, \tag{31}$$

as the direction of the magnetic force is always perpendicular to the velocity. The magnetic force is thus trivially path independent. In fact the magnetic force can be obtained from a potential function $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$.

An example of a non-conservative force is friction (as well as other dissapative forces) which can be expressed as $\mathbf{F}_f = -b\mathbf{v}$ or $-c\mathbf{v}v$ depending on the situation.

3 Calculus of Variations

Previously, we have specified the Euler-Lagrange equations by fiat. However, we shall see that they can be obtained from **Hamilton's principle**, which states that the motion of a system extremizes the **action functional**

$$S = \int_{t_1}^{t_2} L\,\mathrm{d}t, \qquad S = S[q_i(t), \dot{q}_i(t), t]. \tag{32}$$

In almost all cases of pratical interest we are concerned with finding the minimum of the action functional, so this is sometimes called the **principle of least action**.

In order to find an equation which describes the q_i which minimize the action, we need to introduces techniques in the subject of **calculus of variations**. Generally we shall examine a functional

$$J = \int_{x_1}^{x_2} f(y(x), y'(x), x)\,\mathrm{d}x. \tag{33}$$

Given the extremal function $y(x)$, a 'neighboring' function $y(x,\alpha)$ is defined by

$$y(x,\alpha) = y(x) + \alpha\eta(x) \qquad \Rightarrow \qquad y'(x,\alpha) = y'(x) + \alpha\eta'(x), \tag{34}$$

where $\eta(x)$ is an arbitrary smooth function such that $\eta(x_1) = \eta(x_2) = 0$. Since y extremizes J, we have

$$\frac{\partial J}{\partial\alpha} = 0 = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)\mathrm{d}x. \tag{35}$$

We note that the second term can be transformed into

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y'}\eta'\right)\mathrm{d}x = \frac{\partial f}{\partial y'}\eta\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x = -\int_{x_1}^{x_2}\eta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x, \tag{36}$$

where we have used the vanishing of η at the endpoints. Now we substitute Eq. 36 in to Eq. 35 to get

$$\frac{\partial J}{\partial \alpha} = 0 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) \right] \eta \, \mathrm{d}x. \tag{37}$$

In order for the r.h.s. to vanish for all functions η, we demand that the quantity in brakets vanishes, i.e.

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) = 0, \tag{38}$$

which is precisely the Euler-Lagrange equations.

We next derive an alternative but equivalent form of the Euler-Lagrange equation. We first note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'} \right) = y'' \frac{\partial f}{\partial y'} + y' \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \tag{39}$$

and that

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial x} \\ \Rightarrow \qquad \frac{\partial f}{\partial y'} y'' &= \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\partial f}{\partial y} y' - \frac{\partial f}{\partial x} \end{aligned} \tag{40}$$

Substituting this into Eq. 39 and using the Euler-Lagrange equation to eliminate two terms gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'} \right) &= \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\partial f}{\partial x} \\ \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} &= 0. \end{aligned} \tag{41}$$

This is sometimes called the **Hamiltonian form** of the Euler-Lagrange equation. Note that this implies that if $\frac{\partial f}{\partial x} = 0$ (that is if f has no explicit x-dependence), then the quantity

$$H = y' \frac{\partial f}{\partial y'} - f \tag{42}$$

is independent of x. In the context of classical mechanics, this quantity is known as the **Hamiltonian**, and it is conserved (independent of time) if L has no explicit time dependence.

We can apply these variational techniques to the **Brachistichrone problem**, which concerns finding the path of motion which minimizes the time. Specifically we consider an object moving a gravitational field. Choose the x-axis to point in the negative vertical direction (i.e. 'down') and the y-axis in the positive horizontal direction ('to the right'). The path traveled is specified by the endpoints (x_1, y_1) and (x_2, y_2) as well as the curve $y(x)$.

To construct a functional which corresponds to the total time, we first note that since energy is conserved

$$\frac{m}{2}v^2 - mgx = 0 \qquad \Rightarrow \qquad v = \sqrt{2gx} = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|. \tag{43}$$

Next we note that

$$|\mathrm{d}\mathbf{r}| = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1+y'^2} \tag{44}$$

so that the total time is given by

$$t = \int_1^2 \frac{|\mathrm{d}\mathbf{r}|}{v} = \int_1^2 \sqrt{\frac{1+y'^2}{2gx}}\,\mathrm{d}x. \tag{45}$$

Since the integrand has no explicit y-dependence, we know that

$$\frac{\partial f}{\partial y'} = const. = (2a)^{-1/2} \tag{46}$$

$$\begin{aligned}
\Rightarrow \qquad & \left(\frac{1+y'^2}{x}\right)^{-1/2}\frac{y'}{x} = (2a)^{-1/2} \\
\Rightarrow \qquad & \frac{y'^2}{(1+y'^2)x} = \frac{1}{2a} \\
\Rightarrow \qquad & 2ay'^2 = (1+y'^2)x \\
\Rightarrow \qquad & y'^2(2a-x) = x \\
\Rightarrow \qquad & y' = \frac{x}{\sqrt{2ax-x^2}} \\
\Rightarrow \qquad & y = \int \frac{x\,\mathrm{d}x}{\sqrt{2ax-x^2}}.
\end{aligned}$$

We now make the substitution $x = a(1-\cos\theta)$ so that $\mathrm{d}x = a\sin\theta\mathrm{d}\theta$. This leads to

$$y = \int a(1-\cos\theta)\mathrm{d}\theta = a(\theta - \sin\theta) + c. \tag{47}$$

The parametric curve defined by $y = a(\theta-\sin\theta)$, $x = a(1-\cos\theta)$ is a **cycloid**.

We can generalize this derivation of the Euler-Lagrange equation to functions of several dependent variables, namely $f = f\big[y_i(x), y_i'(x); x\big]$. We similarly introduce variations $y_i(x,\alpha) = y_i(x) + \alpha\eta_i(x)$, where the η_i are independent. We now construct the variation of J

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'}\right)\eta_i(x)\,\mathrm{d}x. \tag{48}$$

Since the η_i are independent, the terms in the integrand must vanish seperately for each i

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} = 0, \qquad \forall i = 1, \ldots, n \tag{49}$$

We now consider extremization with constraints. For example, if we wish to find the shortest path between two points on a surface, the path must satisfy the equation of the surface, i.e. $g(y_i; x) = 0$. Specifically, if the surface is a sphere of radius ρ, we have $g = \sum_i x_i^2 - \rho^2 = 0$. Let us examine in detail the case of two dependent variables. Here we have $f = f(y, z, y', z'; x)$ and $g = g(y, z; x) = 0$. As before we introduce the variations

$$y(x,\alpha) = y(x) + \alpha\eta_1(x), \qquad z(x,\alpha) = z(x) + \alpha\eta_2(x) \tag{50}$$

and write the variation of J as

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right)\eta_1 + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right)\eta_2 \right] \mathrm{d}x \tag{51}$$

Now the variation must also satisfy the constraint, so that

$$\frac{\mathrm{d}g}{\mathrm{d}\alpha} = \frac{\partial g}{\partial y}\eta_1 + \frac{\partial g}{\partial z}\eta_2 = 0. \tag{52}$$

We see that because of the constraint, the variations η_1 and η_2 are no longer independent, but are related by

$$\frac{\partial g}{\partial y}\eta_1 = -\frac{\partial g}{\partial z}\eta_2. \tag{53}$$

We now use this to eliminate η_2 in favor of η_1 in Eq. 51

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right)\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \right] \eta_1 \, \mathrm{d}x. \tag{54}$$

Thus in order for the variation of J to vanish for arbitrary η_1, we have

$$\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right)\left(\frac{\partial g}{\partial z}\right)^{-1}. \tag{55}$$

Since this equation must hold for all values of x, we can introduce a separation constant $\lambda(x)$ called the **Lagrange multiplier** $\lambda(x)$ and write

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda\frac{\partial g}{\partial y} = 0 \tag{56}$$

$$\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} + \lambda\frac{\partial g}{\partial z} = 0. \tag{57}$$

These two equations together with the equation of constraint $g = 0$ provide three equations for the three unknowns y, z, λ.

The most general constraints are those in integral form

$$K[y_i] = \int_{x_1}^{x_2} g(y_i, y_i'; x)\,\mathrm{d}x = const. \tag{58}$$

Note that constraints of the form $g = 0$ are a special case of integral constraints. There is a theorem of calculus of variations that states that such constraints are equivalent to extremizing the functional

$$G = \int_{x_1}^{x_2} (f + \lambda g)\,\mathrm{d}x \tag{59}$$

according to the usual methods. This leads to the modified form of the Euler-Lagrange equation in the presence of a constraint

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y'}\right) = 0 \tag{60}$$

Although they are not so common in classical mechanics, a common example of an integral constraint is furnished by the **Dido problem**. The version we shall consider here seeks to maximize the area enclosed by a curve $y(x)$, $y(-a) = y(a) = 0$ and the x-axis subject to the constraint that the length of the curve is fixed ($= \ell$). This constraint can be cast in integral form by noting

$$\ell = \int_{-a}^{a} \mathrm{d}\ell = \int_{-a}^{a} \sqrt{1 + y'^2}\,\mathrm{d}x. \tag{61}$$

The function we are seeking to extremize is the total area given by

$$J = \int_{-a}^{a} y\,\mathrm{d}x \tag{62}$$

so that we may find the extremum by solving the equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y}, \qquad F = f - \lambda g = y - \lambda\sqrt{1 + y'^2}. \tag{63}$$

Note that F has no explicit x dependence so that

$$\begin{aligned}
& y'\frac{\partial F}{\partial y} - F = const. = c_1 \qquad (64) \\
& \Rightarrow y'\left(\frac{-\lambda y'}{\sqrt{1 + y'^2}}\right) - y + \lambda\sqrt{1 + y'^2} = c_1 \\
& \Rightarrow -\lambda y'^2 - y\sqrt{1 + y'^2} + \lambda(1 + y'^2) = c_1\sqrt{1 + y'^2} \\
& \Rightarrow \lambda = (c_1 + y)\sqrt{1 + y'^2} \\
& \Rightarrow y' = \pm\sqrt{\frac{\lambda^2 - (c_1 + y)^2}{(c_1 + y)^2}} \\
& \Rightarrow \int \frac{c_1 + y}{\sqrt{\lambda^2 - (c_1 + y)^2}}\mathrm{d}y = \pm\int dx = \pm x + c_2
\end{aligned}$$

We make the substitution $y + c_1 = \lambda \cos\theta$, so that

$$-\int \lambda \cos\theta = x + c_2 \qquad \Rightarrow \qquad x + c_2 = -\lambda \sin\theta. \tag{65}$$

The equations $y+c_1 = \lambda\cos\theta$ and $x+c_2 = \lambda\sin\theta$ describe a circle, as $(x+c_2)^2+(y+c_1)^2 = \lambda^2$. Because of the constraints, we have that $c_1 = c_2 = 0$ and that $\lambda = a = \frac{\ell}{\pi}$.

We now introduce a short-hand for the variational notation that we have been using

$$\delta J \equiv \frac{\mathrm{d}J}{\mathrm{d}\alpha}, \qquad \delta y \equiv \frac{\mathrm{d}y}{\mathrm{d}\alpha}. \tag{66}$$

Using this notation, the extremum condition can be written

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) \delta y \, \mathrm{d}x = 0 \tag{67}$$

4 Lagrangian Mechanics

We now apply the ideas develop in the previous section to classical mechanics. From the preceeding discussion we see that Hamilton's principle ($\delta S = 0$) directly implies the Euler-Lagrange equations. Also, from the Hamiltonian form of the equations we see that the Hamiltionian

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{68}$$

is conserved if L has no explicit time dependence, i.e. if

$$\frac{\partial L}{\partial t} = 0. \tag{69}$$

We now examine generalized coordinates in more detail. Consider the work done by a force $\mathbf{F}$ due to an infinitesimal displacement $\mathrm{d}\mathbf{x}$

$$\mathrm{d}W = F_i \, \mathrm{d}x_i, \tag{70}$$

where the dot product (i summed over) is over the Cartesian components of the vectors. Introducing generalized coordinates $\{q_j\}$ amounts to expressing the Cartesian coordinates as functions of the generalized coordinates (and possibly time)

$$x_i = x_i(q_j, t) \tag{71}$$

such a transformation is known as a **point transformation**. The infinitesimal work now becomes

$$\mathrm{d}W = F_i \frac{\partial x_i}{\partial q_j} \mathrm{d}q_j \tag{72}$$

$$\mathrm{d}W = Q_j \, \mathrm{d}q_j$$

Where the Q_j are **generalized forces**, expressed as

$$Q_j = F_i \frac{\partial x_i}{\partial q_j}. \tag{73}$$

Note that while the generalized forces appear in the equation for infinitesimal work a similar way to ordinary forces, they don't necessarily have the units of a force. However, the units of the product $Q_j \, \mathrm{d}q_j$ must be units of work. An example of this is torque, where $\mathrm{d}q_i = \mathrm{d}\theta$.

If $\mathbf{F}$ is a conservative force, we have

$$F_i = -\frac{\partial V}{\partial x_i} \tag{74}$$

so that

$$Q_j = -\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{75}$$

Provided that generalized coordinates are such that the kinetic energy only depends on $\dot{q}_i$, we have

$$Q_j = \frac{\partial L}{\partial q_j} \qquad \Rightarrow \qquad Q_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}, \tag{76}$$

where in the last step we have used the Euler-Lagrange equations. This motivates introduction of **generalized momenta**

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{77}$$

so that the generalized forces can be expressed as

$$\dot{p}_i = Q_i \tag{78}$$

which is similar in form to Newton's second law. We see now that if L is independent of q_i, the generalized force $Q_i = 0$. In this situation the generalized momentum is constant $\dot{p}_i = 0$, and q_i is referred to as a **cyclic coordinate**.

An example of this is the simple pendulum (under the influence of gravity) with a freely falling pivot point. We consider the motion in a plane and point the y-axis up and the x-axis to the right. The position of the pendulum bob is given by $(x_s + x, y_s - y)$, where (x_s, y_s) is the position of the pivot point. Note that in the usual way x and y are constrained such that

$$x = \ell \sin\theta, \qquad y = \ell\cos\theta \tag{79}$$

where θ is the angle that the rod makes with the vertical. The kinetic and potential energies can now be expressed as

$$T = \frac{m}{2}\left[(\dot{x}_s + \dot{x})^2 + (\dot{y}_s - \dot{y})^2\right], \qquad V = mg(y_s - y) \tag{80}$$

so that the Lagrangian is

$$L = \frac{m}{2}\left[\dot{x}_s^2 + \dot{y}_s^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{\theta}(\dot{x}_s\cos\theta + \dot{y}_s\sin\theta)\right] - mg(y_s - \ell\cos\theta). \tag{81}$$

Although this lagrangian looks somewhat complicated, there is no x_s-dependence so the generalized momentum is conserved

$$\frac{\partial L}{\partial \dot{x}_s} = const. = m\dot{x}_s + m\ell\dot{\theta}\cos\theta. \tag{82}$$

This is simply the conservation of linear momentum in the x-direction, which is due to the absence of any external forces in that direction.

This formalism may be extendend to include constraints. Recall that for a system with n generalized coordinates and m constraints ($n - m$ degrees of freedom), we must extremize the functional

$$F = L + \sum_{a=1}^{m} \lambda_a f_a, \tag{83}$$

where the λ_a are Lagrange multipliers and the equations of constraint are $f_a(q_i, \dot{q}_i, t) = 0$. The Euler-Lagrange equations that result can be expressed as

$$\begin{aligned} \dot{p}_i &= Q_i + \sum_a \lambda_a \left(\frac{\partial f_a}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_a}{\partial \dot{q}_i}\right) \\ &= Q_i + Q_i'. \end{aligned} \tag{84}$$

Note that we have introduced the **generalized forces of constraint** Q_i'. These forces are the magnitude of the forces required to make the generalized coordinates satisfy the constraints. The direction of these forces is apparent from the problem at hand.

As an example we consider again the simple pendulum. With the y-axis pointed down as before, we have

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta. \tag{85}$$

The constraint is $f = r - \ell = 0$, which must be added to the Lagrangian with a Lagrange multiplier λ. Note that in the previous treatment of the simple pendulum, the constraint was employed from the very beginning, to eliminate r from the Lagrangian. However, since we are now interested in the magnitude of the force of constraint, we leave r in the Lagrangian, but add a Lagrange multiplier together with the constraint.

The Euler-Lagrange equation for θ is unchanged from the previous treatment

$$mr^2\ddot{\theta} + mgr\sin\theta = 0, \tag{86}$$

but now we have an additional Euler-Lagrange equation for r

$$m\ddot{r} + mr\dot{\theta}^2 + mg\cos\theta + \lambda = 0. \tag{87}$$

Together with the equation of constraint, this gives us three equations for the three unknowns r, θ, λ. Applying the constraint equation reduces the θ equation to the well known form

$$\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0 \tag{88}$$

and the r equation to

$$m\ell\dot{\theta}^2 + mg\cos\theta + \lambda = 0. \tag{89}$$

We now note that

$$\ddot{\theta} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\dot{\theta}^2}{2}\right) \tag{90}$$

The equation for θ now becomes

$$\mathrm{d}\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{g}{\ell}\sin\theta\mathrm{d}\theta \qquad \Rightarrow \qquad \frac{\dot{\theta}^2}{2} = \frac{g}{\ell}\cos\theta + c \tag{91}$$

where c is an integration constant that depends on the initial conditions of the problem. Substituting this into the equation for r gives

$$\lambda = 3mg\cos\theta - d \tag{92}$$

where d is a different constant which is set by the initial conditions. Finally to get the magnitude of the tension force of constraint, we note that

$$Q_i' = \lambda\left(\frac{\partial f}{\partial r}\right) = \lambda. \tag{93}$$

Note that in this example we had the choice to either enforce the constraint from the beginning and reduce the number of degrees of freedom or keep both degrees of freedom and introduce the constraint together with a Lagrange multiplier.Clearly, if we are interested in the magnitude of the force of constraint, we should do the latter, while if we care only about the motion of the system we should do the former.

This choice is only possible for a special kind of constraint, called a **holonomic** constraint that can be expressed as

$$f(q_i, t) = 0. \tag{94}$$

The general constraints we have been considering so far are **semi-holonomic** as they can be written

$$f(q_i, \dot{q}_i, t) = 0. \tag{95}$$

Of course, there can be constraints that don't fit into either of these forms (**non-holonomic**) but in practice these are difficult to treat.

Another common type of constraint is illustrated in the following example. Consider a disk of radius R **rolling without slipping** down an inclined ramp of length ℓ which makes and angle α with the horizontal. In general, the degrees of freedom in this problem are the motion of the center of the disk and its rotational motion. rotational motion. The disk is constrained to be on the ramp by the normal force provided by the ramp, in which we are not interested. Therefore we choose degrees of freedom y, which represents the parallel distance of the center of the disk from the top of the ramp, and θ which represents the angle of rotation of the disk from its initial configuration.

The constraint of rolling without slipping relates y and θ

$$f = y - R\theta = 0 \tag{96}$$

which is holonomic. The kinetic and potential energies are

$$T = \frac{m}{2}\dot{y}^2 + \frac{I}{2}\dot{\theta}^2, \qquad V = mq(\ell - y)\sin\alpha \tag{97}$$

Since the constraint is holonomic, we have the option Of substituting the equation of constraint immediately and using one degree of freedom in the Lagrangian or introducing a Lagrange multiplier λ together with the constraint equation to gain information about the magnitude of the force of constraint, which for rolling without slipping is provided by the static frictional force.

Of course, for this example it's not crucial to know about the constraint force. However, for the following example the magnitude of the constraint force is paramount. Consider a mass sliding off the top of a frictionless hemisphere of radius R. Call θ the angle of the position with respect to the vertical. We are interested in determining the angle at which the mass fall off of the hemisphere. The constraint is therefore

$$f = r - R = 0, \tag{98}$$

which is holonomic.

First, it is important to note that in this case the constraint force is provided by the normal force, which acts perpendicular to the surface of the hemisphere. The mass leaves the hemisphere at the instant that this normal force vanishes. The magnitude of the constraint force is certainly needed, so we introduce it together with a Lagrange multiplier, while working with two degrees of freedom (r, θ). The kinetic and potential energies are

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2), \qquad V = mgr\cos\theta \tag{99}$$

The Euler-Lagrange equation for r gives

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{100}$$

while the one for θ gives

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \tag{101}$$

Applying the constraint equation to the equation for θ gives

$$\ddot{\theta} = \frac{g}{R}\sin\theta \tag{102}$$

Applying Eq. 90 and integrating gives

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{R}\cos\theta + c. \tag{103}$$

To fix the integration constant c, we note that at $t = 0$, $\theta = \dot{\theta} = 0$, as the mass starts from rest. Therefore $c = \frac{g}{R}$. Substituting this into the equation for r gives

$$\lambda = mg(3\cos\theta - 2) \tag{104}$$

Since $\frac{\partial f}{\partial r} = 1$, the magnitude of the constraint force is given by λ, and we see that it vanishes when

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \tag{105}$$

which is the angle at which the mass leaves the hemisphere. Note that this is independent of g, as well as the radius of the hemisphere.

We can infer some information about the motion a long time after the mass leaves the hemisphere, assuming that the energy lost in the collision with the ground is negligible, and that the mass slides on the ground without slipping as well. Note that since the Lagrangian has no explicit time dependence, the energy is conserved, so $E_i = E_f$. At the initial time, all the energy is potential energy, while at the final time, it is all kinetic

$$mgR = \frac{m}{2}v^2 \qquad \Rightarrow \qquad v = \sqrt{2gR} \tag{106}$$

We now take a more in-depth look at these conservation laws. Consider a Lagrangian expressed in terms of Cartesian coordinates $L[x_i, \dot{x}_i]$. Furthermore, consider the **space translation** defined by the vector $\boldsymbol{\alpha}$

$$x_i \to x_i + \alpha_i \tag{107}$$

If L is invariant under this transformation we have that $\frac{\partial L}{x_i} = 0$. Therefore, x_i is a cyclic coordinate and

$$\frac{\partial L}{\partial \dot{x}_i} = const. \tag{108}$$

Since $T = \frac{m}{2}\sum_i x_i^2$, we have that $m\dot{x}_i$ is conserved. Therefore, the **invariance of the Lagrangian under space translations results in linear momentum conservation**.

We have already seen that the lack of an explicit time dependence in L means that H is conserved, or that the invariance of L under time translations means the H is conserved. Let us examine this in more detail. Generally, for a system of n particles described by cartesian coordinates, we have that the kinetic energy is

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha i}^2. \tag{109}$$

When we change to generalized coordinates, we express $x_{\alpha i} = x_{\alpha i}(q_j, t)$ so that

$$\dot{x}_{\alpha i} = \frac{\partial x_{\alpha i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha i}}{\partial t} \tag{110}$$

and (no sum over α)

$$\dot{x}_{\alpha i}^2 = \left(\frac{\partial x_{\alpha i}}{\partial q_j} \dot{q}_j \right) \left(\frac{\partial x_{\alpha i}}{\partial q_k} \dot{q}_k \right) + 2 \left(\frac{\partial x_{\alpha i}}{\partial q_j} \dot{q}_j \right) \frac{\partial x_{\alpha i}}{\partial t} + \left(\frac{\partial x_{\alpha i}}{\partial t} \right)^2. \tag{111}$$

Therefore the kinetic energy can be written

$$\begin{aligned} T &= \left(\sum_\alpha \frac{m_\alpha}{2} \frac{\partial x_{\alpha i}}{\partial q_j} \frac{\partial x_{\alpha i}}{\partial q_k} \right) \dot{q}_j \dot{q}_k + \left(\sum_\alpha m_\alpha \frac{\partial x_{\alpha i}}{\partial q_j} \frac{\partial x_{\alpha i}}{\partial t} \right) \dot{q}_j + \sum_\alpha \frac{m_\alpha}{2} \frac{\partial x_{\alpha i}}{\partial t} \\ &= a_{jk} \dot{q}_j \dot{q}_k + b_j \dot{q}_j + c, \end{aligned} \tag{112}$$

where the a, b, and c depend generally on the q_i and t. As an example consider 2D polar coordinates where

$$T = \frac{m}{2} (r^2 \dot{\theta}^2 + \dot{r}^2) \tag{113}$$

The transformation from Cartesian to 2D polar coordinates evidently results in $b_j = c = 0$. This is because the transformation $x_i = x_i(r, \theta)$ does not depend on time, so that the temporal derivatives $\frac{\partial x_{\alpha i}}{\partial t}$ vanish. Transformations of this type $x_i = x_i(q_j)$ with no explicit t-dependence are said to be **scleronomous**. For scleronomous generalized coordinates the kinetic energy can be written as

$$T = a_{jk} \dot{q}_j \dot{q}_k, \tag{114}$$

where the a depend possibly on the q_i. Furthermore, for scleronomous generalized coordinates we have

$$\dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = a_{ik} \dot{q}_i \dot{q}_k + a_{ki} \dot{q}_k \dot{q}_i = 2T. \tag{115}$$

This is a special case of Euler's theorem, which states that if $f(y_k)$ is a homogeneous function of degree n, that is $f(\alpha y_k) = \alpha^n f(y_k)$, then

$$y_k \frac{\partial f}{\partial y_k} = nf. \tag{116}$$

Recall that if L lacks explicit time dependence, the quantity

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = const. \tag{117}$$

If we have scleronomous generalized coordinates and $V = V(q_i)$, then

$$H = \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - (T - V) = 2T - T + V = T + V = const. \tag{118}$$

Therefore, for scleronomous generalized coordinates and $V = V(q_i)$, the Hamiltonian is the total energy of the system, which is conserved if the Lagrangian lacks explicit time dependence. The lack of explicit time dependence is equivalent to the invariance of L under **time translations**

$$t \to t + \alpha \tag{119}$$

so that for scleronomous generalized coordinates and $V = V(q_i)$, **the invariance of the Lagrangian under time translations results in energy conservation**.

Finally, we consider rotations. An infinitesimal rotation on a vector $\mathbf{r}$ by a infinitesimal angle $\delta\theta$ is effected by

$$\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}, \qquad \delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r} \tag{120}$$

The vector $\delta\boldsymbol{\theta}$ has magnitude $\delta\theta$ and a direction given by the right hand rule. Expressing the vector $\delta\mathbf{r}$ in Cartesian coordinates, the invariance of L under rotations implies

$$\begin{aligned} \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i &= 0 \\ \dot{p}_i \delta x_i + p_i \delta \dot{x}_i &= 0 \\ \dot{\mathbf{p}} \cdot (\delta\boldsymbol{\theta} \times \mathbf{r}) + \mathbf{p} \cdot (\delta\boldsymbol{\theta} \times \dot{\mathbf{r}}) &= 0 \\ \delta\boldsymbol{\theta} \cdot (\dot{\mathbf{p}} \times \mathbf{r} + \mathbf{p} \times \dot{\mathbf{r}}) &= 0. \end{aligned} \tag{121}$$

Since the infinitesimal rotation $\delta\boldsymbol{\theta}$ is arbitrary, we must have

$$\dot{\mathbf{p}} \times \mathbf{r} + \mathbf{p} \times \dot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \times \mathbf{r}) = 0 \qquad \Rightarrow \qquad \dot{\mathbf{L}} = 0. \tag{122}$$

Therefore, **if the Lagrangian is invariant under rotations, the angular momentum is conserved**.

The transformations we have discussed (space translations, time translations, and rotations) are continuous transformations as they depend on a set of continuous parameters. We have seen that the invariance of the Lagrangian under these continuous transformations results in a conserved quantity. The most general form of this statement is codified in **Noether's Theorem**, which is one of the most important results in theoretical physics.

To recap, there are **seven possible constants of the motion** in a mechanics problem, the three components of the angular momentum, the three components of the linear momentum, and the energy. In order to examine the transformations which led to these conservation laws in more detail, we differentiate between two viewpoints. The **active transformation** view is that the object in question is actually transformed, while the **passive transformation** viewpoint merely transforms the coordinate axes which are used to describe the object. Both viewpoints are mathematically equivalent.

We have discussed several different types of transformations. **Rotations** transform the cartesian components of a vector according to

$$x_i \to x_i' = \lambda_{ij} x_j, \qquad \lambda_{ij} = \cos\theta_{ij} \tag{123}$$

where (taking the passive viewpoint) θ_{ij} is the angle between the x_i'-axis and the x_j-axis. The matrix λ is called the matrix of **direction cosines**. Clearly, for no transformation $\theta_{ii} = 0$ and $\theta_{ij} = \frac{\pi}{2}$ for $i \neq j$, so that $\lambda_{ij} = \delta_{ij}$.

Since rotations preserve the length of vectors, we have

$$\lambda_{ij}\lambda_{ik} x_j x_k = x_\ell x_\ell \tag{124}$$

so that $\lambda_{ij}\lambda_{ik} = (\lambda^T)_{ji}\lambda_{ik} = \delta_{jk}$. Matricies which satisfy $AA^T = I$ are said to be **orthogonal**.

Additionally, we discussed space and time **translations** in which

$$x_i' = x_i + b_i, \qquad t' = t + c. \tag{125}$$

We can also consider **boosts**, in which the space and time coordinates are transformed into

$$x_i' = x_i + v_i t, \qquad t' = t. \tag{126}$$

This transformation transforms e.g. between a coordinate system at rest and one moving with constant velocity v_i. We have seen that the invariance of the Lagrangian under rotations and translations led to angular momentum, linear momentum and energy conservation. Since $L = T - V$ and the kinetic energy would certainly be changed by a boost, it is unlikely that the Lagrangian will be invariant under a boost.

For example, consider a free particle in one dimension, with $L = \frac{m}{2}\dot{x}^2$. Under a boost the Lagrangian becomes

$$L \to \frac{m}{2}(\dot{x}^2 + v^2 + 2\dot{x}v) \tag{127}$$

This leads to the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x} + v) = 0 \qquad \Rightarrow \qquad m\ddot{x} = 0, \tag{128}$$

which is the same as the original equation of motion! Therefore, the equation of motion was unaffected by the boost, even though L was not invariant. If we view this boost in the passive sense, the equation of motion is the same in both frames of reference. The equivalence of the equations of motion in different reference frames is called **frame covariance**. Reference frames which are connected to each other by rotations, translations, and boosts are called **inertial reference frames**.

With the addition of boosts, we have that the most general transformation is

$$x_i' = \lambda_{ij} x_j + b_i + v_i t, \qquad t' = t + c \tag{129}$$

This transformation is specified by 10 parameters: 3 rotation angles, the 3 components of the space translation, the three components of the boost, and the 1 time translation parameter. These transformations form a group, the **Galilei group**.

We now return to the fact that the boost described about did not leave the Lagrangian unchanged by did not alter the equations of motion. This is actually a more general property. Consider adding a total derivative term to the Lagrangian

$$L \to L' = L(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(q, t) = L + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q}. \tag{130}$$

The equation of motion becomes

$$\frac{\partial L}{\partial q} + \frac{\partial^2 f}{\partial q \partial t} + \frac{\partial^2 f}{\partial q^2}\dot{q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q}\right) = 0. \tag{131}$$

We now note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q}\right) = \frac{\partial^2 f}{\partial q \partial t} + \frac{\partial^2 f}{\partial q^2}\dot{q} \tag{132}$$

so that the equation of motion reduces to the Euler-Lagrange equation for L. Therefore we see that the addition of the total time derivative term had no effect on the equations of motion.

Returning to the boost, we see that it resulted in the addition of a constant v^2 as well as $m\dot{x}v$ to the Lagrangian. The latter can be written as the total time derivative of the function $f = mxv$.

We now turn to other transformations of the Lagrangian that do not affect the equation of motion. Clearly, a rescaling of the Lagrangian by a constant

$$L \to L' = \alpha L \tag{133}$$

has no effect on the equations of motion, as the constant can be factored out of the derivatives. This leads us to the principle of **Mechanical Similarity**. Assume that the potential has the form

$$V(\alpha x_i) = \alpha^k V(x_i). \tag{134}$$

Furthermore, assume that the spatial coordinates and the time are rescaled according to

$$x_i \to \alpha x_i, \qquad t \to \beta t \tag{135}$$

so that $\dot{x}_i \to \frac{\alpha}{\beta}\dot{x}_i$. Under these transformations, the kinetic and potential energies transform as

$$T \to \frac{\alpha^2}{\beta^2} T, \qquad V \to \alpha^k V. \tag{136}$$

Therefore, if

$$\frac{\alpha^2}{\beta^2} = \alpha^k \qquad \Rightarrow \qquad \beta = \alpha^{1-\frac{k}{2}} \tag{137}$$

then the Lagrangian transforms as $L \to \alpha^k L$ and the equations of motion are not affected. To recap, given a solution to the equations of motion $x_i(t)$, an additional solution can be obtained by substituting $t \to t'$ and $x_i \to x_i'$ provided

$$\frac{t'}{t} = \left(\frac{x_i'}{x_i}\right)^{1-\frac{k}{2}}, \tag{138}$$

which results in

$$\frac{\dot{x}_i'}{\dot{x}_i} = \left(\frac{x_i'}{x_i}\right)^{\frac{k}{2}}, \qquad \frac{E'}{E} = \left(\frac{x_i'}{x_i}\right)^{k}, \qquad \frac{L_i'}{L_i} = \left(\frac{x_i'}{x_i}\right)^{1+\frac{k}{2}}. \tag{139}$$

As an example, we consider **small oscillations**, where we expand the potential $V = V(x_i)$ around a local minimum, with $\frac{\partial V}{\partial x_i} = 0$

$$V(x_i) \approx c + d_{ij} x_i x_j. \tag{140}$$

Note that the constant c is irrelevant, so that this potential has $k = 2$. The characteristic time for small oscillations is given by the period p, while the characteristic length is given by the amplitude ℓ. Upon rescaling we have

$$\frac{p'}{p} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{2}{2}} = 1 \tag{141}$$

so that the period of small oscillations is independent of the amplitude.

Another example is a uniform field of force, such as gravity where the potential has $k = 1$. Here we have

$$\frac{t'}{t} = \sqrt{\frac{\ell'}{\ell}} \tag{142}$$

so that e.g., the time of a fall is equal to the square root of the height.

Finally, we consider a potential with $k = -1$, such as the gravitational or electrostatic potential, and

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{3/2} \tag{143}$$

which is simply Kepler's third law of planetary motion, that the period is proportional to the size of the orbit to the 3/2 power.

The idea of mechanical similarity can be used to derive additional results. Recall that from Euler's theorem

$$\dot{x}_i \frac{\partial T}{\partial \dot{x}_i} = 2T = \dot{x}_i p_i = \frac{\mathrm{d}}{\mathrm{d}t}(p_i x_i) - x_i \dot{p}_i. \tag{144}$$

We now define the **time average** of a function to be

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} f(t)\,\mathrm{d}t. \tag{145}$$

Note that if f is the total time derivative of a function F, then $\bar{f} = \lim_{\tau\to\infty} \frac{F(\tau)-F(0)}{\tau}$. If F is bounded function then $\bar{f} = 0$. If the motion of a system is confined to a bounded region of space, $p_i x_i$ bounded so that

$$2\bar{T} = -\overline{x_i \dot{p}_i} = \overline{x_i \frac{\partial V}{\partial x_i}} \tag{146}$$

Therefore, if V is a homogenous function of degree k, we have

$$2\bar{T} = k\bar{V} \tag{147}$$

This result is known as the **Virial Theorem**. Since energy is conserved, we have $\bar{T}+\bar{V} = E$ so that

$$\bar{V} = \frac{2E}{k+2}, \qquad \bar{T} = \frac{kE}{k+2} \tag{148}$$

An important use of the Virial Theorem concerns a gas of N atoms confined to a rigid container of volume V, at a temperature T. We shall employ the **equi-partition theorem** which states

$$\bar{T} = \frac{3}{2} N k_B T, \tag{149}$$

i.e. that the average kinetic energy is equal to $k_B T/2$ times the number of degrees of freedom. If we assume that the gas is dilute, so that the interactions between particles are negligible, employing the virial theorem (with $k = 1$) gives

$$3Nk_B T = -\int \mathbf{F} \cdot \mathrm{d}\mathbf{r} = P \int \mathbf{n} \cdot \mathbf{r} \, \mathrm{d}A, \tag{150}$$

where P is pressure, defined as the force per area. Employing the divergence theorem gives

$$3Nk_B T = P \int \boldsymbol{\nabla} \cdot \mathbf{r} \, \mathrm{d}V = 3VP \tag{151}$$

Thus we have derived the **ideal gas law**

$$PV = Nk_B T. \tag{152}$$

The last general technique we shall discuss concerns the integration of the equations of motion. We consider motion in one dimension, so that the Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 - V(x) \tag{153}$$

Note that the lack of explicit time dependence ensures that energy is conserved.

$$E = \frac{m}{2}\dot{x}^2 + V(x) = const. \qquad \Rightarrow \qquad \dot{x} = \pm\sqrt{2(E-V)/m} \tag{154}$$

This can be cast as an integral equation

$$t - t_0 = \pm\sqrt{\frac{m}{2}} \int \frac{\mathrm{d}x}{\sqrt{E - U}} \tag{155}$$

This equation is sensible only if the argument of square root is positive. That is, the motion can only occur if $E > V(x)$. For a fixed value of E, the points x_i where $E = V(x_i)$ are called **turning points**, as the motion must be bounded by these points.

If the E and V are such that the motion is bounded between two turning points $[x_1, x_2]$, we can determin the period of the motion

$$p(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}. \tag{156}$$

Note that we have equated the motion from x_1 to x_2 with the motion from x_2 to x_1. This is true as the Lagrangian is invariant under **time reversal**

$$t \to -t. \tag{157}$$

Note that the Lagrangian is only invariant under **spatial inversion** (also called a **parity transformation**)

$$x \to -x \tag{158}$$

if $V(\alpha x) = \alpha^k V(x)$, with k even.

As an exmample we consider small oscillations, where the potential is approximated as $V(x) = cx^2$. For a fixed energy E the turning points are $\pm\sqrt{\frac{E}{c}}$, so the period becomes

$$p(E) = \sqrt{\frac{2m}{E}} \int_{-\sqrt{\frac{E}{c}}}^{\sqrt{\frac{E}{c}}} \frac{\mathrm{d}x}{\sqrt{1 - \frac{c}{E}x^2}}. \tag{159}$$

This can be integrated by the substitution

$$x = \sqrt{\frac{E}{c}} \sin\theta \qquad \Rightarrow \qquad \mathrm{d}x = \sqrt{\frac{E}{c}} \cos\theta \, \mathrm{d}\theta \tag{160}$$

so that

$$p = \pi\sqrt{\frac{2m}{c}} \tag{161}$$

Note that the period is independent of the energy, and that the familiar result for Hooke's Law is obtained for $c = \frac{k}{2}$.