Problems for MA3412
Hilary Term 2010
Solutions concerning Commutative Algebra

1. *Factorize the following Gaussian integers as products of irreducible elements of the ring* $\mathbb{Z}[\sqrt{-1}]$*:* 5, $3+4i$, 14, $7+6i$.

2. *Let* R *be a unital commutative ring.*

   (a) *Let* I, J *and* K *be ideals of* R*. Verify that*

$$I + J = J + I, \qquad IJ = JI,$$

$$(I + J) + K = I + (J + K), \qquad (IJ)K = I(JK),$$

$$(I + J)K = IK + JK, \qquad I(J + K) = IJ + IK.$$

   Explain why the set of ideals of a ring R *is not itself a unital commutative ring with respect to these operations of addition and multiplication.*

   (b) *Let* I *and* J *be ideals of* R *satisfying* $I + J = R$*. Show that* $(I + J)^n \subset I + J^n$ *for all natural numbers* n *and hence prove that* $I + J^n = R$ *for all* n*. Thus show that* $I^m + J^n = R$ *for all natural numbers* m *and* n*. (The ideal* J^n *is by definition the set of all elements of* R *that can be expressed as a finite sum of elements of* R *of the form* $a_1 a_2 \cdots a_n$ *with* $a_i \in J$ *for* $i = 1, 2, \ldots, n$*.)*

   (c) *Let* I *and* J *be ideals of* R *satisfying* $I + J = R$*. By considering the ideal* $(I \cap J)(I + J)$*, or otherwise, show that* $IJ = I \cap J$*.*

3. *Let* R *be a unital commutative ring, and let* M *and* N *be* R*-modules. Show that the set* $\mathrm{Hom}_R(M, N)$ *of all* R*-module homomorphisms from* M *to* N *is itself an* R*-module (where the algebraic operations on* $\mathrm{Hom}(M, N)$ *are defined in an obvious natural way).*

4. (a) *Show that the cubic curve* $\{(t, t^2, t^3) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ *is an algebraic set.*

   Solution. The given set is the algebraic set

$$\{(x, y, z) \in \mathbb{A}^3(\mathbb{R}) : y - x^2 = 0 \text{ and } z - x^3 = 0\}.$$

(b) *Show that the cone* $\{(s\cos t, s\sin t, s) \in \mathbb{A}^3(\mathbb{R}) : s, t \in \mathbb{R}\}$ *is an algebraic set.*

Solution. The given set is the algebraic set

$$\{(x, y, z) \in \mathbb{A}^3(\mathbb{R}) : x^2 + y^2 - z^2 = 0\}.$$

(c) *Show that the unit sphere* $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ *in* $\mathbb{A}^2(\mathbb{C})$ *is not an algebraic set.*

Solution. The affine one-dimensional subspace

$$\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : w = 0\}$$

intersects this unit sphere in the circle

$$\{(z, 0) \in \mathbb{A}^2(\mathbb{C}) : z \in \mathbb{C} \text{ and } |z| = 1\}.$$

This circle is an infinite set that is not the whole of the affine one-dimensional subspace. Thus the affine one-dimensional subspace $w = 0$ is neither contained in the unit sphere, nor does it intersect the unit sphere in a finite set of points. One of these alternatives would have to hold, were the unit sphere an algebraic set. It follows that the unit sphere cannot be an algebraic set.

(d) *Show that the curve* $\{(t\cos t, t\sin t, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ *is not an algebraic set.*

Solution. This set is a curve, which is a helix or spiral in $\mathbb{R}^3$. It intersects the line

$$\{(x, y, z) \in \mathbb{A}^3(\mathbb{R}) : x = 1 \text{ and } y = 0\}$$

in an infinite set

$$\{(1, 0, 2\pi n) : n \in \mathbb{Z}\}$$

that is a proper subset of the line. It follows that the helix is not an algebraic set.

5. *Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.*

   Let V and W be algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Show that the Cartesian product $V \times W$ of V and W is an algebraic set in $\mathbb{A}^{m+n}$, where

$$\begin{aligned} V \times W &= \{(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \in \mathbb{A}^{m+n} : \\ &\qquad (x_1, x_2, \ldots, x_m) \in V \textit{ and } (y_1, y_2, \ldots, y_n) \in W\}. \end{aligned}$$

6. *Give an example of a proper ideal I in $\mathbb{R}[X]$ with the property that $V[I] = \emptyset$. [Hint: consider quadratic polynomials in X.]*

   Solution. Let I be the ideal of $\mathbb{R}[X]$ generated by the polynomial $X^2 + 1$.

7. *Let I be the ideal of the polynomial ring $\mathbb{R}[X, Y, Z]$ generated by the polynomials XY, YZ and XZ. Determine the corresponding algebraic set $V(I)$.*

   Solution. Let $(x, y, z) \in V(I)$. Then $xy = 0$, $yz = 0$ and $xz = 0$. If $x \neq 0$ then $y = 0$ and $z = 0$. If $y \neq 0$ then $x = 0$ and $z = 0$. If $z \neq 0$ then $x = 0$ and $y = 0$. It follows that the algebraic set is the union of the three lines $\{(x, 0, 0) : x \in \mathbb{R}\}$, $\{(0, y, 0) : y \in \mathbb{R}\}$, $\{(0, 0, z) : z \in \mathbb{R}\}$ corresponding to the coordinate axes.

8. *Let R be an integral domain, and let F be a free module of rank n over R (where n is some positive integer). Let* $\mathrm{Hom}_R(F, R)$ *be the R-module consisting of all R-module homorphisms from F to R. Prove that* $\mathrm{Hom}_R(F, R)$ *is itself a free module of rank n.*

   Solution. Let $f_1, f_2, \ldots, f_n$ be a free basis for the R-module F. Then, for each integer j between 1 and n there is an R-module homomophism ϵ_j defined such that

   $$\epsilon_j(r_1 f_1 + r_2 f_2 + \cdots + r_n f_n) = r_j$$

   for all $r_1, r_2, r_3, \ldots, r_n$.
   Let $\theta \colon F \to R$ be an R-module homomorphism from F to R, and let $s_j = \theta(f_j)$ for $j = 1, 2, \ldots, n$. Let x be an element of F. Then there exist uniquely determined elements $r_1, r_2, \ldots, r_n$ of R such that

   $$x = r_1 f_1 + r_2 f_2 + \cdots + r_n f_n.$$

   Then $r_j = \epsilon_j(x)$ for $j = 1, 2, \ldots, n$, and therefore

   $$\begin{aligned} \theta(x) &= \theta(r_1 f_1 + r_2 f_2 + \cdots + r_n f_n) \\ &= r_1 \theta(f_1) + r_2 \theta(f_2) + \cdots + r_n \theta(f_n) \\ &= r_1 s_1 + r_2 s_2 + \cdots + r_n s_n \\ &= s_1 \epsilon_1(x) + s_2 \epsilon_2(x) + \cdots + s_n \epsilon_n(x) \\ &= (s_1 \epsilon_1 + s_2 \epsilon_2 + \cdots + s_n \epsilon_n)(x). \end{aligned}$$

It follows that

$$\theta = s_1\epsilon_1 + s_2\epsilon_2 + \cdots + s_n\epsilon_n.$$

Moreover $s_1, s_2, \ldots, s_n$ are uniquely determined by θ. Indeed if

$$\theta = s'_1\epsilon_1 + s'_2\epsilon_2 + \cdots + s'_n\epsilon_n$$

for some $s'_1, s'_2, \ldots, s'_n \in R$, then

$$s_j = \theta(f_j) = \sum_{k=1}^{n} s'_k\epsilon_k(f_j) = s_j,$$

since $\epsilon_j(f_j) = 1_R$ and $\epsilon_k(f_j) = 0_R$ when $j \neq k$. We conclude that $\mathrm{Hom}_R(F, R)$ is a free module of rank n over R which is freely generated by $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.