564 - Stochastic Methods
Michelmas Term - 2014-2015
Homework 3 - Due Dec. 3rd, 2014

1. Consider the population dynamics of a predator-prey system. At a given time, there are x predators and y prey animals. The time evolution of the system is modeled by a Markov Chain such that

$$P(X_t = x', Y_t = y' | X_{t-1} = x, Y_{t-1} = y) = \frac{1}{c(x,y)} \times \begin{cases} s, & (x', y') = (x, y), \\ \alpha y, & (x', y') = (x, y+1), \\ \beta xy, & (x', y') = (x, y-1), \\ \gamma xy, & (x', y') = (x+1, y), \\ \delta x, & (x', y') = (x-1, y), \\ 0, & \text{otherwise}, \end{cases}$$

where $s, \alpha, \beta, \gamma, \delta > 0$.

   (a) Calculate the normalization factor $c(x, y)$.

   (b) Is this Markov chain irreducible?

   (c) Describe the long-time behaviour of the each of the populations if $\beta = \gamma = 0$.

   (d) Implement this Markov chain and calculate the average number of time steps until either of the species becomes extinct. Use $s, \alpha, \beta, \gamma, \delta = 0.1$ and initial populations of $x = 100$, $y = 1000$. Run many simulations and estimate the error on your result.

2. Consider a queue which consists of customers waiting in line to be served. We shall consider the number customers immediately after the nth service, Q_n. Assume service takes a fixed amount of time $d = 1$, and that the number of arrivals during a time interval d is Poisson distributed with intensity λ.

   (a) Show that $Q_{n+1} = Q_n + p - h(Q_n)$, where p is drawn from a Poisson distribution of intensity λ and

$$h(x) = \begin{cases} 0, & x = 0, \\ 1, & \text{otherwise}. \end{cases}$$

   (b) Implement this Markov chain and discuss the behavior for the two cases $\lambda < 1$ and $\lambda > 1$.

(c) Modify the chain to treat the case where the service times are exponentially distributed with mean $d = 1$. This is called the $M/M/1$ queue while the system discussed above is the $M/D/1$ queue. Consider the heavy traffic limit, where $\lambda \to 1$ from below. Try $\lambda \approx 0.95 - 0.99$ and plot a histogram of $(1 - \lambda)Q$ for both queues for many iterations. How are the histograms different? Hint: They should both be exponential distributions, but with different parameters.