MA2341 - Advanced Mechanics 1
Michelmas Term - 2015-2016
Homework 5 - Due Nov. 17th, 2015

1. Consider two particles of equal masses moving in one dimension subject to the potential $V(x_1^2 + x_2^2)$.

   (a) Find the Lagrangian and equation of motion.

   (b) Discuss any continuous symmetries and conserved quantities.

   (c) In particular, verify that the Lagrangian is invariant under rotations in the two-dimensional space in which x_1 and x_2 are the horizontal and vertical axes, respectively. Using Noether's theorem, construct the conserved quantity which corresponds to this 'internal' symmetry. Explicitly demonstrate that this quantity is indeed conserved from the equations of motion.

2. Properties of 3×3 rotation matrices:

   (a) In order preserve the magnitude of three-dimensional vectors, rotation matrices must be orthogonal (why?). Prove that the determinant of an orthogonal matrix is 1 or -1. Argue why orthogonal matrices with determinant -1 do not describe rotations.

   (b) Consider rotations around the x-, y-, and z-axes. Determine the rotation matrices that effect each of these rotations for finite as well as infinitesimal angles. Write each of the infinitesimal rotation matrices in the form

   $$R_i = I + \epsilon L_i, \quad i = 1, 2, 3$$

   where I is the 3×3 identity matrix and ϵ is the infinitesimal rotation angle. The 3×3 matrices L_i are called 'generators'. Prove that they satisfy the commutation relations inidicative of the Lie group $SO(3)$:

   $$[L_i, L_j] = \epsilon_{ijk} L_k.$$

   Note that the commutator of two matrices is $[A, B] = AB - BA$.

   (c) Do two infinitesimal rotations commute, i.e. can they be performed in any order? What about two finite rotations?

3. Consider a generic system of n particles described by the position vectors $\vec{r}_\alpha$, $\alpha = 1, \ldots n$. Furthermore assume the motion is periodic in some way, so that after a time τ the positions and velocities of the particles return to their original configuration.

(a) Consider the quantity

$$S = \sum_\alpha \vec{p}_\alpha \cdot \vec{r}_\alpha$$

and show that time average of the total time derivative of S vanishes when taken over a period, i.e. show that

$$\langle \frac{dS}{dt} \rangle = \frac{1}{\tau} \int_0^\tau \left(\frac{dS}{dt} \right) dt = 0$$

(b) Use this to prove the virial theorem:

$$\langle T \rangle = -\frac{1}{2} \langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \rangle,$$

where $\vec{F}_\alpha$ is the total force acting on particle α and T is the kinetic energy.

(c) Consider a particle moving in periodically in one dimension under the influence of a power-law potential of the form $V(x) = kx^{n+1}$. Prove (using the virial theorem) that

$$\langle T \rangle = \frac{n+1}{2} \langle V \rangle.$$