MA3412, Hilary Term 2010.
Problems concerning Rings, Modules and Algebraic Sets

1. Factorize the following Gaussian integers as products of irreducible elements of the ring $\mathbb{Z}[\sqrt{-1}]$: 5, $3+4i$, 14, $7+6i$.

2. Let R be a unital commutative ring.

   (a) Let I, J and K be ideals of R. Verify that

   $$I + J = J + I, \qquad IJ = JI,$$

   $$(I + J) + K = I + (J + K), \qquad (IJ)K = I(JK),$$
   $$(I + J)K = IK + JK, \qquad I(J + K) = IJ + IK.$$

   Explain why the set of ideals of a ring R is *not* itself a unital commutative ring with respect to these operations of addition and multiplication.

   (b) Let I and J be ideals of R satisfying $I + J = R$. Show that $(I + J)^n \subset I + J^n$ for all natural numbers n and hence prove that $I + J^n = R$ for all n. Thus show that $I^m + J^n = R$ for all natural numbers m and n. (The ideal J^n is by definition the set of all elements of R that can be expressed as a finite sum of elements of R of the form $a_1 a_2 \cdots a_n$ with $a_i \in J$ for $i = 1, 2, \ldots, n$.)

   (c) Let I and J be ideals of R satisfying $I + J = R$. By considering the ideal $(I \cap J)(I + J)$, or otherwise, show that $IJ = I \cap J$.

3. Let R be a unital commutative ring, and let M and N be R-modules. Show that the set $\mathrm{Hom}_R(M, N)$ of all R-module homomorphisms from M to N is itself an R-module (where the algebraic operations on $\mathrm{Hom}(M, N)$ are defined in an obvious natural way).

4. (a) Show that the cubic curve $\{(t, t^2, t^3) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is an algebraic set.

   (b) Show that the cone $\{(s \cos t, s \sin t, s) \in \mathbb{A}^3(\mathbb{R}) : s, t \in \mathbb{R}\}$ is an algebraic set.

   (c) Show that the unit sphere $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ in $\mathbb{A}^2(\mathbb{C})$ is not an algebraic set.

(d) Show that the curve $\{(t\cos t, t\sin t, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is not an algebraic set.

5. Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.

   Let V and W be algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Show that the Cartesian product $V \times W$ of V and W is an algebraic set in $\mathbb{A}^{m+n}$, where

$$\begin{aligned} V \times W \quad = \quad & \{(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \in \mathbb{A}^{m+n} : \\ & \qquad (x_1, x_2, \ldots, x_m) \in V \text{ and } (y_1, y_2, \ldots, y_n) \in W\}. \end{aligned}$$

6. Give an example of a proper ideal I in $\mathbb{R}[X]$ with the property that $V[I] = \emptyset$. [Hint: consider quadratic polynomials in X.]

7. Let I be the ideal of the polynomial ring $\mathbb{R}[X, Y, Z]$ generated by the polynomials XY, YZ and XY. Determine the corresponding algebraic set $V(I)$.

8. Let R be an integral domain, and let F be a free module of rank n over R (where n is some positive integer). Let $\text{Hom}_R(F, R)$ be the R-module consisting of all R-module homorphisms from F to R. Prove that $\text{Hom}_R(F, R)$ is itself a free module of rank n.